NON-KÄHLER COMPLEX STRUCTURES ON R^4 II

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ABSTRACT. We follow our study of non-Kähler complex structures on $\mathbf{R}^4$ that we defined in [4]. We prove that these complex surfaces do not admit any smooth complex compactification. Moreover, we give an explicit description of their meromorphic functions. We also prove that the Picard groups of these complex surfaces are uncountable, and give an explicit description of the canonical bundle. Finally, we show that any connected non-compact oriented 4-manifold admits complex structures without Kähler metrics.

1. INTRODUCTION

In [4] we constructed a two-parameters family of pairwise not biholomorphic complex structures $J(\rho_1, \rho_2)$ on $\mathbf{R}^4$, where $1 < \rho_2 < \rho_1^{-1}$. It is remarkable that these complex surfaces cannot be tamed by any symplectic structure. In particular, they have no compatible Kähler metrics. In addition, there is a surjective map $f\colon \mathbf{R}^4 \to \mathbf{CP}^1$, which is holomorphic with respect to any $J(\rho_1, \rho_2)$. We denote by $E(\rho_1, \rho_2)$ the complex surface $(\mathbf{R}^4, J(\rho_1, \rho_2))$. By following [4], we give the following definition.

DEFINITION 1. A complex manifold M is said to be of *Calabi-Eckmann type* if there is a holomorphic immersion $k\colon X \to M$, with X a compact complex manifold of positive dimension, such that k is null-homotopic as a continuous map. Such a compact complex immersed submanifold is said to be *homotopically trivial*.

Notice that if M is of Calabi-Eckmann type, then it cannot be tamed by a symplectic form, and hence M cannot be endowed with a Kähler metric.

The motivation for this definition is that Calabi and Eckmann in [2] constructed such complex structures on $\mathbf{R}^{2n}$, for $n \geq 3$, arising as open subsets of certain compact complex n-manifolds (which are diffeomorphic to the product of two odd-dimensional spheres).

On the other hand, the manifolds $E(\rho_1, \rho_2)$ are the first examples of Calabi-Eckmann type complex surfaces [4]. However, as the following main theorem states, our surfaces cannot be realised as analytic subsets of any compact complex non-singular surface.

THEOREM 2. *$E(\rho_1, \rho_2)$ cannot be holomorphically embedded in any compact complex surface.*

For a complex manifold X, let $\mathcal{M}(X)$ denote the field of meromorphic functions of X. Moreover, we denote by $\mathrm{Pic}(X)$ the Picard group of X, and by ω_X the canonical line bundle of X.

Let $\mathcal{O}_{\mathbf{CP}^1}(k)$ be the holomorphic line bundle on $\mathbf{CP}^1$ with first Chern number $k \in \mathbf{Z}$, and consider the induced holomorphic line bundle $L_k = f^*(\mathcal{O}_{\mathbf{CP}^1}(k))$ on $E(\rho_1, \rho_2)$. By abusing notation, we denote by f^* also the pullback homomorphisms determined by f

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between the groups that we consider in the next theorem. This shall not be misleading, being the exact meaning of the symbol f^* clear from the context.

Theorem 3. *The following properties hold:*

(1) $f^*\colon \mathcal{M}(\mathbf{CP}^1) \to \mathcal{M}(E(\rho_1, \rho_2))$ *is an isomorphism.*
(2) $f^*\colon \mathrm{Pic}(\mathbf{CP}^1) \cong \mathbf{Z} \to \mathrm{Pic}(E(\rho_1, \rho_2))$ *is injective and maps* $\mathcal{O}_{\mathbf{CP}^1}(-2)$ *to* $\omega_{E(\rho_1,\rho_2)}$*, namely* $\omega_{E(\rho_1,\rho_2)} = L_{-2}$.
(3) $\mathrm{Pic}(E(\rho_1, \rho_2))$ *is uncountable.*

Theorem 4. *The total space of a holomorphic vector bundle of rank n over $E(\rho_1, \rho_2)$ determines a Calabi-Eckmann type complex structure on $\mathbf{R}^{2n+4}$. Moreover, the total spaces of the Whitney sums $L_{k_1} \oplus L_{k_2} \oplus \cdots \oplus L_{k_n}$, with $k_1 \le k_2 \le \cdots \le k_n$, are pairwise not biholomorphic to each other for all choices of the parameters $1 < \rho_2 < \rho_1^{-1}$ and $(k_1, k_2, \cdots, k_n) \in \mathbf{Z}^n$.*

Remark 5. The complex structures on $\mathbf{R}^{2n}$ that arise as in the statement of Theorem 4 are not biholomorphic to those constructed by Calabi and Eckmann in [2]. Indeed, by our construction, $\mathbf{R}^{2n}$ admits an immersed rational curve with one node, while the only compact curves in the Calabi and Eckmann examples are smooth elliptic curves.

Finally, we give a general result for Calabi-Eckmann type complex surfaces.

Theorem 6. *Any non-compact connected oriented smooth 4-manifold M admits a complex structure of Calabi-Eckmann type, which is compatible with the given orientation.*

The paper is organized as follows. In Section 2 we recall the construction of the complex manifolds $E(\rho_1, \rho_2)$ as it is given in [4], and we also fix the notations. In Section 3, we recall some classification results for compact complex surfaces that are needed in the proof of Theorem 2. In Section 4 we prove Theorems 2 and 3. Finally, in Section 5 we prove Theorems 4 and 6.

2. Non-Kähler complex structures on $\mathbf{R}^4$

Consider the complex annulus $\Delta(r_0, r_1) = \{z \in \mathbf{C} \mid r_0 < |z| < r_1\}$ and the disk $\Delta(r) = \{z \in \mathbf{C} \mid |z| < r\}$. Fix positive numbers ρ_0, ρ_1 and ρ_2 such that $1 < \rho_2 < \rho_1^{-1} < \rho_0^{-1}$.

Let $f_1\colon W_1 \to \Delta(\rho_1)$ be a relatively minimal elliptic holomorphic Lefschetz fibration with one singular fiber. In other words, W_1 is a fibered tubular neighborhood of a singular fiber of type I_1 in an elliptic complex surface by Kodaira results [10, 12].

Following Kodaira [11], we can represent f_1 as follows. First, we consider the quotient $W_1' = (\mathbf{C}^* \times \Delta(0, \rho_1))/\mathbf{Z}$, where the action is given by $n \cdot (z, w) = (zw^n, w)$. The projection on the second factor $W_1' \to \Delta(0, \rho_1)$ is a holomorphic elliptic fibration that is equivalent to f_1 restricted over $\Delta(0, \rho_1)$. Therefore, we can consider W_1' as an open subset of W_1 (precisely, as the complement of the singular fiber).

Let $W_2 = \Delta(1, \rho_2) \times \Delta(\rho_0^{-1})$ be endowed with the product complex structure, and let $f_2\colon W_2 \to \Delta(\rho_0^{-1})$ be the canonical projection. Consider also the open subset $V_2 = \Delta(1, \rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset W_2$.

Now, consider the multi-valued holomorphic function $\varphi\colon \Delta(\rho_0, \rho_1) \to \mathbf{C}^*$ defined by

$$\varphi(w) = \exp\left(\frac{1}{4\pi i}(\log w)^2 - \frac{1}{2}\log w\right).$$

As it can be easily checked, the image of the map $\Phi\colon \Delta(1, \rho_2) \times \Delta(\rho_0, \rho_1) \to \mathbf{C}^* \times \Delta(0, \rho_1)$ defined by $\Phi(z, w) = (z\varphi(w), w)$, is invariant under the above $\mathbf{Z}$-action. Therefore, the composition of Φ with the projection map $\pi\colon \mathbf{C}^* \times \Delta(0,\rho_1) \to W_1' \subset W_1$ determined by the $\mathbf{Z}$-action, is a single-valued holomorphic embedding, and let $V_1 \subset W_1$ be the image of $\pi \circ \Phi$. Notice that $\pi \circ \Phi$ gives an identification of V_1 with the product $\Delta(1,\rho_2) \times \Delta(\rho_0,\rho_1)$.

We are now ready to glue W_1 and W_2 together by identifying V_1 and V_2 via the biholomorphism $j\colon V_1 = \Delta(1,\rho_2) \times \Delta(\rho_0,\rho_1) \to V_2$ defined by $j(z,w) = (z, w^{-1})$. We obtain the complex manifold $E(\rho_1,\rho_2) = W_1 \cup_j W_2$. The result does not depend on the parameter ρ_0 up to biholomorphisms, since ρ_0 determines only the size of the gluing region.

Moreover, we can define a surjective holomorphic map $f\colon E(\rho_1,\rho_2) \to \mathbf{CP}^1$ by putting $f(x) = f_1(x)$ if $x \in W_1$ and $f(x) = f_2(x)$ if $x \in W_2$, where $\mathbf{CP}^1$ is regarded as the Riemann sphere resulting from the gluing $\Delta(\rho_1) \cup_h \Delta(\rho_0^{-1})$, with $h\colon \Delta(\rho_0,\rho_1) \to \Delta(\rho_1^{-1},\rho_0^{-1})$ defined by $h(w) = w^{-1}$.

It can be proved that $E(\rho_1,\rho_2)$ is diffeomorphic to $\mathbf{R}^4$ essentially by means of the Kirby calculus applied to Lefschetz fibrations, see Gompf and Kirby [7] and Apostolakis, Piergallini and Zuddas [1]. In [4] this has been done by relating the decomposition of $E(\rho_1,\rho_2)$ given in its definition with a decomposition of $\mathbf{R}^4 \subset S^4$ determined by the Matsumoto-Fukaya fibration, which is a genus-one achiral Lefschetz fibration $S^4 \to S^2$ [16]. Actually, the map f above coincides with the restriction of this fibration on $\mathbf{R}^4 \subset S^4$.

It follows that the fibers of W_1 have null-homotopic inclusion map in the contractible space $E(\rho_1,\rho_2)$, namely they are homotopically trivial. Then, the complex manifold $E(\rho_1,\rho_2) \cong \mathbf{R}^4$ is of Calabi-Eckmann type by Definition 1.

Moreover, the only compact holomorphic curves in $E(\rho_1,\rho_2)$ are the compact fibers of f, see [4, Proposition 5].

3. The classification of compact complex surfaces

According to Kodaira [15, Theorem 55] (see also [10, 14]), compact complex surfaces are classified into the following seven classes, up to blow ups and blow downs.

(I) $\mathbf{CP}^2$ or $\mathbf{CP}^1$-bundles over a compact complex curve;
(II) $K3$ surfaces;
(III) complex tori;
(IV) Kähler elliptic surfaces;
(V) algebraic surfaces of general type;
(VI) elliptic surfaces whose first Betti number b_1 is odd and greater than 1;
(VII) surfaces with $b_1 = 1$.

Surfaces of class I to V are Kähler, while surfaces of classes VI and VII are non-Kähler, because of the following theorem.

Theorem 7 (Miyaoka [17] and Siu [19]). *A compact complex surface is Kähler if and only if the first Betti number is even.*

We recall also the following theorem.

Theorem 8 (Chow and Kodaira [3] and Kodaira [10]). *A compact complex surface is algebraic if the algebraic dimension is two, and it is elliptic if the algebraic dimension is one.*

Thus, the algebraic dimension of a non-Kähler surface is 0 or 1. Hence it is 1 for class VI. On the other hand, class VII is divided into the two cases where the algebraic dimension is 0 or 1. If it is 0, then X has at most finitely many compact holomorphic curves by Kodaira [10]. Otherwise, X is an elliptic surface. Such elliptic surfaces are further classified according as if they are minimal or not minimal. If X is a minimal elliptic surface of class

VII, then X is obtained from the product $\mathbf{CP}^1 \times E$ by a finite sequence of logarithmic transformations, where E is a smooth elliptic curve [13]. If X is not minimal, then it is obtained from a minimal one by blow ups.

4. The proofs of Theorems 2 and 3

Proof of Theorem 2. The proof is by contradiction. So, suppose that $E(\rho_1, \rho_2)$ is embedded as a domain in a compact complex surface X, and let $i\colon E(\rho_1, \rho_2) \to X$ be a holomorphic embedding.

Recall that $E(\rho_1, \rho_2)$ is not Kähler. Hence, X cannot be Kähler, and so it is of class VI or VII. Moreover, the algebraic dimension is 0 or 1. If it is 0, then X contains at most finitely many compact holomorphic curves, while $E(\rho_1, \rho_2)$ does contain infinitely many, which is impossible. Hence, the algebraic dimension must be 1. Therefore, X has an elliptic fibration $g\colon X \to \Sigma$ over a compact complex curve.

Notice that any compact fiber F of $f\colon E(\rho_1, \rho_2) \to \mathbf{CP}^1$ is homotopically trivial in $E(\rho_1, \rho_2)$, because $E(\rho_1, \rho_2)$ is contractible. Then $(g \circ i)_{|F}\colon F \to \Sigma$ is holomorphic and null-homotopic as a continuous map, and so it is of zero degree, implying that $g \circ i$ is constant on F. Indeed, a holomorphic non-constant map between compact Riemann surfaces is open (hence surjective), and it has positive degree because it is orientation-preserving at the regular points.

Therefore, by compactness, $i(F)$ is a fiber of g. In other words, i is fiber-preserving in the elliptic part of $E(\rho_1, \rho_2)$.

It follows that $g \circ i$ is constant on any fiber of f, even in the non-compact ones, because the vertical derivative of $g \circ i$, that is the tangent map restricted to the kernel of Tf, is null in the elliptic part of $E(\rho_1, \rho_2)$, which is an open subset of $E(\rho_1, \rho_2)$, and so, by analyticity, it must be null everywhere. Therefore, $i\colon E(\rho_1, \rho_2) \to X$ is a fiberwise embedding. This also implies that Σ must be $\mathbf{CP}^1$, and we can regard $f\colon E(\rho_1, \rho_2) \to \mathbf{CP}^1$ as a restriction of the fibration $g\colon X \to \mathbf{CP}^1$ by identifying $E(\rho_1, \rho_2)$ with its image $i(E(\rho_1, \rho_2)) \subset X$.

Now, we first consider the case when X is of class VI. In this case, X is an elliptic surface over $\mathbf{CP}^1$ such that $b_1(X) \geq 3$. Since $\pi_1(T^2) = \mathbf{Z}^2$ and $\mathbf{CP}^1$ is simply connected, the first Betti number $b_1(X)$ must be smaller than 3. This is a contradiction.

Next, we consider X of class VII. Suppose that X is a minimal elliptic surface. Such surfaces are classified as in Section 3. There is an elliptic fibration $\pi\colon X \to \mathbf{CP}^1$ having only finitely many multiple fibers that are smooth elliptic curves, and no singular fibers.

On the other hand, the fibration $f\colon E(\rho_1, \rho_2) \to \mathbf{CP}^1$ has a singular fiber which is an immersed sphere with one node, and by the above argument we have $f = \pi \circ i$. Thus, π has a singular fiber, which is a contradiction.

Finally, we consider the case where X is a non-minimal elliptic surface. Let $C \subset X$ be an exceptional curve of the first kind. Since the rational curve C cannot be a fiber of $\pi\colon X \to \mathbf{CP}^1$, it must be a branched multi-section of π. Then, C has a positive intersection number with an elliptic fiber of $f\colon E(\rho_1, \rho_2) \to \mathbf{CP}^1$. This is a contradiction, since $E(\rho_1, \rho_2)$ is contractible. □

Remark 9. Ichiro Enoki, in a private communication, told us that there is no compact complex surface of Calabi-Eckmann type. It is proven as follows. If a compact complex surface contains a compact holomorphic curve representing a homologically trivial 2-cycle, then it must be a surface of Class VI, a Hopf surface, a parabolic Inoue surface, or an Enoki surface [5]. In each case, the holomorphic curve contains a loop representing a generator of the fundamental group of the surface. This fact leads to an alternative proof of Theorem 2.

Proof of Theorem 3. We begin with statement (1). Let h be a meromorphic function on $E(\rho_1, \rho_2)$ and let $(h) = (h)_0 - (h)_\infty$ be the associated divisor, see Griffiths and Harris [8, pages 36, 131]. Then, $(h)_0$ and $(h)_\infty$ are codimension-1 analytic subsets and the indeterminacy set $N = (h)_0 \cap (h)_\infty$ is an analytic subset of codimension greater than 1, implying that it is a finite set.

The restriction of the meromorphic function h to $E(\rho_1, \rho_2) - N$ is a holomorphic map $h_| \colon E(\rho_1, \rho_2) - N \to \mathbf{CP}^1$. All but finitely many compact fibers of f are contained in $E(\rho_1, \rho_2) - N$, and they are also homotopically trivial (cf. the end of Section 2) in the complement of N.

By the same argument in the proof of Theorem 2, we see that h is constant on the compact fibers of f, and then it is constant on all fibers of f. Therefore, h is the pullback of a meromorphic function s on $\mathbf{CP}^1$, implying that $f^* \colon \mathcal{M}(\mathbf{CP}^1) \to \mathcal{M}(E(\rho_1, \rho_2))$ is surjective. On the other hand, f^* is clearly injective, and this concludes the proof of (1).

Now, we prove the statement (2). As in the Introduction, let $\mathcal{O}_{\mathbf{CP}^1}(k)$ denote the holomorphic line bundle over $\mathbf{CP}^1$ with first Chern number $k \in \mathbf{Z}$. It is well known that $\mathcal{O}_{\mathbf{CP}^1}(k)$ is holomorphically trivial if and only if $k = 0$.

Consider the decomposition $E(\rho_1, \rho_2) = W_1 \cup_j W_2$ as in Section 2. So, W_1 and W_2 are the preimages by f of two open disks $\Delta_1 = \Delta(\rho_1)$ and $\Delta_2 = \Delta(\rho_0^{-1})$ whose union is $\mathbf{CP}^1$. We have to show that if $L_k = f^*(\mathcal{O}_{\mathbf{CP}^1}(k))$ is trivial, then $k = 0$.

Since $\mathcal{O}_{\mathbf{CP}^1}(k)$ is holomorphically trivial over each of the disks Δ_1 and Δ_2, there are nowhere vanishing holomorphic sections σ_i of $\mathcal{O}_{\mathbf{CP}^1}(k)$ over Δ_i, for $i = 1, 2$. Then, the pullback section $f^*(\sigma_i)$ is a trivialization of L_k over W_i, $i = 1, 2$.

If L_k is holomorphically trivial, there is a nowhere vanishing global holomorphic section τ. Hence, there are holomorphic functions $\tau_i \in \mathcal{O}^*(W_i)$, $i = 1, 2$, such that

$$\tau_{|W_i} = \tau_i f^*(\sigma_i)$$

Now, we show that τ_1 and τ_2 are the pullbacks of holomorphic functions on Δ_1 and Δ_2. On W_1, the fibers of f are all compact holomorphic curves. Hence, τ_1 is constant along the fibers of f. Thus, τ_1 is the pullback $f^*(u_1)$ of some holomorphic function u_1 on Δ_1. Then, on the intersection $V = W_1 \cap W_2$, we have

$$f^*(u_1 \sigma_1) = \tau_2 f^*(\sigma_2).$$

Hence, τ_2 is fiberwise constant on V. By analyticity, it is fiberwise constant over W_2. Hence, there is a holomorphic function u_2 on Δ_2 such that $\tau_2 = f^*(u_2)$. Then, $u_1\sigma_1$ and $u_2\sigma_2$ define a nowhere vanishing holomorphic global section of $\mathcal{O}_{\mathbf{CP}^1}(k)$. Hence $\mathcal{O}_{\mathbf{CP}^1}(k)$ is trivial, and so $k = 0$.

Next, we prove that $\omega_{E(\rho_1,\rho_2)} = L_{-2}$.

Set $Y = E(\rho_1, \rho_2)$. The canonical line bundle ω_Y is by definition the determinant of the contangent bundle T^*Y, that is $\omega_Y = \Lambda^2(T^*Y)$.

Let (z, w) be the coordinates of $\mathbf{C}^* \times \Delta(\rho_1)$. An easy computation shows that the holomorphic 2-form

$$\sigma = \frac{\mathrm{d}z \wedge \mathrm{d}w}{z}$$

passes to quotient $\mathbf{C}^* \times \Delta(0, \rho_1)/\mathbf{Z}$, where, as above, the action is given by $n \cdot (z, w) = (zw^n, w)$.

Since σ is defined on $\mathbf{C}^* \times \Delta(\rho_1)$, it induces a regular holomorphic 2-form on

$$\big(\mathbf{C}^* \times \Delta(0,\rho_1)/\mathbf{Z}\big) \cup \mathbf{C}^* = W_1 - \{q\}\,,$$

where q is the nodal singularity of the singular elliptic fiber. By Hartog's theorem, it can be extended over the point q. Hence, σ determines a regular holomorphic 2-form on the whole elliptic fibration W_1.

Let (u,t) be the coordinates of $\Delta(1,\rho_2) \times \Delta(\rho_0,\rho_1)$. By the above identification of V with $\Delta(1,\rho_2) \times \Delta(\rho_0,\rho_1)$ we see that:

$$\sigma = \frac{\mathrm{d}u \wedge \mathrm{d}t}{u}$$

on $\Delta(1,\rho_2) \times \Delta(\rho_0,\rho_1)$. By means of the biholomorphism j we get that

$$\sigma = -\frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}s}{s^2}$$

where here (z,s) are coordinates of $\Delta(1,\rho_2) \times \Delta(\rho_0^{-1})$.

Thus, σ gives rise to a meromorphic section of the canonical bundle of Y. The polar set of σ is $2F_2$, where F_2 is an annulus fiber of the map f. Hence we obtain that

$$\omega_Y = f^*(\mathcal{O}_{\mathbf{CP}^1}(-2)).$$

Finally, we prove statement (3). For a complex manifold Y, it is well known that the exponential function gives rise to the following short exact sequence of sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0\ .$$

For $Y = E(\rho_1,\rho_2)$, being diffeomorphic to $\mathbf{R}^4$, we obtain the following isomorphism from the associated long exact sequence of cohomology groups

$$\mathrm{Pic}(Y) = \mathrm{H}^1(Y,\mathcal{O}_Y^*) \cong \mathrm{H}^1(Y,\mathcal{O}_Y).$$

By definition, $\mathrm{H}^1(Y,\mathcal{O}_Y)$ is a complex vector space. Hence, $\mathrm{Pic}(E(\rho_1,\rho_2))$ is isomorphic to the additive group of a complex vector space. We have already proved that $\mathrm{Pic}(E(\rho_1,\rho_2))$ is not trivial. Therefore, it is uncountable. $\square$

5. The proofs of Theorems 4 and 6

Proof of Theorem 4. The first sentence of the statement is straightforward, because the Calabi-Eckamnn type surface $E(\rho_1,\rho_2)$ is holomorphically embedded in the total space as the zero section of the bundle. We prove the second sentence.

Let $Q^{n+2}(\rho_1,\rho_2,k) \cong \mathbf{R}^{2n+4}$ be the total space of the holomorphic vector bundle $L_{k_1} \oplus \dots \oplus L_{k_n}$, and let π_k be its projection map, that is $\pi_k\colon Q^{n+2}(\rho_1,\rho_2,k) \to E(\rho_1,\rho_2)$, with $k = (k_1,\dots,k_n)$.

Put $\xi_k = \mathcal{O}_{\mathbf{CP}^1}(k_1) \oplus \dots \oplus \mathcal{O}_{\mathbf{CP}^1}(k_n)$. We have $\pi_k = f^*(\xi_k)$. Then,

$$Q^{n+2}(\rho_1,\rho_2,k) = \{(x,y) \in E(\rho_1,\rho_2) \times E(\xi_k) \mid f(x) = \xi_k(y)\},$$

where we denote by $E(\xi_k)$ the total space of ξ_k. The map π_k is the projection on the first factor, and we denote by $\tilde{f}\colon Q^{n+2}(\rho_1,\rho_2,k) \to E(\xi_k)$ the projection on the second factor, as it is shown in the following commutative diagram.

$$\begin{tikzcd} Q^{n+2}(\rho_1,\rho_2,k) \arrow[r, "\tilde f"] \arrow[d, "\pi_k"'] & E(\xi_k) \arrow[d, "\xi_k"] \\ E(\rho_1,\rho_2) \arrow[r, "f"'] & \mathbf{CP}^1 \end{tikzcd}$$

Next, we classify compact connected holomorphic curves on $Q^{n+2}(\rho_1,\rho_2,k)$. Let S be a compact Riemann surface and let $\iota\colon S \to Q^{n+2}(\rho_1,\rho_2,k)$ be a holomorphic immersion. The holomorphic map $\xi_k \circ \tilde f \circ \iota\colon S \to \mathbf{CP}^1$ is null-homotopic, because $Q^{n+2}(\rho_1,\rho_2,k)$ is contractible, and hence it is constant (cf. the proof of Theorem 2 again).

Therefore, $(\tilde f \circ \iota)(S)$ is contained in a fiber of ξ_k, which is biholomorphic to $\mathbf{C}^n$. This implies that $\tilde f \circ \iota$ is constant. Hence, $\pi_k \circ \iota\colon S \to E(\rho_1,\rho_2)$ must be a holomorphic immersion. By the classification of compact holomorphic curves on $E(\rho_1,\rho_2)$, $(\pi_k \circ \iota)(S)$ is a compact fiber of f.

It follows that compact connected holomorphic curves in $Q^{n+2}(\rho_1,\rho_2,k)$ are of the form $F_p \times \{y\}$ where $F_p = f^{-1}(p)$ is a compact fiber of f, and $\xi_k(y) = p$, for some $p \in \mathbf{CP}^1$.

Now, suppose that there exists a biholomorphism $\Phi : Q^{n+2}(\rho_1,\rho_2,k) \to Q^{n+2}(\rho_1',\rho_2',k')$, where $k' = (k_1',\dots,k_n') \in \mathbf{Z}^n$ is another n-tuple with not decreasing components, and $1 < \rho_2' < (\rho_1')^{-1}$. We are going to show that $(\rho_1,\rho_2,k) = (\rho_1',\rho_2',k')$.

By keeping the above notation, we consider the map $f'\colon E(\rho_1',\rho_2') \to \mathbf{CP}^1$ of the construction in Section 2, the bundle $\pi_{k'}$, and the projection map $\tilde f'\colon Q^{n+2}(\rho_1',\rho_2',k') \to E(\xi_{k'})$ such that $\xi_{k'} \circ \tilde f' = f' \circ \pi_{k'}$.

Let $F_p \times \{y\}$ be a compact connected holomorphic curve in $Q^{n+2}(\rho_1,\rho_2,k)$. Then, $\Phi(F_p \times \{y\}) = F'_{p'} \times \{y'\}$ for certain $p' \in \mathbf{CP}^1$ and $y' \in E(\xi_{k'})$ such that $\xi_{k'}(y') = p'$, with $F'_{p'} = (f')^{-1}(p')$.

By the construction of Section 2, there is a biholomorphism $F_p \cong F'_{p'}$ if and only if $p = p'$ (by considering the moduli of elliptic fibers).

By analyticity, the equality $\Phi(F_p \times \{y\}) = F'_p \times \{y'\}$ must hold for all $p \in \mathbf{CP}^1$ and for all $y \in \xi_k^{-1}(p)$, where $y' \in \xi_{k'}^{-1}(p)$ is uniquely determined by p and y. Therefore, there are two biholomorphisms $\varphi\colon E(\rho_1,\rho_2) \to E(\rho_1',\rho_2')$ and $\psi\colon E(\xi_k) \to E(\xi_{k'})$, such that $\Phi(x,y) = (\varphi(x),\psi(y))$ for all $(x,y) \in Q^{n+2}(\rho_1,\rho_2,k)$. An application of the main theorem of [4] yields $(\rho_1,\rho_2) = (\rho_1',\rho_2')$.

Note that ψ is a fiberwise biholomorphism (that is, $\xi_{k'} \circ \psi = \xi_k$), but is not necessarily a linear bundle isomorphism. So, we take the fiber derivative of ψ evaluated along the zero section of ξ_k. By identifying the fibers of ξ_k and $\xi_{k'}$ with the corresponding tangent spaces at a suitable point, we obtain a linear isomorphism $\xi_k \cong \xi_{k'}$ of holomorphic vector bundles. Then, the Birkhoff-Grothendieck theorem [9] tells us that $k = k'$. □

Proof of Theorem 6. Teichner and Vogt proved in an unpublished paper that any oriented 4-manifold admits a spin^c-structure (see Gompf and Stipsicz [7, Remark 5.7.5] for a sketch of their proof). By Gompf [6], a spin^c-structure is a homotopy class of complex structures over the 2-skeleton that are extendable over the 3-skeleton. Since M is non-compact, it has the homotopy type of a 3-complex. It follows that M admits an almost complex structure that is compatible with the given orientation.

There is a nowhere zero vector field on M, because M is non-compact. Thus, the tangent bundle TM, regarded as a rank-two complex vector bundle, splits as the Whitney sum of a complex line bundle ξ and a complex trivial line bundle ε^1, that is $TM = \xi \oplus \varepsilon^1$.

Let $t\colon M \to \mathbf{CP}^\infty$ be a classifying map for ξ, so that ξ is the pullback of the tautological line bundle over $\mathbf{CP}^\infty$. Up to homotopy we can assume that t takes values in the 3-skeleton of $\mathbf{CP}^\infty$, which is $\mathbf{CP}^1$.

Then, TM is isomorphic, as a real vector bundle, to a pullback of an oriented non-trivial rank-four real vector bundle over S^2. Since $\pi_1(\mathrm{SO}(4)) = \mathbf{Z}_2$, there is only one such bundle up to isomorphisms, which is equivalent to the restriction to $\mathbf{CP}^1 \subset \overline{\mathbf{CP}}^2 - \{\mathrm{pt}\}$ of $T(\overline{\mathbf{CP}}^2 - \{\mathrm{pt}\})$.

Hence, there is a monomorphism of real vector bundles $G\colon TM \to T(\overline{\mathbf{CP}}^2 - \{\mathrm{pt}\})$. By Phillips theorem [18], there is an orientation-preserving immersion $g\colon M \to \overline{\mathbf{CP}}^2 - \{\mathrm{pt}\}$.

Notice that $\overline{\mathbf{CP}}^2 - \{\mathrm{pt}\}$ admits a Calabi-Eckmann type complex structure, obtained by taking the blow up of $E(\rho_1, \rho_2)$ at a point (cf. [4, Corollary 6]). Let us denote by $P(\rho_1, \rho_2) \cong \overline{\mathbf{CP}}^2 - \{\mathrm{pt}\}$ such blow up. It is still true that any holomorphic torus in $P(\rho_1, \rho_2)$ away from the blow up point is contained in a 4-ball.

Now, take a 4-ball $D \subset M$ where the restriction of g is an embedding. Up to isotopy we can assume that $g(D)$ contains a holomorphic torus of $P(\rho_1, \rho_2)$. Then, the complex structure on M induced by g is of Calabi-Eckmann type. $\square$

Remark 10. The proof of Theorem 6 can be slightly modified to show that any non-compact connected oriented 4-manifold M can be immersed into $\mathbf{CP}^2$, implying the known fact that M admits a Kähler complex structure. With Theorem 6, we have shown that any non-compact connected oriented 4-manifold admits both Kähler and non-Kähler complex structures.

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