Weakly nonlinear Fully relativistic
vs.
Weakly relativistic Fully nonlinear
Approaches in Cosmology

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June 27th - July 1st (KIAS rm 1114)

Newton's gravity:

- Non-relativistic (no c)
  - Action at a distance, violates causality
  - $c \rightarrow \infty$ limit of Einstein gravity ⟵ Post Newtonian approximation
  - No horizon
  - Static nature
- No strong pressure allowed
- No strong gravity allowed
- No gravitational waves
- Incomplete and inconsistent

Einstein's gravity:

- Relativistic gravity
- Strong gravity, dynamic
- Simplest

★ **The two theories give the same descriptions for the cosmological world model and its linear structures.**

World model: spatially homogeneous and isotropic world model

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\mu - \frac{\text{const.}}{a^2} + \frac{\Lambda}{3}, \quad \mu \propto a^{-3}.$$

- **Relativistic (Friedmann 1922)** [5]
- **Newtonian (Milne-McCrea 1933)** [6]

Structures: linear perturbations

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0.$$

- **Relativistic (Lifshitz 1946)** [7]
- **Newtonian (Bonnor 1957)** [8]

"It is curious that it took so long for these dynamic models to be discovered after the (more complex) general relativity models were known."

G. F. R. Ellis (1989) [9]

★ **In fact, the known "Newtonian cosmology" is a GR guided version!**

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[5] Friedmann A. A., 1922, Zeitschrift für Physik, **10**, 377; translated in Bernstein J., Feinberg G., eds, 1986, Cosmological-constants: papers in modern cosmology, Columbia Univ. Press, New York, p. 49

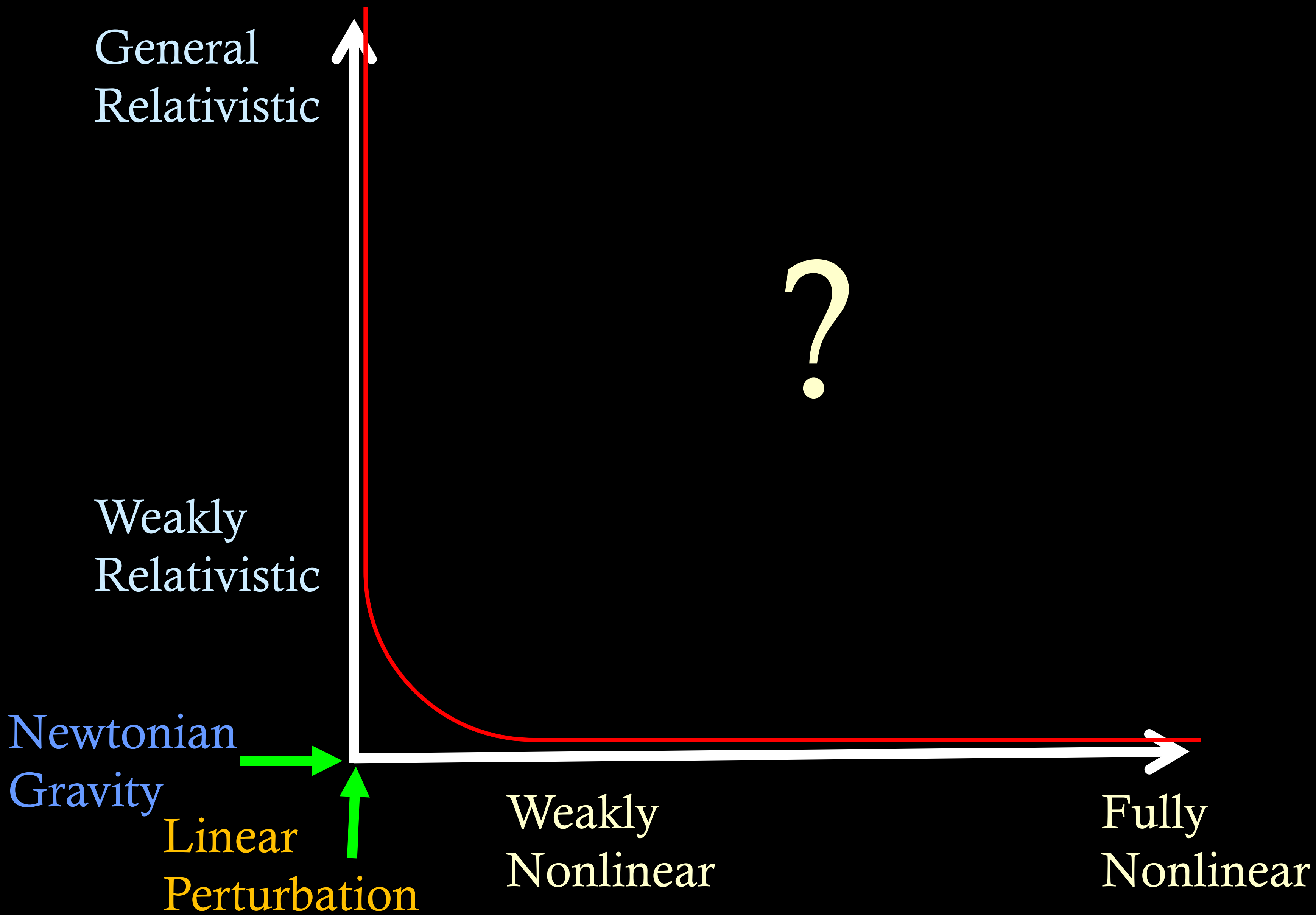
[6] Milne E. A., 1934, Quart. J. Math., **5**, 64; McCrea W. H., Milne E. A., 1934, Quart. J. Math., **5**, 73

[7] Lifshitz E. M., 1946, J. Phys. (USSR), **10**, 116

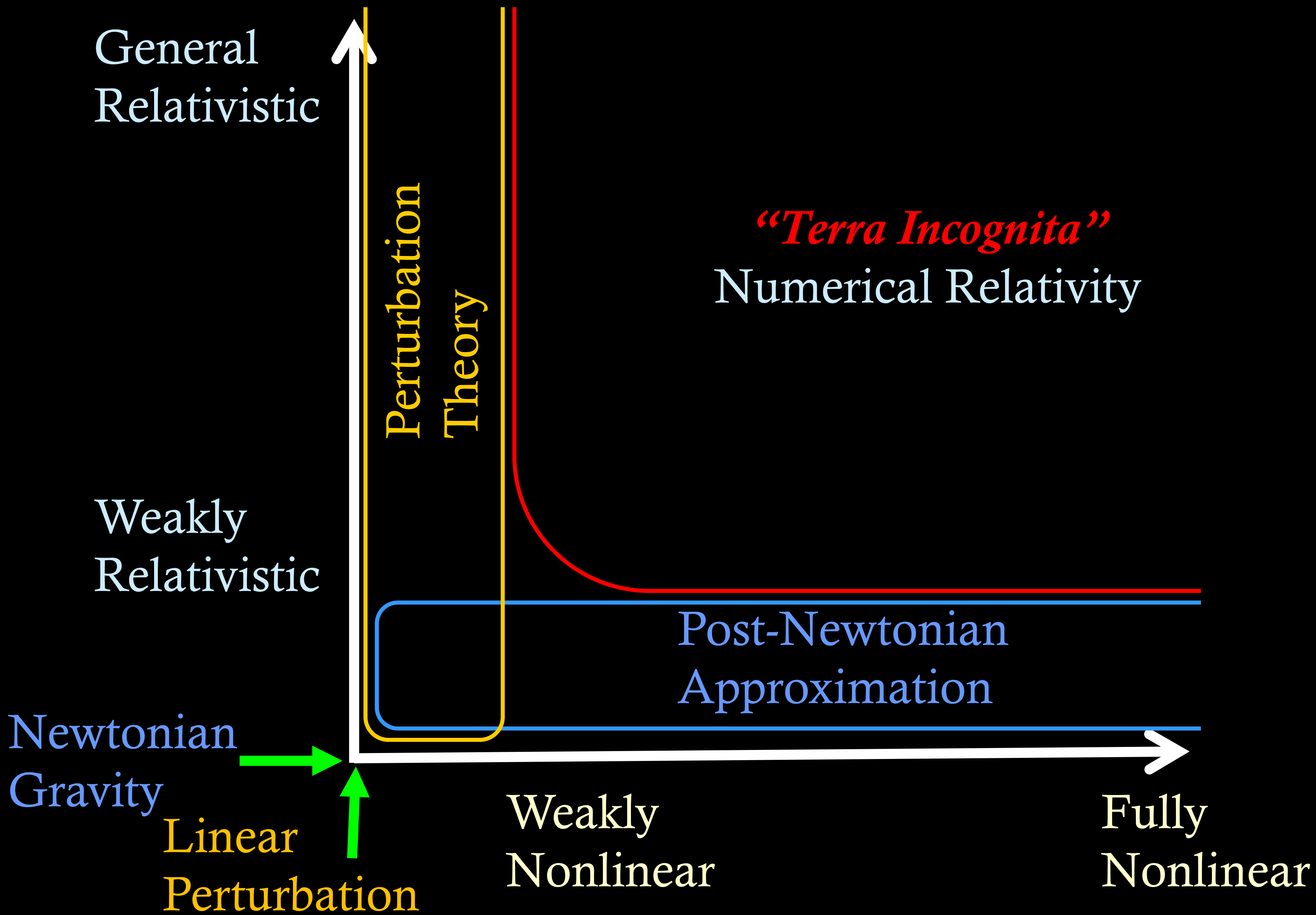
[8] Bonnor W. B., 1957, MNRAS, **117**, 104

[9] Ellis, G. F. R., 1989, in Einstein and the history of general relativity, ed. D. Howard and J. Stachel (Berlin, Birkhäuser), 367

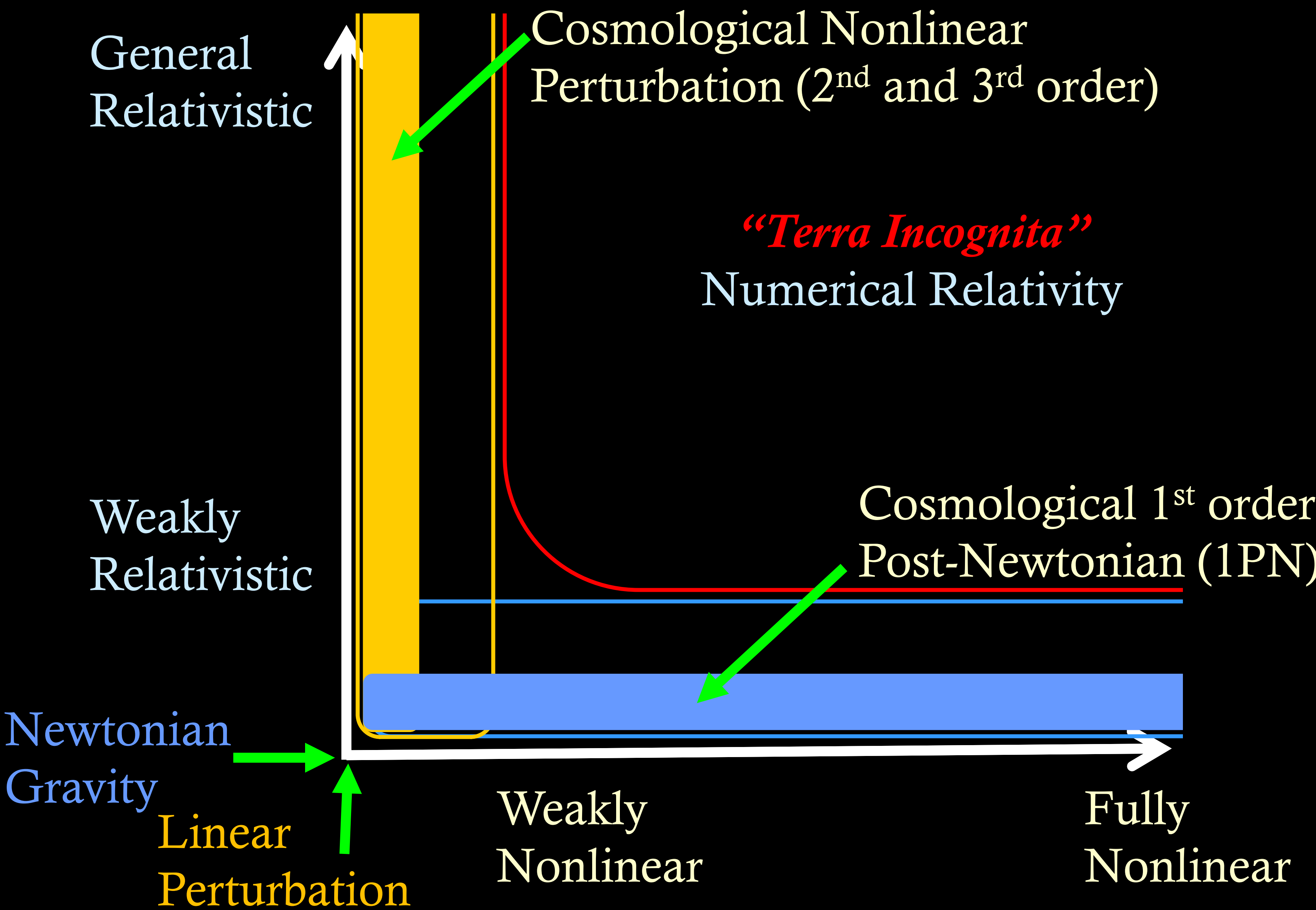
Studies of Large-scale Structure



Perturbation Theory *vs.* Post-Newtonian



Cosmology and Large-Scale Structure



Perturbation method:

- Perturbation expansion.
- All perturbation variables are small.
- Weakly nonlinear.
- Strong gravity; fully relativistic!
- Valid in all scales!

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time.
- Provide GR correction terms in the Newtonian equations of motion.
- Expansion in $\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} << 1$
- Fully nonlinear!
- No strong gravity situation; weakly relativistic.
- Valid far inside horizon

Weakly nonlinear
Fully relativistic
Approach

perturbation theory

Relativistic / Newtonian correspondence includes Λ, but assumes:

1. Flat Friedmann background
2. Zero-pressure
3. Irrotational
4. Single component fluid
5. No gravitational waves
6. Second order in perturbations

Relaxing any of these assumptions could lead to pure general relativistic effects!

Linear order: Lifshitz (1946)/Bonnor(1957) (comoving-synchronous gauge)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Second order: Peebles (1980)/Noh-JH (2004) (K=0, comoving gauge)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}),$$

Third order: JH-Noh (2005)

Curvature perturbation in the comoving gauge

$\sim 10^{-5}$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$

$$+ \frac{1}{a^2}\frac{\partial}{\partial t}\left\{a\left[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)\right]\cdot\nabla\delta\right\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$

$$+ \frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla(\Delta^{-1}X)\right] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},$$

$$X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi\right].$$

Pure General Relativistic corrections

Physical Review D **72** 044012 (2005)

Power spectrum

$$\delta \equiv \delta_1 + \delta_2 + \delta_3 + \cdots$$

$$\langle \delta(\mathbf{k})\delta(\mathbf{q})\rangle \equiv (2\pi)^3 \delta^D(\mathbf{k}+\mathbf{q})P(\mathbf{k})$$

$$P \equiv P_{11} + P_{22} + P_{13} + \cdots$$

General Relativistic contribution!

$$\delta_3 = \delta_{3,Newton} + \delta_{3,Einstein}$$

$$P_{13} = P_{13,Newton} + P_{13,Einstein}$$

Pure Einstein

Relativistic / Newtonian

D. Jeong, J. Gong, H. Noh, JH, ApJ (2011)

The Astrophysical Journal, 727:22 (5pp), 2011 January 20 doi:10.1088/0004-637X/727/1/22

GENERAL RELATIVISTIC EFFECTS ON NONLINEAR POWER SPECTRA

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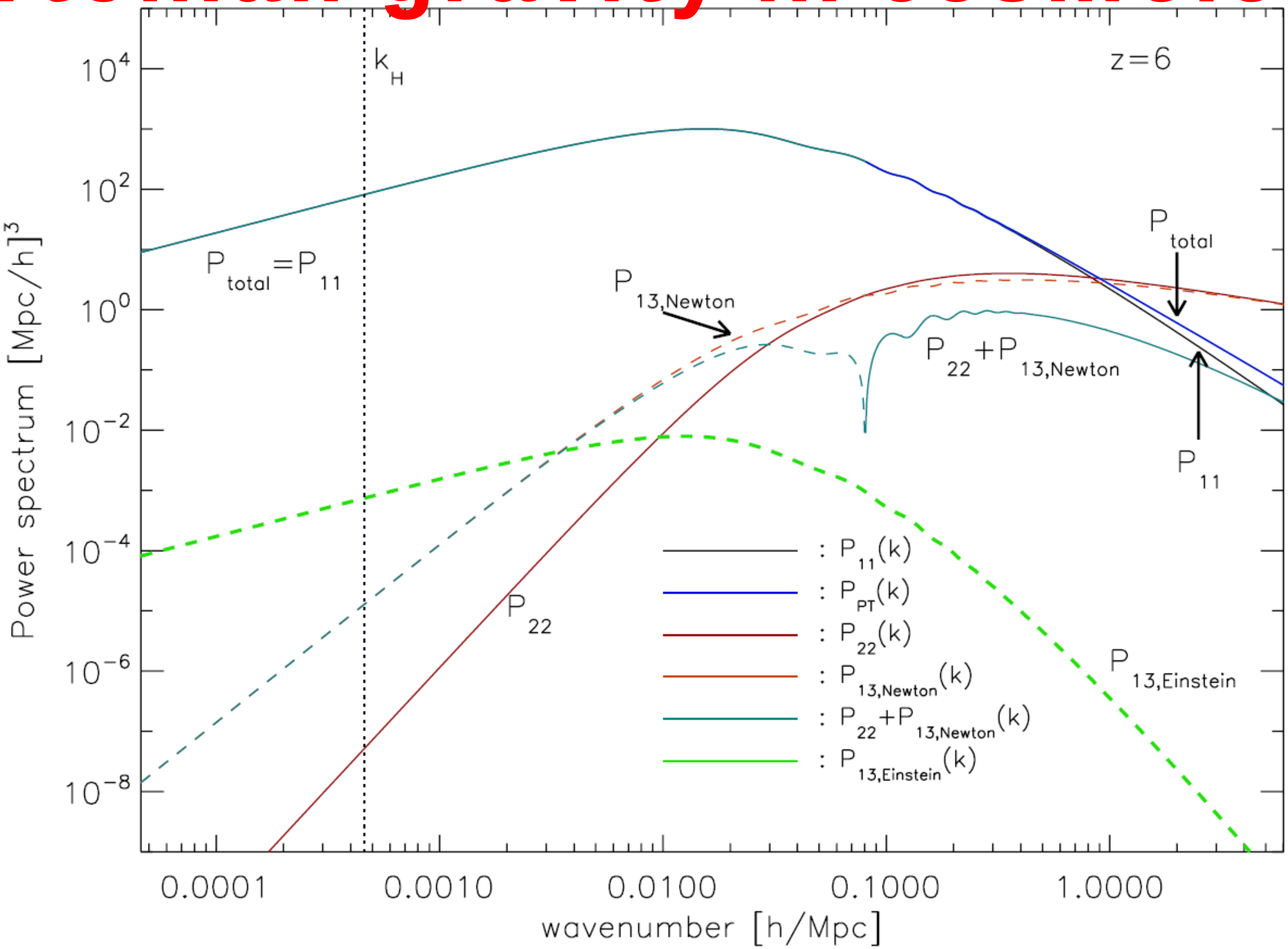
ABSTRACT

The nonlinear nature of Einstein's equation introduces genuine relativistic higher order corrections to the usual Newtonian fluid equations describing the evolution of cosmological perturbations. We study the effect of such novel nonlinearities on the next-to-leading order matter and velocity power spectra for the case of a pressureless, irrotational fluid in a flat Friedmann background. We find that pure general relativistic corrections are negligibly small over all scales. Our result guarantees that, in the current paradigm of standard cosmology, one can safely use Newtonian cosmology even in nonlinear regimes.

The unreasonable effectiveness of Newtonian gravity in cosmology!



D. Jeong, J. Gong, H. Noh, JH, ApJ (2011)

Weakly relativistic

Fully nonlinear

Approach

Post-Newtonian approximation

4. Cosmological post-Newtonian Approach

Perturbation method:

- Perturbation expansion.
- All perturbation variables are small.
- Weakly nonlinear.
- Strong gravity; fully relativistic!
- Valid in all scales!

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time.
- Provide GR correction terms in the Newtonian equations of motion.
- Expansion in v/c:

$$\frac{GM}{\lambda c^2} \sim \left(\frac{v}{c}\right)^2 \ll 1.$$

- Fully nonlinear!
- No strong gravity situation; weakly relativistic.
- Valid far inside horizon $\frac{GM}{\lambda c^2} \sim \left(\frac{\lambda}{c/H}\right)^2 \ll 1.$

Complementary!

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JCAP03(2008)010

Cosmological non-linear hydrodynamics with post-Newtonian corrections

Jai-chan Hwang[1], Hyerim Noh[2] and Dirk Puetzfeld[3]

Metric:

Newtonian limit:

$$\tilde{g}_{00} = -\left(1 - \frac{1}{c^2} 2U\right), \quad \tilde{g}_{0i} = 0, \quad \tilde{g}_{ij} = \delta_{ij}. \tag{37}$$

Newtonian gravitational potential

1PN metric [16]:

$$\tilde{g}_{00} = -\left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6},$$
$$\tilde{g}_{0i} = -\frac{1}{c^3} P_i + \mathcal{O}^{-5},$$
$$\tilde{g}_{ij} = \left(1 + \frac{1}{c^2} 2U\right)\delta_{ij} + \mathcal{O}^{-4}. \tag{38}$$

Minkowski background

Cosmological 1PN metric [17]:

$$\tilde{g}_{00} \equiv -\left[1 - \frac{1}{c^2} 2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] + \mathcal{O}^{-6},$$
$$\tilde{g}_{0i} \equiv -\frac{1}{c^3} a^2 P_i + \mathcal{O}^{-5},$$
$$\tilde{g}_{ij} \equiv a^2\left(1 + \frac{1}{c^2} 2V\right)\gamma_{ij} + \mathcal{O}^{-4}. \tag{39}$$

Robertson-Walker background

[16] Chandrasekhar, S., 1965, ApJ, **142**, 1488.
[17] Preprint, astro-ph/0507085.

JH, Noh, Puetzfeld, JCAP (2008)

Energy-momentum tensor:

Covariant decomposition:

$$\tilde{T}_{ab} = \tilde{\varrho} c^2 \left(1 + \frac{1}{c^2} \tilde{\Pi} \right) \tilde{u}_a \tilde{u}_b + \tilde{p} \left(\tilde{u}_a \tilde{u}_b + \tilde{g}_{ab} \right) + 2 \tilde{q}_{(a} \tilde{u}_{b)} + \tilde{\pi}_{ab}, \tag{40}$$

where $\tilde{q}_a \tilde{u}^a \equiv 0$, $\tilde{\pi}_{ab} \tilde{u}^b \equiv 0$, $\tilde{\pi}^c_c \equiv 0$, and $\tilde{\pi}_{ab} \equiv \tilde{\pi}_{ba}$.
Fluid four vector, $\tilde{u}_a$, follows from $\tilde{u}^a \tilde{u}_a \equiv -1$ and $\tilde{u}^i \equiv \frac{v^i}{c} \tilde{u}^0$.
We introduce

$$\tilde{\varrho} \equiv \varrho, \quad \tilde{\Pi} \equiv \Pi, \quad \tilde{p} \equiv p, \quad \tilde{q}_i \equiv \frac{1}{c} Q_i, \quad \tilde{\pi}_{ij} \equiv \Pi_{ij}. \tag{41}$$

Newtonian limit: **Newtonian, indeed!**

$$\frac{1}{a^3} \left(a^3 \varrho \right)^{\cdot} + \frac{1}{a} \nabla_i \left(\varrho v^i \right) = 0, \tag{42}$$

$$\frac{1}{a} \left(a v_i \right)^{\cdot} + \frac{1}{a} v^j \nabla_j v_i + \frac{1}{a \varrho} \left(\nabla_i p + \nabla_j \Pi^j_i \right) - \frac{1}{a} \nabla_i U = 0, \tag{43}$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{a} \mathbf{v} \cdot \nabla \right) \Pi + \left(3 \frac{\dot{a}}{a} + \frac{1}{a} \nabla \cdot \mathbf{v} \right) \frac{p}{\varrho} + \frac{1}{\varrho a} \left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i} \right) = 0, \tag{44}$$

$$\frac{\Delta}{a^2} U + 4 \pi G \left(\varrho - \varrho_b \right) = 0. \tag{45}$$

★ No gauge condition used!

★ We subtract the Friedmann background equation.

JH, Noh, Puetzfeld, JCAP (2008)

1PN equations:

For $K = 0$, we have $V = U$. In a gauge-ready form (assuming an ideal fluid):

1PN order

determined by Einstein's equation

$$\frac{1}{a^3}\left(a^3\varrho^*\right)^\cdot + \frac{1}{a}\left(\varrho^* v^i\right)_{|i} = 0, \tag{46}$$

$$\frac{1}{a}\left(a v_i^*\right)^\cdot + \frac{1}{a} v^*_{i|j} v^j = -\frac{1}{a}\left(1 + \frac{1}{c^2} 2U\right)\frac{p_{,i}}{\varrho^*} + \frac{1}{a}\left[1 + \frac{1}{c^2}\left(\frac{3}{2}v^2 - U + \Pi + \frac{p}{\varrho}\right)\right] U_{,i} + \frac{1}{c^2}\frac{1}{a}\left(2\Phi_{,i} - v^j P_{j|i}\right), \tag{47}$$

where

$$\varrho^* \equiv \varrho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right)\right], \quad v_i^* \equiv v_i + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + 3U + \Pi + \frac{p}{\varrho}\right)v_i - P_i\right]. \tag{48}$$

Metric variables (potentials) U, Φ and P_i are determined by

$$\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) + \frac{1}{c^2}\left\{\frac{1}{a^2}\left[2\Delta\Phi - 2U\Delta U + \left(a P^i{}_{|i}\right)^\cdot\right] + 3\ddot{U} + 9\frac{\dot{a}}{a}\dot{U} + 6\frac{\ddot{a}}{a}U + 8\pi G\left[\varrho v^2 + \frac{1}{2}\left(\varrho\Pi - \varrho_b\Pi_b\right) + \frac{3}{2}\left(p - p_b\right)\right]\right\} = 0, \tag{49}$$

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i + \frac{1}{a}\left(\frac{1}{a}P^j{}_{|j} + 4\dot{U} + 4\frac{\dot{a}}{a}U\right)_{,i}. \tag{50}$$

★ We can impose a temporal gauge condition on $P^i{}_{|i}$.

★ 1PN correction terms are $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \sim 10^{-5}$ order **smaller** than the Newtonian terms.

JH, Noh, Puetzfeld, JCAP (2008)

Gauge choice

Table 1. Gauge conditions to 1PN order.

Temporal gauge	Definition	
General gauge	$\frac{1}{a}P^i{}_{\vert i} + n\dot{U} + m\frac{\dot{a}}{a}U = 0$	
Chandrasekhar's gauge	$n = 3$, $m =$ arbitrary	~ Coulomb gauge
Uniform-expansion gauge	$n = m = 3$	
Transverse-shear gauge	$n = m = 0$	
Harmonic gauge	$n = 4$, $m =$ arbitrary	~ Lorenz gauge

Kinematic quantities based on $\tilde{n}^a$ become

$$\tilde{\theta} = \frac{1}{c}3\frac{\dot{a}}{a} + \frac{1}{c^3}\left(3\dot{V} + 3\frac{\dot{a}}{a}U + \frac{1}{a}P^i{}_{\vert i}\right) + L^{-1}\mathcal{O}^{-5},$$

$$\tilde{\sigma}_{ij} = \frac{1}{c^3}a\left(P_{(i\vert j)} - \frac{1}{3}P^k{}_{\vert k}\gamma_{ij}\right) + L^{-1}\mathcal{O}^{-5},$$

$$\tilde{\omega}_{ij} = 0,$$

$$\tilde{a}_i = -\frac{1}{c^2}U_{,i} - \frac{1}{c^4}2\Phi_{,i} + L^{-1}\mathcal{O}^{-6}.$$

$$\tilde{g}^{ab}\tilde{\Gamma}^0_{ab} = -\frac{1}{\sqrt{-\tilde{g}}}\left(\sqrt{-\tilde{g}}\tilde{g}^{0a}\right)_{,a} = \frac{1}{c}3\frac{\dot{a}}{a} + \frac{1}{c^3}\left(\frac{1}{a}P^i{}_{\vert i} + 4\dot{U} + 6\frac{\dot{a}}{a}U\right) + L^{-1}\mathcal{O}^{-5}.$$

JH, Noh, Puetzfeld, JCAP (2008)

Propagation speed issue

Apparently, the wave speed of the metric (potential) can take an arbitrary value depending on the temporal gauge condition that we choose. For example, if we take

$$\frac{1}{a} P^i{}_{|i} + n\dot{U} + m\frac{\dot{a}}{a}U \equiv 0, \tag{210}$$

as the gauge condition, with n and m real numbers, equations (119) and (120) give

$$\frac{\Delta}{a^2} P_i = -16\pi G \varrho v_i - \frac{1}{a}\left[(n-4)\dot{U} + (m-4)\frac{\dot{a}}{a}U\right]_{,i}, \tag{211}$$

$$\begin{aligned} &\frac{\Delta}{a^2}U + 4\pi G(\varrho - \varrho_b) \\ &\quad + \frac{1}{c^2}\left\{2\frac{\Delta}{a^2}\Phi - (n-3)\ddot{U} - (2n+m-9)\frac{\dot{a}}{a}\dot{U} + \left[(6-m)\frac{\ddot{a}}{a} - m\frac{\dot{a}^2}{a^2}\right]U \right. \\ &\quad \left. + 8\pi G\left[\varrho v^2 + \tfrac{1}{2}(\varrho\Pi - \varrho_b\Pi_b) + U(\varrho - \varrho_b) + \tfrac{3}{2}(p - p_b)\right]\right\} = 0. \end{aligned} \tag{212}$$

In this case, for $n \geq 3$, the speed of propagation corresponds to

$$\frac{c}{\sqrt{n-3}}, \tag{213}$$

which can take an *arbitrary* value depending on our choice of the value of n. It becomes c for $n = 4$ (e.g., the harmonic gauge), and infinity for $n = 3$ (e.g., Chandrasekhar's gauge and the uniform-expansion gauge). In the case of the transverse-shear gauge we have $n = 0$; thus equation (202) is no longer a wave equation. At this point one may

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Resolution using Weyl tensor:

$$\tilde{E}_{ab} \equiv \tilde{C}_{acbd}\tilde{u}^c\tilde{u}^d, \qquad \tilde{H}_{ab} \equiv \tfrac{1}{2}\tilde{\eta}_{ac}{}^{ef}\tilde{C}_{efbd}\tilde{u}^c\tilde{u}^d,$$

$$\tilde{E}^i_j = -\tilde{C}^i{}_{00j} = -\frac{1}{c^2}\frac{1}{a^2}\left[\frac{1}{2}\left(U+V\right)^{|i}{}_j - \frac{1}{6}\Delta\left(U+V\right)\delta^i_j\right] + L^{-2}\mathcal{O}^{-4},$$

$$\tilde{E}_{ij} \equiv E_{ij}, \qquad \tilde{H}_{ij} \equiv H_{ij},$$

where the indices of E_{ij} and H_{ij} are based on γ_{ij} as the metric.

$$\begin{aligned}
&\ddot{E}_{ij} + 3\frac{\dot{a}}{a}\dot{E}_{ij} + \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)E_{ij} \\
&\quad = \frac{c^2}{a^2}\left[\Delta E_{ij} - E^k_{(i|j)k} - \frac{1}{2}E_{k(i}{}^{|k}{}_{j)} + \frac{1}{2}\left(E^{kl}{}_{|kl} - \Delta E^k_k\right)\gamma_{ij} + \frac{1}{2}E^k_{k|ij}\right] \\
&\quad + \frac{1}{a^2}\left\{\left[(V+2U)_{,(i}E^k_{j)}\right]_{|k} - \left[(V+2U)^{,k}E_{ij}\right]_{|k} + \frac{1}{2}\left[(V+2U)^{,k}E_{k(i}\right]_{|j)}\right. \\
&\quad \left. - \frac{1}{2}\left[(V+2U)^{,l}E^k_l\right]_{|k}\gamma_{ij}\right\} + \frac{4\pi G}{c^2}\left(-\Delta\Pi_{ij} + \Pi^k_{(i|j)k}\right), \qquad (225)
\end{aligned}$$

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- PN corrections: $\frac{GM}{Rc^2} \sim \frac{\delta\Phi}{c^2} \sim \frac{v^2}{c^2} \sim 10^{-6} - 10^{-4}$
- Secular effects? Require numerical study.

- Newtonian: action at a distance (Laplacian)

  PN: propagation with speed of light (D'Alembertian)
- Propagation speed depends on the gauge choice!

  Propagation speed of the (electric and magnetic parts of) Weyl tensor is "c". → Gravity's propagation speed is "c"!
- Exists electromagnetic analogy:

  Propagation speed of potential depends on the gauge choice:

  Coulomb gauge *vs.* Lorentz gauge

  Propagation speed of the field is "c".

JH, Noh, Puetzfeld, JCAP (2008)

Conclusion

- **Perturbation method:** Fully relativistic but weakly nonlinear.
  Relativistic/Newtonian correspondence for zero-pressure to 2^{nd} .
  General relativistic corrections appear, otherwise.
  Pure relativistic third-order corrections: $\varphi_v \sim \frac{\delta\Phi}{c^2} \sim 10^{-5}$
- Pure GR correction in matter power spectrum: negligible
- Leading NL correction to inflation PS: negligible

- **PN approximation:** Fully nonlinear but weakly relativistic.
  PN corrections:
  $$\frac{GM}{Rc^2} \sim \frac{\delta\Phi}{c^2} \sim \frac{v^2}{c^2} \sim 10^{-6} - 10^{-4}$$
- Newtonian theory looks quite reliable in cosmological dynamics. (gravitational lensing requires PN effect!)
- Quantitative effects require numerical study.

Cosmology and Large-Scale Structure