Riv. Mat. Univ. Parma (6) **4** (2001), 71-86

A. LOI and D. ZUDDAS (*)

# Some remarks on Bergmann metrics (**)

## 1 - Introduction

Let $L$ be a holomorphic line bundle on a compact complex manifold $M$. A Kähler metric on $M$ is *polarized* with respect to $L$ if the Kähler form $\omega_g$ associated to $g$ represents the Chern class $c_1(L)$ of $L$. Recall that if in a complex coordinate system $(z_1, \ldots, z_n)$ of $M$ the metric $g$ is expressed by a tensor $(g_{j\bar{k}})_{1 \leq j, \bar{k} \leq n}$ then $\omega_g$ is the $d$-closed $(1, 1)$-form defined by $\frac{i}{2\pi} \sum_{j, \bar{k} = 0}^{n} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$.

The line bundle $L$ is called a *polarization* of $(M, g)$. In terms of cohomology classes, a Kähler manifold $(M, g)$ admits a polarization if and only if $\omega_g$ is integral, i.e. its cohomology class $[\omega_g]_{dR}$ in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. The integrality of $\omega_g$ implies, by a well-known theorem of Kodaira, that $M$ is a projective algebraic manifold. This mean that $M$ admits a holomorphic embedding into some complex projective space $\mathbb{C}P^N$. In this case a polarization $L$ of $(M, g)$ is given by the restriction to $M$ of the hyperplane line bundle on $\mathbb{C}P^N$. Given a polarized Kähler metric $g$ with respect to $L$, one can find a hermitian metric $h$ on $L$ with its Ricci curvature form

(*) A. LOI: Struttura Dipartimentale di Matematica e Fisica, Università, Via Vienna 2, 07100 Sassari, Itlay, e-mail: loi@ssmain.uniss.it; D. ZUDDAS: Scuola Normale Superiore, Classe di Scienze, Piazza dei Cavalieri 7, 56126 Pisa, Italy, e-mail: zuddas@cibs.sns.it.

(**) Received April 4, 2001. AMS classification 53 C 55, 58 F 06.

$\mathrm{Ric}(h) = \omega_g$ (see Lemma 1.1 in [**12**]). Here $\mathrm{Ric}(h)$ is the 2-form on $M$ defined by the equation:

$$\mathrm{Ric}(h) = -\frac{i}{2\pi}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)), \tag{1}$$

for a trivializing holomorphic section $\sigma : U \subset M \to L \setminus \{0\}$ of $L$.

For each positive integer $k$, we denote by $L^{\otimes k}$ the $k$-th tensor power of $L$. It is a polarization of the Kähler metric $kg$ and the hermitian metric $h$ induces a natural hermitian metric $h^k$ on $L^{\otimes k}$ such that $\mathrm{Ric}(h^k) = kg$.

Denote by $H^0(M, L^{\otimes k})$ the space of global holomorphic sections of $L^{\otimes k}$. It is in a natural way a complex Hilbert space with respect to the norm

$$\|s\|_{h^k} = \langle s, s\rangle_{h^k} = \int_M h^k(s(x), s(x))\frac{\omega_g^n(x)}{n!} < \infty,$$

for $s \in H^0(M, L^{\otimes k})$.

For sufficiently large $k$ we can define a holomorphic embedding of $M$ into a complex projective space as follows. Let $(s_0, \ldots, s_{N_k})$, be a orthonormal basis for $(H^0(M, L^{\otimes k}), \langle\cdot, \cdot\rangle_{h^k})$ and let $\sigma : U \to L$ be a trivialising holomorphic section on the open set $U \subset M$. Define the map

$$\varphi_\sigma : U \to \mathbb{C}^{N_k+1}\setminus\{0\} : x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \ldots, \frac{s_{N_k}(x)}{\sigma(x)}\right). \tag{2}$$

If $\tau : V \to L$ is another holomorphic trivialisation then there exists a non-vanishing holomorphic function $f$ on $U \cap V$ such that $\sigma(x) = f(x)\,\tau(x)$. Therefore one can define a holomorphic map

$$\varphi_k : M \to \mathbb{C}P^{N_k}, \tag{3}$$

whose local expression in the open set $U$ is given by (2). It follows by the above mentioned Theorem of Kodaira that, for $k$ sufficiently large, the map $\varphi_k$ is an embedding into $\mathbb{C}P^{N_k}$ (see, e.g. [**6**] for a proof).

Let $g_{FS}^{N_k}$ be the Fubini–Study metric on $\mathbb{C}P^{N_k}$, namely the metric whose associated Kähler form is given by

$$\omega_{FS}^{N_k} = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{N_k}|z_j|^2 \tag{4}$$

for a homogeneous coordinate system $[z_0, \ldots, z_{N_k}]$ in $\mathbb{C}P^{N_k}$. This restricts to a Kähler metric $g_k = \varphi_k^* g_{FS}^{N_k}$ on $M$ which is cohomologous to $k\omega_g$ and is polarized with respect to $L^{\otimes k}$. In [**12**] Tian christined the set of normalized metrics $\frac{1}{k} g_k$ as the *Bergmann* metrics on $M$ with respect to $L$ and he proves that the sequence $\frac{1}{k} g_k$ converges to the metric $g$ in the $C^2$-topology (see Theorem $A$ in [**12**]). This theorem was further generalizes by Ruan [**10**] who proved that the sequence $\frac{1}{k} g_k$ $C^\infty$-converges to the metric $g$ (see also [**13**]).

The aim of this paper is twofold. On one hand, in Section **2** we study, the polarized metrics $g$ on $M$ satisfying the equation

$$g_k = kg \tag{5}$$

(for some natural number $k$) which we call *self-Bergmann* metrics of degree $k$. If our Kähler manifold $(M, g)$ is homogeneous and simply connected then the metric $g$ is self-Bergmann of degree $k$ for all sufficiently large $k$ (for a proof see Theorem 2.1 below and cf. also [**2**]). In Theorem 2.4 and 2.6 we prove a sort of converse of Theorem 2.1 in the case of self-Bergmann metrics of degree 2 on $\mathbb{C}P^1$ induced by the Veronese map and in the case of self-Bergmann metrics of degree 1 on $\mathbb{C}P^1 \times \mathbb{C}P^1$ induced by the Segre map.

On the other hand, in Section **3**, we consider the polarizations on non-compact Kähler manifolds $(M, g)$. In particular we deal with the case of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ equipped with the complete Kähler metric $g^*$ whose associated Kähler form is given by

$$\omega^* = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2}$$

and the polarization $L$ given by the trivial bundle $L = \mathbb{C}^* \times \mathbb{C}$.

Our main results are contained in Theorem 3.5 where we describe all the hermitian metrics $h^k$ on $L^{\otimes k} = L$ such that $\mathrm{Ric}(h) = \omega^*$ (in other words all the geometric quantizations on $(\mathbb{C}^*, \omega^*)$ (see Remark 2)). Moreover in Theorem 3.6 we calculate the set of Bergmann metrics $\frac{g_k}{k}$ and we prove that the sequence $\frac{g_k}{k}$ $C^\infty$-converges to the metric $g^*$ on every compact set $K \subset M$.

## 2 - Self-Bergmann metrics

As we pointed our in the introduction a large class of self-Bergmann metrics is given by the following:

Theorem 2.1 (cfr. [2]). *Let $L$ be a polarization of a homogeneous and simply-connected compact Kähler manifold $(M, g)$. Then $g$ is self-Bergmann of degree $k$ for every sufficiently large positive integer $k$.*

Proof. Recall that a Kähler manifold $(M, g)$ is homogeneous if the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$ acts transitively on $M$, where $\mathrm{Aut}(M)$ denotes the group of holomorphic diffeomorphisms of $M$ and $\mathrm{Isom}(M, g)$ the isometry group of $M$. Let $k$ be large enough in such a way that the map $\varphi_k \colon M \to \mathbb{C}P^{N_k}$ given by (3) is an embedding. An easy calculation shows that

$$\omega_{g_k} = \varphi_k^*(\omega_{FS}^{N_k}) = k\omega_g + \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{N_k} h^k(s_j, s_j) \tag{6}$$

where $\{s_0, \ldots, s_{N_k}\}$ is the orthonormal basis for $(H^0(M, L^{\otimes k}, \langle\cdot, \cdot\rangle_{h^k})$, and where $\omega_{g_k}$, in accordance with out notation, is the Kähler form associated to $g_k$. It turns out the if the manifold $M$ is symply-connected then the holomorphic line bundle $f^*L$ is isomorphic to $L$ for any $f \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$. Moreover the smooth function $\sum_{j=0}^{N_k} h^k(s_j, s_j)$ is invariant under the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$. Therefore, if $(M, g)$ is assumed to be homogeneous then this function is constant which, by formula (6), implies that the metric $g$ is self-Bergmann of degree $k$. ■

Remark 2.2. Note that the condition of simply-connectedness in Theorem 2.1 can not be relaxed. In fact the $n$-dimensional complex torus $M$ can be naturally endowed with a polarized flat metric $g$ invariant by translation, making $(M, g)$ into a homogeneous Kähler manifold. On the other hand the flat metric can not be the pull-back of the Fubini-Study metric via a holomorphic map (see Lemma 2.2 in [**11**] for a proof) and hence in particular condition (5) can not hold for any $k$ (cf. also [8]).

Remark 2.3. In the terminology of quantization of a Kähler manifold $(M, g)$ a pair $(L, h)$ satisfying $\mathrm{Ric}(h) = \omega_g$ is called a *geometric quantization* of $(M, g)$. In the work of Cahen-Gutt-Rawnsley the function $\sum_{j=0}^{N_k} h^k(s_j, s_j)$ is the central object of the theory (see [**2**], [**3**], [**4**], [**5**]). Infact it is one of the main ingredient needed to apply a procedure called *quantization by deformation* introduced by Berezin in his foundational paper [**1**]. Observe also that our definition of self-Bergmann metrics above is equivalent to the *regularity* of a quantization as defined in [**2**] and [**3**].

In view of Theorem 2.1 the following question naturally arises: *Let $(M, g)$ be a homogenous and simply connected Kähler manifold (and hence $g$ is self-Bergmann of degree $k$ for $k$ large) and let $\tilde{g}$ be a Kähler metric on $M$ which is self-Bergmann of degree $k$. Can we conclude that also $\tilde{g}$ is homogeneous, namely there exists $f \in \mathrm{Aut}(M)$ such that $\tilde{g} = f^* g$?*

When $M = \mathbb{C}P^N$, $g = g_{\omega_{FS}^N}$ and $L$ is the hyperplane bundle, then the space $H^0(M, L)$ consisting of global holomorphic sections of $L$ can be identified with the space of degree 1 homogeneous polynomials in the variables $\{z_0, \ldots, z_n\}$ (see, e.g. [**6**]). Let $\tilde{g}$ be a self-Bergmann metric of degree $k = 1$ then $N_k = \dim H^0(M, L) - 1 = N$ and the embedding $\varphi_1$ given by (3) goes from $\mathbb{C}P^N$ to $\mathbb{C}P^N$. By the very definition of self-Bergmann metrics $\varphi_1^* g = \tilde{g}$ and since $\varphi_1$ belongs to the group $\mathrm{Aut}(\mathbb{C}P^N) = PGL(N+1, \mathbb{C})$ we deduce that the previous question has a positive answer for $M = \mathbb{C}P^N$, $g = g_{\omega_{FS}^N}$ and $k = 1$.

The case of self-Bergmann metrics of any degree $k \geq 2$ on $\mathbb{C}P^N$ is much more complicated to handle even when $N = 1$. Nevertheless we prove the following:

Theorem 2.4. *Let $\tilde{g}$ be a self-Bergmann metric of degree 2 on $\mathbb{C}P^1$ induced by the Veronese map:*

$$\varphi : \mathbb{C}P^1 \to \mathbb{C}P^2 : [z_0, z_1] \mapsto [a z_0^2, b z_0 z_1, c z_1^2], \quad a, b, c \in \mathbb{C}^*, \tag{7}$$

*then there exists $f \in PGL(2, \mathbb{C})$ such that $f^*(2g) = \tilde{g}$, where $g = g_{\omega_{FS}^1}$.*

Proof. Under the action of $f \in PGL(2, \mathbb{C})$, we can suppose that the map (7) is given by

$$\varphi([z_0, z_1]) = [z_0^2, \alpha z_0 z_1, z_1^2],$$

for $\alpha \in \mathbb{C}^*$ $\left(\text{one simply defines } f([z_0, z_1]) = \left[\frac{1}{\sqrt{a}} z_0, \frac{1}{\sqrt{c}} z_1\right]\right)$.

Observe that if $|\alpha|^2 = A = 2$ then $\varphi^* g_{FS}^2 = \varphi_2^* g_{FS}^2 = 2g$ which is self-Bergmann of degree $k$ for large $k$ by Theorem 2.1. Hence it is enough to show that if $\tilde{g}$ is self-Bergmann of degree 2 then $A = 2$. Let $\tilde{h}$ denote the hermitian structure on $H^0(M, L^{\otimes 2})$ such that $\mathrm{Ric}(\tilde{h}) = \omega_{\tilde{g}}$. Since $H^0(M, L^{\otimes 2})$ can be identified with the space homogeneous polynomials of degree 2 in $z_0$ and $z_1$, in order to prove our Theorem we need to show that if $\{z_0^2, \alpha z_0 z_1, z_1^2\}$ is a othonormal basis for $(H^0(M, L^{\otimes 2}), \langle \cdot, \cdot \rangle_{\tilde{h}})$ then $A = 2$.

In the chart $U_0 = \{z_0 \neq 0\}$, equipped with coordinate $z = \dfrac{z_1}{z_0}$, the Kähler form

$\omega_{\tilde{g}}$ associated to $\tilde{g} = \varphi^* g_{FS}^2$ is given by:

$$\omega_{\tilde{g}} = \varphi^*(\omega_{FS}^2) = \frac{i}{2\pi}\,\partial\bar{\partial}\log(1+A|z|^2+|z|^4) = \frac{i}{2\pi}\,\frac{A+4|z|^2+A|z|^4}{(1+A|z|^2+|z|^4)^2}\,dz\wedge d\bar{z}\,.$$

Let $P(z_0, z_1)$ and $Q(z_0, z_1)$ be homogeneous polynomials of degree 2 in $z_0$ and $z_1$. We denote by small letter $p$ and $q$ their expression in $U_0$, namely $p(z) = P\left(1, \frac{z_1}{z_0}\right)$ and $q(z) = Q\left(1, \frac{z_1}{z_0}\right)$. With the above notation the hermitian structure $\tilde{h}$ on $U_0$ is given by:

$$\tilde{h}(P, Q) = \frac{p(z)\, q(\bar{z})}{1+A|z|^2+|z|^4}\,.$$

Hence,

$$\langle P, Q\rangle_{\tilde{h}} = \int_{CP^1} \tilde{h}(P, Q)\,\omega_{\tilde{g}} = \int_{C} \frac{(A+4|z|^2+A|z|^4)\,p(z)\,q(\bar{z})}{(1+A|z|^2+|z|^4)^3}\,\frac{i}{2\pi}\,dz\wedge d\bar{z}\,.$$

This can be written in polar coordinates $z = re^{i\theta}$ as

$$\langle P, Q\rangle_{\tilde{h}} = \frac{1}{\pi}\int_{r=0}^{+\infty} \frac{(A+4r^2+Ar^4)\,p(re^{i\theta})\,q(re^{-i\theta})}{(1+Ar^2+r^4)^3}\,r\,dr\,d\theta\,.$$

By the change of variable $r^2 = \varrho$, one obtains:

$$\langle P, Q\rangle_{\tilde{h}} = \frac{1}{2\pi}\int_{\varrho=0}^{+\infty} \frac{(A+4\varrho+A\varrho^2)\,p(\sqrt{\varrho}e^{i\theta})\,q(\sqrt{\varrho}e^{-i\theta})}{(1+A\varrho+\varrho^2)^3}\,d\varrho\,. \tag{8}$$

It follows immediately by (8) that $\{z_0^2, z_0z_1, z_2^2\}$ (which on $U_0$ is given by $\{1, z, z^2\}$) is an orthogonal basis of $(H^0(M, L^{\otimes 2}), \langle\cdot,\cdot\rangle_{\tilde{h}})$. Furthermore,

$$\|z_0\|_{\tilde{h}}^2 = \int_{\varrho=0}^{+\infty} \frac{(A+4\varrho+A\varrho^2)}{(1+A\varrho+\varrho^2)^3}\,d\varrho\,,$$

$$\|az_0z_1\|_{\tilde{h}}^2 = A\int_{\varrho=0}^{+\infty} (A\varrho+4\varrho^2+A\varrho^3)(1+A\varrho+\varrho^2)^3 d\varrho\,,$$

$$\|z_2^2\|_{\tilde{h}}^2 = \int_{\varrho=0}^{+\infty} \frac{(A\varrho^2+4\varrho^3+A\varrho^4)}{(1+A\varrho+\varrho^2)^3}\,d\varrho\,.$$

A direct calculation, using Lemma 2.5 below gives:

$$\|z_0\|^2_{\tilde{h}} = \left(\frac{A^3}{4} - A\right) I_3 + \frac{A}{4} I_2 + 1 - \frac{A^2}{8}, \tag{9}$$

$$\|\alpha z_0 z_1\|^2_{\tilde{h}} = \left(\frac{A^3}{2} - \frac{A^5}{8}\right) I_3 + \left(A - \frac{3A^3}{8}\right) I_2 + \frac{A^4}{16}, \tag{10}$$

$$\|z_2^2\|^2_{\tilde{h}} = \left(\frac{A^5}{16} - \frac{A^3}{4}\right) I_3 + \left(\frac{3A^3}{8} - \frac{5A}{4}\right) I_2 + \frac{3A}{8} I_1 + 1 - \frac{3A^2}{16} - \frac{A^4}{32}. \tag{11}$$

Hence it remains to show that if $A \neq 2$, then either $\|z_0\|^2_{\tilde{h}} \neq A\|z_0 z_1\|^2_{\tilde{h}}$, or $\|z_0\|^2_{\tilde{h}} \neq \|z_2^2\|^2_{\tilde{h}}$. Indeed we prove that $\|z_0\|^2_{\tilde{h}} \neq A\|z\|^2_{\tilde{h}}$. Suppose, by a contradiction that $\|z_0\|^2_{\tilde{h}} = A\|z_0 z_1\|^2_{\tilde{h}}$. By subtracting (9) from (10) one obtains:

$$-32 + 6A^2 + 3A^4 - 12AI_1 + (72A - 24A^3) I_2 + 6A^3(A^2 - 4) I_3 = 0. \tag{12}$$

We distinguish two cases: $0 < A < 2$ and $A > 2$.

For $0 < A < 2$, we easily obtain:

$$I_1 = \frac{\pi}{\sqrt{4-A^2}} - \frac{2}{\sqrt{4-A^2}} \arctan \frac{A}{\sqrt{4-A^2}},$$
$$I_2 = \frac{2\pi}{(\sqrt{4-A^2})^3} - \frac{A}{4-A^2} - \frac{4}{(\sqrt{4-A^2})^3} \arctan \frac{A}{\sqrt{4-A^2}},$$
$$I_3 = \frac{6\pi}{(\sqrt{4-A^2})^5} + \frac{A^3 - 10A}{2(4-A^2)^2} - \frac{12}{(\sqrt{4-A^2})^5} \arctan \frac{A}{\sqrt{4-A^2}}.$$

By (12) one gets:

$$-(8 + A^2)\sqrt{4-A^2} + 6A\pi - 12A \arctan \frac{A}{\sqrt{4-A^2}} = 0,$$

which can be easily seen to be impossible for $0 < A < 2$. Indeed the function $F(A) = -(8 + A^2)\sqrt{4-A^2} + 6A\pi - 12A \arctan \frac{A}{\sqrt{4-A^2}}$ satisfies $F(0) = -16$, $\lim_{A \to 2^-} F(A) = 0$, $F'(0) = 6\pi$, $\lim_{A \to 2^-} F'(A) = 0$ and $F''(A) = -6\sqrt{4-A^2}$ which implies that $F(A) < 0$ on the interval $(0, 2)$.

For $A > 2$, we get:

$$I_1 = -\frac{1}{\sqrt{A^2-4}} \log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}},$$

$$I_2 = \frac{A}{A^2-4} + \frac{2}{(\sqrt{A^2-4})^3} \log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}},$$

$$I_3 = \frac{A^3-10A}{2(A^2-4)^2} - \frac{6}{(\sqrt{A^2-4})^5} \log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}.$$

By (12) one gets:

$$(8+A^2)\sqrt{A^2-4} + 6A \log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}} = 0,$$

which can not hold for $A > 2$.

Indeed the function $G(A) = (8+A^2)\sqrt{A^2-4} + 6A \log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}$ satisfies $\lim_{A\to 2^+} F(A) = \lim_{A\to 2^+} F'(A) = 0$, $\lim_{A\to +\infty} F(A) = \lim_{A\to +\infty} F'(A) = +\infty$, and $F''(A) = 6\sqrt{A^2-4}$ which implies that $F(A) > 0$ on $(2, +\infty)$. ■

Lemma 2.5. *The following equalities hold:*

$$\int_{\varrho=0}^{+\infty} \frac{\varrho}{(1+A\varrho+\varrho^2)^3}\, d\varrho = \frac{1}{4} - \frac{A}{2} I_3;$$

$$\int_{\varrho=0}^{+\infty} \frac{\varrho}{(1+A\varrho+\varrho^2)^3}\, d\varrho = \frac{1}{4} I_2 + \frac{A^2}{4} I_3 - \frac{A}{8};$$

$$\int_{\varrho=0}^{+\infty} \frac{\varrho}{(1+A\varrho+\varrho^2)^3}\, d\varrho = \frac{1}{4} + \frac{A^2}{16} - \frac{3A}{8} I_2 - \frac{A^3}{8} I_3;$$

$$\int_{\varrho=0}^{+\infty} \frac{\varrho}{(1+A\varrho+\varrho^2)^3}\, d\varrho = \frac{3}{8} I_1 + \frac{3A^2}{8} I_2 + \frac{A^4}{16} I_3 - \frac{5A}{16} - \frac{A^3}{32},$$

*where*

$$I_n = \int_{\varrho=0}^{+\infty} \frac{1}{(1 + A\varrho + \varrho^2)^n} d\varrho, \quad n = 1, 2, 3.$$

Proof. Direct calculation integrating by parts. ■

Let consider now $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with the metric $g = g_{FS}^1 + g_{FS}^1$ which we know to be self-Bergmann of degree $k$ for all $k$ (compare Theorem 2.1). In this case the map $\varphi_1$ (given by 3)) (which satisfies $\varphi_1^* g_{FS}^3 = g$) is given by:

$$\varphi_1 : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^3 : ([z_0, z_1], [w_0, w_1]) \mapsto [z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1].$$

The polarization $L$ on $M$ is the restriction to $M$ of the hyperplane bundle on $\mathbb{C}P^3$ via the map $\varphi_1$ and a basis of $H^0(M, L)$ is $\{z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$.

Theorem 2.6. *Let $\tilde{g}$ be a self-Bergmann metric of degree $k = 1$ on $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ induced by the* Segree embedding $\varphi : M \to \mathbb{C}P^3$ *given by:*

(13) $\varphi([z_0, z_1], [w_0, w_1]) \mapsto [a z_0 w_0, b z_0 w_1, c z_1 w_0, d z_1 w_1]$, $a, b, c, d \in \mathbb{C}^*$.

*Then there exists $f \in \mathrm{Aut}(M) = PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})$ such that $f^* g = \tilde{g}$.*

Proof. The proof follows the same pattern of that of Theorem 2.4. First of all under the action of $f \in \mathrm{Aut}(M)$, we can suppose that the map (13) is given by

$$\varphi([z_0, z_1], [w_0, w_1]) = [\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1],$$

for $\alpha \in \mathbb{C}^*$. Indeed one takes $f([z_0, z_1], [w_0, w_1]) = \left[\frac{1}{b} z_0, \frac{1}{d} z_1\right], \left[\frac{d}{c} w_0, w_1\right]$. Hence it is enough to show that if $\tilde{g} = \varphi^* g_{FS}^3$ is a self-Bergmann metric of degree 1 then $A = |\alpha|^2 = 1$. Let $\tilde{h}$ be the hermitian structure on $H^0(M, L)$ such that $\mathrm{Ric}(\tilde{h}) = \omega_{\tilde{g}}$. In order to prove our Theorem it suffices to show that if $\{\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$ is a othonormal basis for $(H^0(M, L), \langle \cdot, \cdot \rangle_{\tilde{h}})$ then $A = 1$. Let $U \cong \mathbb{C}^2$ be the chart on $M$ defined by $(z_0, w_0) \neq (0, 0)$ equipped with coordinates $(z, w) = \left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)$. We can easily calculate the Kähler form $\omega_{\tilde{g}} = \varphi^*(\omega_{FS}^3)$ on $U$ and obtain:

$$\omega_{\tilde{g}}^2 = \omega_g \wedge \omega_g = \frac{A(1 + |z|^2 + |w|^2) + |z|^2 |w|^2}{(A + |z|^2 + |w|^2 + |z|^2 |w|^2)^3} d\nu,$$

where $d\nu = \left(\frac{i}{2\pi}\right)^2 dz \wedge d\bar{z} \wedge dw \wedge d\overline{w}$.

Let $P \in H^0(M, L) = \mathrm{span}\,\{z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$. We denote by small letter $p$ its expression in the chart $U$, namely $p(z, w) = P\left(1, \frac{w_1}{w_0}, \frac{z_1}{z_0}, \frac{z_1}{z_0}\frac{w_1}{w_0}\right)$. With the above notation the hermitian structure $\tilde{h}$ on $U$ is given by:

$$\tilde{h}(P, Q) = \frac{p(z, w)\, q(\bar{z}, \overline{w})}{A + |z|^2 + |w|^2 + |z|^2\,|w|^2}\,.$$

Hence,

$$\langle P, Q\rangle_{\tilde{h}} = \int_M \tilde{h}(P, Q)\,\frac{\omega_g^2}{2!} = \frac{1}{2}\int_{\mathbb{C}^2} \frac{(A(1 + |z|^2 + |w|^2) + |z|^2\,|w|^2)\, p\bar{q}}{(A + |z|^2 + |w|^2 + |z|^2\,|w|^2)^4}\, d\nu\,,$$

for $P, Q \in H^0(M, L)$.

It follows that $\{\alpha z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1\}$ (which on $U$ is given by $\{\alpha, w, z, zw\}$) is a othogonal basis of $(H^0(M, L), \langle\cdot, \cdot\rangle_{\tilde{h}})$. By passing in polar coordinates, a straightforward calculation gives:

$$\|\alpha z_0 w_0\|_{\tilde{h}}^2 = \|z_1 w_1\|_{\tilde{h}}^2 = \frac{1 - 3A + 2A^2 - A\log A}{48(A-1)^2} \tag{14}$$

and

$$\|z_0 w_1\|_{\tilde{h}}^2 = \|z_1 w_0\|_{\tilde{h}}^2 = \frac{2 - 3A + A^2 + A\log A}{48(A-1)^2}\,. \tag{15}$$

It is now easy to see that (14) and (15) are equal if and only if $A = 1$ which concludes the proof of our theorem. ■

## 3 - Quantizations and Bergmann metrics of $(\mathbb{C}^*, g^*)$

In this section we consider the case of a complete Kähler manifold $(M, g)$. Let $L$ be a holomorphic line bundle on $M$ endowed with an hermitian structure $h$. Following Tian (Sect. 4 in [**12**]) we denote by $H^0_{(2)}(M, L^{\otimes k})$ the Hilbert space consisting of all $L^2$ integrable global holomorphic sections of $L^{\otimes k}$, namely

$$s \in H^0_{(2)}(M, L^{\otimes k}) \Leftrightarrow \langle s, s\rangle_{h^k} = \int_M h^k(s(x), s(x))\,\frac{\omega_g^n(x)}{n!} < \infty\,.$$

Let $\{s_j\}_{j \geqslant 0}$ be an orthonormal basis of $(H^0_{(2)}(M, L^{\otimes k}), \langle\cdot, \cdot\rangle_{h^k})$. One if his main result, which generalizes the above mentioned Theorem A, is summarized in the following:

Theorem 3.1. *(Tian) Let $M$ be a complete Kähler manifold with a polarized Kähler metric $g$ and let $L$ be a holomorphic line bundle with hermitian metric $h$ such that its Ricci curvature form satisfies:* $\mathrm{Ric}(h) = \omega_g$. *Then for any compact set $K \subset M$ and $k$ sufficiently large*

$$\omega_k = \frac{i}{2\pi} \partial \bar{\partial} \log \sum_{j=0}^{+\infty} |s_j|^2 \tag{16}$$

*defines a Kähler form on $K$. Moreover if $g_k$ denotes the Kähler metric on $K$ associated to $\omega_k$ (i.e. $\omega_{g_k} = \omega_k$) then the sequence of metrics $\frac{g_k}{k}$ $C^2$-converges to the Kähler metric $g$ on $K$.*

As in the compact case, a geometric quantization of a complete Kähler manifold $(M, g)$ is given by a pair $(L, h)$, where $L$ is a holomorphic line bundle on $M$ equipped with a hermitian metric $h$ such that $\mathrm{Ric}(h) = \omega_g$ (see Remark 2.3)). The metrics $\frac{g_k}{k}$ (defined only on compact sets $K \subset M$) are called the *Bergmann metrics* on $(M, g)$.

Remark 3.2. In analogy with the compact case, we say that a Kähler metric on a complete manifold is *self-Bergmann* of degree $k$ if $g_k = kg$. Observe that this implies that $g_k$ is globally defined on $M$ and not only in a compact set $K \subset M$. A slight modification of Theorem 2.1 shows that in a homogeneous and simply-connected Kähler manifold $(M, g)$ then the metric $g$ is self-Bergmann of degree $k$ for all $k$. Therefore, for example, the flat metric on the complex Euclidean space $\mathbb{C}^n$ is self-Bergmann of degree $k$.

In order to describe all the geometric quantizations of a Kähler manifold $(M, g)$ one gives the following (cf. e.g. [**9**]):

Definition 3.3. *Two holomorphic hermitian line bundles $(L_1, h_1)$ and $(L_2, h_2)$ on a Kähler manifold $(M, g)$ are called equivalent if there exists an isomorphism of holomorphic line bundles $\psi : L_1 \rightarrow L_2$ such that $\psi^* h_2 = h_1$.*

Let us denote by $[L, h]$ the equivalence class of $(L, h)$ and by $\mathcal{L}(M, g)$ the set of equivalence classes. We refer the reader to [**2**] for the proof of the following:

Theorem 3.4. *The group* $\mathrm{Hom}(\pi_1(M), S^1)$ *acts transitively on the set of equivalence classes* $\mathcal{L}(M, g)$.

In Theorem 3.5 below we describe this action in the case of $(\mathbb{C}^*, g^*)$. We first observe that any holomorphic line bundle $L$ on $\mathbb{C}^*$ is holomorphically trivial. Let $h$ be the hermitian metric on $L$ given by:

$$h(f(z), f(z)) = e^{\frac{-\pi}{2}\log^2|z|^2}|f(z)|^2.$$

for a holomorphic function $f$ on $\mathbb{C}^*$. It is easily seen that $\mathrm{Ric}(h_0) = \omega^*$ and hence $L$ is a quantization of $(\mathbb{C}^*, g^*)$. We can prove now the first result of this section:

Theorem 3.5. *The group*

$$\mathrm{Hom}(\pi_1(\mathbb{C}^*), S^1) = \mathrm{Hom}(\mathbb{Z}, S^1) \cong S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$$

*acts on the set of equivalence classes* $\mathcal{L}(\mathbb{C}^*, g^*)$ *by defining*:

$$[\lambda]\cdot(L, h) = (L, h_\lambda), \tag{17}$$

*where* $[\lambda]$ *denotes the equivalence class of* $\lambda$ *in* $S^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$ *and* $h_\lambda$ *is the hermitian metric on* $L$ *defined by:*

$$h_\lambda(f(z), f(z)) = |z|^{2\lambda} h(f(z), f(z)), \tag{18}$$

*for a holomorphic function* $f$ *on* $\mathbb{C}^*$.

Proof. Let $\lambda$ and $\mu$ be real numbers such that $\lambda - \mu \in \mathbb{Z}$. It is easy to see that the map

$$\psi : (L, h_\mu) \to (L, h_\lambda) : (z, t) \mapsto (z, z^{\nu-\lambda} t)$$

is a holomorphic automorphism of the trivial bundle and $\psi^*(h_\lambda) = h_\nu$, namely $[L_0, h_\mu] = [L_0, h_\lambda]$. Furthermore, if $\lambda - \mu \notin \mathbb{Z}$ then $[L, h_\lambda] \neq [L, h_\mu]$. Indeed, suppose that $\psi : L \to L$ is a holomorphic automorphism of the trivial bundle, such that $\psi^* h_\lambda = h_\mu$. It follows that $\psi(z, t) = (z, f(z)\, t)$, where $f$ is a holomorphic function on $\mathbb{C}^*$, satisfying $|f(z)|^2 = |z|^{2(\mu - \lambda)}$. This is impossible unless $\lambda - \mu$ is an integer. ■

Given a natural number $k$ it follows immediately that the trivial bundle $L$ endowed with the hermitian structure

$$h^k(f(z), f(z)) = e^{\frac{-k\pi}{2} \log^2 |z|^2} |f(z)|^2$$

defines a quantization of $(\mathbb{C}^*, kg^*)$. By Theorem 3.5 we know that every class in $\mathcal{L}(\mathbb{C}^*, kg^*)$ can be represented by a pair $(L, h^k_\lambda)$, where

$$h^k_\lambda(f(z), f(z)) := e^{\frac{-k\pi}{2} \log^2 |z|^2} |z|^{2\lambda} |f(z)|^2, \tag{19}$$

and two such pairs $(L, h^k_\lambda)$ and $(L, h^k_\mu)$ are equivalent iff $[\lambda] = [\mu]$. In what follows, to simplify the notation, we consider the class corresponding to $\lambda = 0$, namely the trivial bundle $L$ on $\mathbb{C}^*$ endowed with the hermitian metric

$$h^k(f(z), f(z)) := e^{\frac{-k\pi}{2} \log^2 |z|^2} |f(z)|^2.$$

It follows that the space $(H^0_{(2)}(\mathbb{C}^*, L), \langle \cdot, \cdot \rangle_{h^k})$, which we will denote by $\mathcal{H}_k$, equals the space of holomorphic functions $f$ in $\mathbb{C}^*$ such that

$$\|f\|^2_{h^k} = \langle f, f \rangle_{h^k} = \int_{\mathbb{C}^*} e^{\frac{-k\pi}{2} \log^2 |z|^2} |f(z)|^2 k \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2} < +\infty .$$

One can check that the functions $z^j$, with $j \in \mathbb{Z}$, form an orthogonal system for $\mathcal{H}_k$. Since every holomorphic function in $\mathbb{C}^*$ can be expanded in Laurent series, it follows that $z^j$ are in fact a complete orthogonal system. Their norms are given by

$$\|z^j\|^2_{h^k_0} = k \int_{\mathbb{C}^*} e^{\frac{-k\pi}{2} \log^2 |z|^2} |z|^{2j} \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2}$$

$$= k\pi \int_0^{+\infty} e^{\frac{-k\pi}{2} \log^2 r^2} r^{2j} \frac{2r}{r^2} dr .$$

By the change of variable $e^{\varrho} = r^2$ one gets

$$\|z^j\|^2_{h^k} = k\pi \int_{-\infty}^{+\infty} e^{\frac{-k\pi}{2}\varrho^2} e^{j\varrho} d\varrho = k\pi e^{\frac{j^2}{2k\pi}} \int_{-\infty}^{+\infty} e^{-\left(\sqrt{\frac{k\pi}{2}}\varrho - \sqrt{\frac{1}{2k\pi}}j\right)^2}$$

$$= k\pi e^{\frac{j^2}{2k\pi}} \sqrt{\frac{2}{k\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{2k}\pi e^{\frac{j^2}{2k\pi}}.$$

Then a orthonormal basis for $\mathcal{H}_k$ is given by

$$s_j = \left(\frac{1}{\sqrt{2k\pi}} e^{-\frac{j^2}{2k\pi}}\right)^{\frac{1}{2}} z^j$$

and by formula (16) we get:

$$\omega_k = \frac{i}{2\pi} \partial\bar{\partial} \log \sum_{j \in \mathbb{Z}} e^{-\frac{j^2}{2k\pi}} |z|^{2j}. \tag{20}$$

Let $\frac{g_k}{k}$ be the corresponding sequence of Bergmann metrics (which are defined, by Theorem 3.1, on every compact set $K \subset \mathbb{C}^*$ for $k$ sufficiently large). The following Theorem extends Tian's theorem 3.1 in the case of the punctured plane endowed with the metric $g^*$.

Theorem 3.6. *Let $\mathbb{C}^*$ be endowed with the complete metric $g^*$. Then the sequence of Bergmann metrics $\frac{g_k}{k}$ $C^\infty$-converges to the metric $g^*$ on every compact set $K \subset \mathbb{C}^*$.*

Proof. By formula (20) it is enough to show that the sequence of functions

$$f_k(x) = \frac{1}{k} \log\left(\sum_{j \in \mathbb{Z}} e^{\frac{-j^2}{2k\pi}} x^j\right) \tag{21}$$

(defined on $\mathbb{R}^+$) $C^\infty$-converges to the function $f(x) = \frac{\pi}{2}\log^2 x$ on every compact set $C \subset \mathbb{R}^+$. In order to prove it we apply the Poisson summation formula (see p. 347, Theorem 24 in [7]) to the function $f(j) = e^{\frac{-j^2}{2k\pi}} x^j = e^{\frac{-j^2}{2k\pi} + j\log x}$. Namely, one has: $\sum_{j \in \mathbb{Z}} f(j) = \sum_{j \in \mathbb{Z}} \hat{f}(j)$, where $\hat{f}(j) = \int_{-\infty}^{+\infty} e^{-2\pi i j\nu} f(\nu)$. By a straightforward calculation one gets:

$$\widehat{f}(j) = e^{k\frac{\pi}{2}(2\pi ij - \log x)^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\pi k}(\nu + 2\pi^2 ijk - \pi k \log x)^2}$$

$$= 2\pi\sqrt{k} e^{k\frac{\pi}{2}\log^2 x} e^{-2k\pi^2 j(\pi j - i \log x)}.$$

Thus

$$\lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} f(j) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} \widehat{f}(j)$$

$$= \frac{\pi}{2}\log^2 x + \lim_{k\to\infty} \frac{1}{k} \log \sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)}.$$

It is now immediate to see that the sequence $\sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)}$ $C^\infty$-converges to the constant function 1 on every compact set $C \subset \mathbb{R}^+$, which concludes the proof of our Theorem. Indeed,

$$\left| \sum_{j\in\mathbb{Z}} e^{-2k\pi^2 j(\pi j - i\log x)} \right| \leqslant 1 + \sum_{j\in\mathbb{Z}\setminus\{0\}} e^{-2k\pi^3 j^2} < 1 + \int_{-\infty}^{+\infty} e^{-2k\pi^3 t^2} dt = 1 + \frac{1}{\sqrt{2k\pi}}.$$

## References


[1] F. A. Berezin, *Quantization*, Izv. Akad. Nauk. SSSR Ser. Mat. 8 (1974), 1116-1175.

[2] M. Cahen, S. Gutt and J. H. Rawnsley, *Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization*, J. Geom. Phys. **7** (1990), 45-62.

[3] M. Cahen, S. Gutt and J. H. Rawnsley, *Quantization of Kähler manifolds II*, Trans. Amer. Math. Soc. **337** (1993), 73-98.

[4] M. Cahen, S. Gutt and J. H. Rawnsley, *Quantization of Kähler manifolds III*, Lett. Math. Phys. **30** (1994), 291-305.

[5] M. Cahen, S. Gutt and J. H. Rawnsley, *Quantization of Kähler manifolds IV*, Lett. Math. Phys. **34** (1995), 159-168.

[6] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons Inc, New York 1978.

[7] R. Godement, *Analyse mathématique II*, Springer, Berlin 1998.

[8] A. LOI, *The function epsilon for complex tori and Riemann surfaces*, Bull. Belg. Math. Soc. Simon Stevin **7** (2000), 229-236.

[9] J. H. RAWNSLEY, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford Ser. **28** (1977), 403-415.

[10] W. D. RUAN, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6** (1998), 589-631.

[11] M. TAKEUCHI, *Homogeneous Kähler manifolds in complex projective space*, Japan J. Math. **4** (1978), 171-219.

[12] G. TIAN, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99-130.

[13] S. ZELDTICH, *Szegö kernel and a theorem of Tian*, J. Differential Geom. **32** (1990), 99-130.



## Abstract

*In this paper we study the set of self-Bergmann metrics on the Riemann sphere endowed with the Fubini-study metric and we extend a theorem of Tian to the case of the punctured plane endowed with a natural flat metric.*


* * *