Journal of Geometry and Physics 61 (2011) 1928–1931

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Embedding almost-complex manifolds in almost-complex euclidean spaces

Antonio J. Di Scala [a,*], Daniele Zuddas [b]

[a] Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

[b] Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

ARTICLE INFO

Article history:
Received 20 August 2010
Received in revised form 3 May 2011
Accepted 6 May 2011
Available online 13 May 2011

MSC:
32Q60
32H02

Keywords:
Embedding
Almost-complex structure
Manifold
Pseudo-holomorphic embedding

ABSTRACT

We show that any compact almost-complex manifold (M, J) of complex dimension m can be pseudo-holomorphically embedded in $\mathbb{R}^{6m}$ equipped with a suitable almost-complex structure $\widetilde{J}$.

1. Introduction

An almost-complex structure on a $2n$-dimensional smooth manifold M is a tensor $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$. If M is oriented we say that J is *positive* if the orientation induced by J on M agrees with the given one. An almost-complex structure is called *integrable* if it is induced by a holomorphic atlas. In dimension two any almost-complex structure is integrable, while in higher dimension this is far from true. A smooth map $f : N \to M$ between two almost-complex manifolds (N, J'), (M, J) is called *pseudo-holomorphic* if $J \circ Tf = Tf \circ J'$, where $Tf : TN \to TM$ is the tangent map of f. When the map f is an embedding, (N, J') is said to be an *almost-complex submanifold* of (M, J). In this case we can identify N with its image $f(N) \subset M$ and the almost-complex structure J' with the restriction of J to $TN \cong T(f(N)) \subset TM$.

If we equip $\mathbb{R}^{2n}$ with the canonical complex structure, that is to say $\mathbb{R}^{2n} \cong \mathbb{C}^n$, then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds pseudo-holomorphically embedded in $\mathbb{R}^{2n}$ equipped with an integrable or non-integrable almost-complex structure.

In [1] Calabi and Eckmann constructed the first examples of compact, simply connected complex manifolds $M_{p,q}$ which are not algebraic. Topologically $M_{p,q}$ is the product $S^{2p+1} \times S^{2q+1}$. Then by deleting a point on each factor one obtains a complex structure J on $\mathbb{R}^{2p+2q+2}$. In Section 5 of [1] it was shown that when $p, q > 1$ there exists a complex torus as a complex submanifold of $(\mathbb{R}^{2p+2q+2}, J)$ [1, p. 499]. It follows that the Calabi–Eckmann complex structure J on $\mathbb{R}^{2n}$ cannot be tamed by any symplectic form and in particular cannot be Kähler. Calabi and Eckmann also observed that

* Corresponding author.
E-mail addresses: antonio.discala@polito.it (A.J. Di Scala), d.zuddas@gmail.com (D. Zuddas).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2011.05.002

the only holomorphic functions on $(\mathbb{R}^{2p+2q+2}, J)$ are the constants answering negatively to a question raised by Bochner about the uniformization of complex structures on $\mathbb{R}^{2n}$. In [2] Bryant constructed pseudo-holomorphic non-constant maps $\varphi : M^2 \to S^6$ for any compact Riemann surface M^2, where S^6 is equipped with the almost-complex structure induced by the octonion multiplication. These maps realize compact Riemann surfaces as pseudo-holomorphic singular curves in S^6.

In [3] it was shown that any almost-complex torus $\mathbb{T}^n = \mathbb{R}^{2n}/\Lambda$ can be pseudo-holomorphically embedded into $(\mathbb{R}^{4n}, J_\Lambda)$ for a suitable almost-complex structure J_Λ. It follows that any compact Riemann surface can be realized as a pseudo-holomorphic curve of some $(\mathbb{R}^{2n}, J)$, where J is a suitable almost-complex structure.

In this paper we prove the following general theorem.

Theorem 1. *Any compact almost-complex manifold (M, J) of real dimension $2m$ can be pseudo-holomorphically embedded in $(\mathbb{R}^{6m}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$.*

In particular, any compact Riemann surface can be realized as a pseudo-holomorphic curve in $(\mathbb{R}^6, \widetilde{J})$. In [3] was shown that the torus is the only compact Riemann surface that can be pseudo-holomorphically embedded in $(\mathbb{R}^4, \widetilde{J})$ for some $\widetilde{J}$.

2. Preliminaries

The space of positive linear complex structures on $\mathbb{R}^{2n}$ is diffeomorphic to the homogeneous space $\widetilde{\mathfrak{J}}(n) = GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ and is homotopy equivalent to $\mathfrak{J}(n) = SO(2n)/U(n)$. So, an almost-complex structure J on $\mathbb{R}^{2n}$ can be regarded as a smooth map $J : \mathbb{R}^{2n} \to \widetilde{\mathfrak{J}}(n)$.

Lemma 2. *Let $M \subset \mathbb{R}^{2n}$ be a closed submanifold and let $J : M \to \widetilde{\mathfrak{J}}(n)$ be a smooth map. Then there exists a smooth extension $\widetilde{J} : \mathbb{R}^{2n} \to \widetilde{\mathfrak{J}}(n)$ if and only if J is homotopic to a constant.*

Proof. The 'only if' part follows immediately from the fact that $\mathbb{R}^{2n}$ is contractible.

Let us prove the 'if' part. Consider a smooth homotopy $H : M \times [0, 1] \to \widetilde{\mathfrak{J}}(2n)$ such that $H_0(x) = J_0$ for all $x \in M$, and $H_1 = J$ where $H_t(x) = H(x, t)$ and $J_0 \in \widetilde{\mathfrak{J}}(n)$. We can extend H to $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n} \times [0, 1]$ by setting $H(x, 0) = J_0$ for any $x \in \mathbb{R}^{2n}$. By the homotopy extension property [4, Chapter 0] there exists $\widetilde{H} : \mathbb{R}^{2n} \times [0, 1] \to \widetilde{\mathfrak{J}}(n)$ which extends H. We conclude the proof by setting $\widetilde{J} = \widetilde{H}_1$. □

Let (M, J) be an almost-complex manifold. The strategy to prove Theorem 1 will be to choose an arbitrary embedding $f : M \hookrightarrow \mathbb{R}^{6m}$, which exists for the weak Whitney embedding theorem, and to show that J extends to the pullback $f^*(T\mathbb{R}^{6m})$ and this extension is null-homotopic.

Consider the standard filtration $SO(1) \subset SO(2) \subset \cdots$. Since $SO(n-1)$ contains the $(n-2)$-skeleton of $SO(n)$ (because the standard fibration $SO(n) \to S^{n-1}$) it follows that the k-skeleton of $SO(n)$ is contained on $SO(k+1)$ for $0 \le k \le n-2$.

Since $SO(n) \subset U(n)$ it follows that $U(n)$ contains the $(n-1)$-skeleton of $SO(2n)$ for $n \ge 1$. Then the homomorphism induced by the inclusion $i_* : \pi_j(U(n)) \to \pi_j(SO(2n))$ is an isomorphism for $j \le n-2$ and is an epimorphism for $j = n-1$.

From the homotopy exact sequence of the fibre bundle $SO(2n) \to \mathfrak{J}(n)$ given by the projection map it follows that $\pi_j(\widetilde{\mathfrak{J}}(n)) \cong \pi_j(\mathfrak{J}(n)) \cong 0$ for $j \le n-1$.

Definition 3. A space X is said to be n-connected if $\pi_j(X) \cong 0$ for all $j \le n$.

In particular, 0-connected means path-connected.

From the above considerations we have that $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected. The following proposition is well-known in the theory of CW-complexes.

Proposition 4. *If X is n-connected then any map $Y \to X$ defined on a CW-complex Y of dimension $\le n$ is homotopic to a constant.*

Also the following proposition is standard, and we give only the idea of the proof.

Proposition 5. *Let $\xi : E \to M$ be an oriented real vector bundle of rank $2k$ over an m-manifold M. If $k \ge m$ then ξ admits a positive complex structure.*

Proof. Consider the bundle $\xi^{\mathfrak{J}} : \widetilde{\mathfrak{J}}(E) \to M$ with fibre $\widetilde{\mathfrak{J}}(k)$ induced by ξ. Namely, for any $p \in M$ the fibre of $\xi^{\mathfrak{J}}$ over p is the space of positive linear complex structures on $\xi^{-1}(p)$. Since $\widetilde{\mathfrak{J}}(k)$ is $(k-1)$-connected, it follows that $\xi^{\mathfrak{J}}$ admits a section if $k \ge m$, see [5, Part III]. This section is a positive complex structure on ξ. □

Let $f : M \to \mathbb{R}^N$ be an immersion. The normal bundle $\nu_f(M)$ is, as usual, the orthogonal complement of TM in $f^*(T\mathbb{R}^N)$, that is to say:

$$f^*(T\mathbb{R}^N) = TM \oplus \nu_f(M).$$

If M is oriented then the normal bundle can be equipped with a canonical orientation, namely that which makes the splitting of $f^*(T\mathbb{R}^N)$ into a Whitney sum of oriented fibre bundles, where $\mathbb{R}^N$ is considered with the standard orientation.

3. Proof of the main results

Theorem 6. *Let $M \subset \mathbb{R}^{2n}$ be a submanifold of even dimension endowed with an almost-complex structure J. If the normal bundle of M in $\mathbb{R}^{2n}$ admits a positive complex structure with respect to the canonical orientation, then for any $k \geq \max(0, \dim_{\mathbb{R}} M - n + 1)$ there exists an almost-complex structure $\widetilde{J}$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ such that $M \times \{0\} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2k}$ is an almost-complex submanifold.*

Proof. Let us choose a positive complex structure on the normal bundle of M. Then by taking the Whitney sum with the almost-complex structure on M we get a complex structure on $(T\mathbb{R}^{2n})_{|M}$. So we obtain a smooth map $J : M \to \widetilde{\mathfrak{J}}(n)$.

In view of Lemma 2 our target is to get a J null-homotopic. This is so if $\dim_{\mathbb{R}} M \leq n - 1$ because $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected and Proposition 4.

If $\dim_{\mathbb{R}} M > n - 1$ we take the product $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$, where $\mathbb{R}^{2k}$ is endowed with the standard complex structure, and we embed M as $M \times \{0\}$. We get a complex structure on the normal bundle of M in $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ in the obvious way. So we obtain a map $J_k : M \to \widetilde{\mathfrak{J}}(n+k)$. It follows that J_k is homotopic to a constant if $k \geq \dim_{\mathbb{R}} M - n + 1$. In this case J_k extends on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ by Lemma 2. □

It follows that if (M, J) is contained in $\mathbb{C}^n$ with a complex normal bundle and if $n \geq 2\dim_{\mathbb{C}} M + 1$, then there is a positive almost-complex structure $\widetilde{J}$ on $\mathbb{C}^n$ which makes (M, J) an almost-complex submanifold of $(\mathbb{C}^n, \widetilde{J})$.

Proof of Theorem 1. Let $f : M \hookrightarrow \mathbb{R}^{6m}$ be any embedding. The normal bundle $\nu_f(M)$ has rank $4m$ and is orientable. By Proposition 5 there is a complex structure on the normal bundle and then we conclude by an application of Theorem 6 with $k = 0$. □

In some cases we can construct an embedding in an euclidean space of lower dimension. Recall that an *s-inverse* of the tangent bundle TM is a vector bundle ξ such that $TM \oplus \xi$ is a trivial vector bundle. Observe that if $f : M \to \mathbb{R}^N$ is an immersion then the normal bundle $\nu_f(M)$ is a real s-inverse of the tangent bundle TM. The converse also holds and is a Theorem of Hirsch [6], and is given as follows.

Theorem 7 (*Hirsch [6]*)**.** *Any s-inverse of TM is the normal bundle of some immersion $f : M \to \mathbb{R}^N$.*

Let ξ be a complex s-inverse of (TM, J) of complex rank k, namely $TM \oplus \xi$ is trivial as a real vector bundle. Now Hirsch's Theorem 7 implies that there exists an immersion $f : M \to \mathbb{R}^{2(m+k)}$ such that ξ is isomorphic to $\nu_f(M)$ as real vector bundles. So $\nu_f(M)$ carries a complex structure.

Up to a product with some $\mathbb{R}^{2h}$, we can assume that $k \geq m + 1$, and then f is regularly homotopic, namely homotopic through immersions, to an embedding $f_1 : M \to \mathbb{R}^{2(m+k)}$. It follows that $\nu_{f_1}(M) \cong \nu_f(M)$ carries a complex structure. Now apply Theorem 6 to get $\widetilde{J}$.

If the rank of ξ satisfies $m + 1 \leq k \leq 2m - 1$ we get a pseudo-holomorphic embedding in an euclidean space of complex dimension $m + k < 3m$.

Let (S^6, J) be the six-dimensional sphere equipped with the standard almost-complex structure J obtained from the octonion multiplication. Theorem 1 implies that (S^6, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^{18}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$. Using the existence of a low-dimensional s-inverse of (TS^6, J) we have the following result.

Corollary 8. *The almost-complex sphere (S^6, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^{14}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$.*

Proof. Since S^6 is embedded in $\mathbb{R}^8$ with trivial normal bundle we conclude by an application of Theorem 6 with $k = 3$. □

Notice that (S^6, J) cannot be pseudo-holomorphically embedded in $(\mathbb{R}^{12}, \widetilde{J})$. In fact, the Euler class of the normal bundle of any embedding of S^6 in $\mathbb{R}^{12}$ is zero by a theorem of Whitney, see [7, p. 138]. On the other hand, if S^6 is contained pseudo-holomorphically in $(\mathbb{R}^{12}, \widetilde{J})$, by a straightforward computation with the Chern class, we obtain for the Euler class $e(\nu(S^6)) = c_3(\nu(S^6)) = -2\lambda \neq 0$, which is a contradiction, where $\lambda \in H^6(S^6)$ is the standard generator.

We conclude with a question. Since our construction is essentially homotopy-theoretic, we are unable to control the integrability of the almost-complex structure $\widetilde{J}$ of Theorem 1. So the following question is very natural.

Question 9. *Let (M, J) be an integrable complex manifold. Is there an embedding of (M, J) into an integrable $(\mathbb{R}^{2n}, \widetilde{J})$?*

Acknowledgements

We thank Simon Salamon for useful comments and remarks. This work was supported by the Project M.I.U.R. PRIN 05 "Riemannian Metrics and Differentiable Manifolds" and by G.N.S.A.G.A. of I.N.d.A.M. The second author was supported by Regione Autonoma della Sardegna with funds from PO Sardegna FSE 2007–2013 and L.R. 7/2007 "Promotion of scientific research and technological innovation in Sardinia".

References

[1] E. Calabi, B. Eckmann, A class of compact, complex manifolds which are not algebraic, Ann. of Math. 58 (1953) 494–500.
[2] R.L. Bryant, Submanifolds and special structures on the octonians, J. Differential Geom. 17 (2) (1982) 185–232.
[3] A.J. Di Scala, L. Vezzoni, Complex submanifolds of almost complex Euclidean spaces, Quart. J. Math. 61 (4) (2010) 401–405.
[4] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[5] N. Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951.
[6] M.W. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93 (1959) 242–276.
[7] M.W. Hirsch, Differential Topology, in: Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York-Heidelberg, 1976.