Quantum field theory in curved spacetimes: What, why and how?

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Outline

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What?
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- Quantum field theory in curved spacetimes

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- Examples
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Motivations

Physical motivations:

- Gravity exists! we are not really in Minkowski spacetime.
- We do not have a working theory of quantum gravity.
- Important effects and predictions.
- Curved spacetimes and spontaneous particle creation:
  - Cosmology.
  - Black hole radiation.
- Flat spacetimes and spontaneous particle creation:
  - Unruh effect.
  - Casimir effect.
- What is a particle?

Mathematical motivations:

- What is the mathematical structure of quantum field theory?
- Are spacetime isometries and Hamiltonians necessary for defining QFT?
- What are states and observables in curved spacetimes?

QFT in flat spacetime

Minkowski: $(\mathbb{R}^4, \eta)$, $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

The theory consists of

- A Hilbert space (usually many particle Fock space), $\mathcal{H}$.
- Unitary operators $\Lambda(\alpha, L)$ representing Poincaré transformations (Minkowski isometries) ($+$ certain spectral conditions).
- A distinguished Poincaré-invariant state $\Omega \in \mathcal{H}$ – the vacuum.
- Observables O localised in Minkowski spacetime generated by quantum fields.
- O transform covariantly under Poincaré, e.g. $O(\lambda \mathrm{x}) = \Lambda(\alpha, L)^{-1} O(\mathrm{x}) \Lambda(\alpha, L)$.
- (Anti-)commutation relations $[O_1, O_2] \neq 0$.
- Some PDE – the field equation (KG, Dirac, Maxwell, YM...).

QFT in flat spacetimes

The Klein-Gordon fied

We heuristically know that:

- Fock space: $\mathcal{H} = \mathbb{C} \otimes_{\mathrm{sym}} \oplus_{n=1}^{\infty} L^2(\mathbb{R}^3)^{\otimes_{\mathrm{sym}} n}$
- Minkowski vacuum: $\Omega = (1, 0, 0, \ldots)$.
- Creation and annihilation operators, $\hat{a}_k^*, \hat{a}_k$:

$$[\hat{a}_k, \hat{a}_p] = 0 \quad [\hat{a}_k^*, \hat{a}_p^*] = 0 \quad [\hat{a}_k, \hat{a}_p^*] = \mathrm{i}\delta(k, p)\hat{\mathbb{1}}$$

  such that $\hat{a}_k \Omega = 0$.
- KG field obeying $(\Box - m^2)\hat{\Phi} = (-\partial_t^2 + \triangle - m^2)\hat{\Phi} = 0$,

$$\hat{\Phi}(t, x) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\omega_k)^{1/2}} \left(\mathrm{e}^{-\mathrm{i}\omega_k t} \psi_k(x) \hat{a}_k + \mathrm{e}^{\mathrm{i}\omega_k t} \overline{\psi_k(x)} \hat{a}_k^* \right).$$

with $\omega_k^2 = k \cdot k + m^2$, $\psi_k(x) = (2\pi)^{-3/2} \mathrm{e}^{\mathrm{i}k \cdot x}$ eigenvalues and generalised eigenfunctions of $-\triangle + m^2$.

QFT in curved spacetimes

- Suppose one attempts the previous construction in general, globally hyperbolic (M, g), such that curvature of g is non-trivial?

In General relativity the curvature of g is gravity, i.e.,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G_{\mathrm{N}}}{c^4} T_{ab}, \tag{1}$$

$$\text{Matter field equation.} \tag{2}$$

- R_{ab} and R: Ricci tensor and scalars, resp.
- Constants: Λ, G_{N}, c.
- Stress-energy tensor of matter: T_{ab}.
- Field equation, e.g. KG: $(\Box - m^2 + \xi R)\Phi = (g^{ab}\nabla_a \nabla_b - m^2 + \xi R)\Phi = 0.$

QFT in curved spacetimes

Example: KG in ultrastatic spacetime

Assume spacetime is ultrastatic: $g = \mathrm{d}t^2 + h$.

- Fock space: $\mathcal{H} = \mathbb{C} \otimes_{\mathrm{sym}} \oplus_{n=1}^{\infty} L^2(\Sigma, \mathrm{dvol}_h)^{\otimes_{\mathrm{sym}} n}$
- Distinguished vacuum: $\Omega = (1, 0, 0, \ldots)$.
- Creation and annihilation operators, $\hat{a}_j^*, \hat{a}_j$:

$$[\hat{a}_m, \hat{a}_n] = 0 \quad [\hat{a}_m^*, \hat{a}_n^*] = 0 \quad [\hat{a}_m, \hat{a}_n^*] = \mathrm{i}\delta_h(m, n)\hat{\mathbb{1}}$$

  such that $\hat{a}_m \Omega = 0$.
- KG field obeying $(\Box - m^2)\hat{\Phi} = (-\partial_t^2 + \triangle_h - m^2)\hat{\Phi} = 0$,

$$\hat{\Phi}(t, x) = \int_\sigma \frac{\mathrm{d}\mu(j)}{(2\omega_j)^{1/2}} \left(\mathrm{e}^{-\mathrm{i}\omega_j t} \psi_j(x) \hat{a}_j + \mathrm{e}^{\mathrm{i}\omega_j t} \overline{\psi_j(x)} \hat{a}_j^* \right).$$

with $(-\triangle_h + m^2)\psi_j = \omega_j^2 \psi_j$.

QFT in curved spacetimes

- In previous case there exists distinguished notion of time, generated by global timelike KVF, ∂_t.
- Not every spacetime has timelike KVF.
- Assume a spacetime has asymptotic KVF, v in the remote future and w in the remote past.

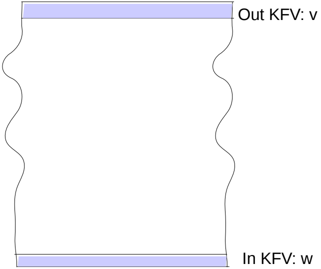


Figure: Spacetime where QFT admits in and out construction.

QFT in curved spacetimes

- Two distinguished constructions: $\mathcal{H}_{\rm in}$ and $\mathcal{H}_{\rm out}$.
- Φ admits representation as op. in $\mathcal{H}_{\rm in}$ and as op. in $\mathcal{H}_{\rm out}$, namely

$$\hat{\Phi}_{\rm in}(t,x) = \int \frac{{\rm d}\mu_{\rm in}(i)}{(2\omega_{{\rm in},i})^{1/2}} \left({\rm e}^{-{\rm i}\omega_{{\rm in},i} t_1} \psi_{{\rm in},i}(x) \hat{a}_i + {\rm e}^{{\rm i}\omega_{{\rm in},i} t_1} \overline{\psi_{{\rm in},i}(x)} \hat{a}_i^* \right),$$

$$\hat{\Phi}_{\rm out}(t,x) = \int \frac{{\rm d}\mu_{\rm out}(j)}{(2\omega_{{\rm out},j})^{1/2}} \left({\rm e}^{-{\rm i}\omega_{{\rm out},j} t_2} \psi_{{\rm out},j}(x) \hat{b}_j + {\rm e}^{{\rm i}\omega_{{\rm out},j} t_2} \overline{\psi_{{\rm out},j}(x)} \hat{a}_j^* \right)$$

- Hamiltonians: $\hat{H}_{\rm in} = \int {\rm d}\mu_{{\rm in},i} \omega_{{\rm in},i} \hat{a}_i^* \hat{a}_i$, $\hat{H}_{\rm out} = \int {\rm d}\mu_{{\rm out},j} \omega_{{\rm out},j} \hat{b}_j^* \hat{b}_j$.
- Formally $\hat{b}_j = \int {\rm d}\mu_{\rm in}(i) \left(\alpha_{ij} \hat{a}_i + \beta_{ij} \hat{a}_i^* \right)$.
- $\langle \Omega_{\rm in} | \hat{H}_{\rm in} \Omega_{\rm in} \rangle = 0 = \langle \Omega_{\rm out} | \hat{H}_{\rm out} \Omega_{\rm out} \rangle$, but

$$\langle \Omega_{\rm in} | \hat{H}_{\rm out} \Omega_{\rm in} \rangle = \int {\rm d}\mu_{\rm out}(j) \omega_{{\rm out},j} \langle \Omega_{\rm in} \hat{b}_j^* \hat{b}_j \Omega_{\rm in} \rangle$$

$$= \int {\rm d}\mu_{\rm out}(j) \omega_{{\rm out},j} \int {\rm d}\mu_{\rm in}(i) |\beta_{ij}|^2 \neq 0 \Rightarrow$$ $\Omega_{\rm in}$ **not vacuum in** $\mathcal{H}_{\rm out}$**!**

QFT in curved spacetimes

Previous discussion implies that spacetime curvature **produces spontaneous particle creation!**
Question: In previous example, is the vacuum state empty or not? Who is right? Answer: Everyone is right! Relativity's lesson: space and time notions are observer-dependent. **QFT in CS shows that the notion of particle is observer-dependent too!**

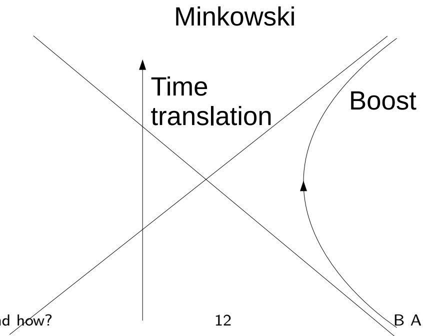
- Relevant e.g. in cosmology.
- In black holes Hawking radiation is understood similarly, but the situation is even more interesting when considering backreaction.
- Emphasis: QFT is a theory of **fields** not of **particles**. The quantum field is the **fundamental** object; particles (like beauty) are in the **eye of the beholder**.

Back to flat spacetime

- In general curved spacetimes the notion of particle is abiguous.
- In absence of symmetries, there is no **distinguished** notion of particle (but fields and correlation functions are unambiguously defined).
- In fact, in Minkowski spacetime there are also different notions of particles.
- The Unruh effect shows that for a linearly uniformly accelerated observer in Minkowski spacetime the Minkowski vacuum looks like a thermal state at temperature proportional to their acceleration.

Unruh effect

- Consider Minkowski spacetime and divide in four so-called Rindler wedges.
- On right Rindler wedge Lorentz boosts are timelike KVF, $\xi = a(x\partial_t + t\partial_x)$, $a > 0$.
- Quantisation w.r.t. time-notion generated by ξ: Fulling-Rindler vacuum, Ω_{FR}.
- Minkowski vacuum looks thermal in FR representation.

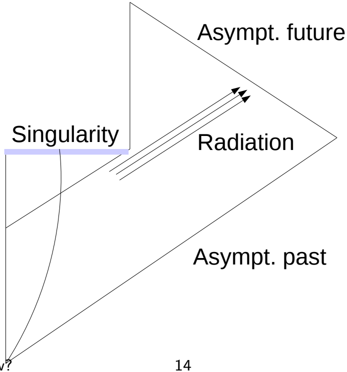


Unruh effect

- A linearly uniformly accelerated observer with acceleration a, perceives the Minkowski vacuum at a temperature $T_{\mathrm{U}} = \frac{\hbar a}{2\pi c k_{\mathrm{B}}}$ – the Unruh temperature.
- ¡T_{U} is very cold! E.g., $10^{20} m/s$, $T_{\mathrm{U}} \sim 1K$.
- Detecting the Unruh temperature is a massive experimental challenge. (Detection $\Rightarrow$ Nobel prize.)
- In Minkowski spacetime the notion of particle is also observer-dependent!
- More generally, an observer following an arbitrary trajectory, will generally too detect particles when interacting with a field in the Minkowski vacuum, although not in a thermal distribution. This can be understood in terms of so-called particle detectors in QFT in CS.

Evaporation of black holes

- The gravitational effects during black hole formation produce quantum radiation.
- Radiation makes black holes lose mass. M decreases.
- Radiation is thermal at $T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G_{\mathrm{N}} k_{\mathrm{B}} M}$.
- We expect black holes to theoretically radiate off all of their mass and disappear.
- Information loss puzzle.



Mathematical structure of QFT

In the absense of distinguished representations for the fields and vacua, what is really QFT?

Example: KG theory in CS

Let (M, g) be a globally hyperbolic spacetime (iff admits a Cauchy surface). We call $\mathcal{A}(M)$ the Klein-Gordon field algebra the unital *-algebra generated by *smeared fields* $\Phi(f)$ with $f \in C_0^\infty(M)$ subject to the relations

1. Linearity: $f \mapsto \Phi(f)$ is linear,
2. Hermiticity: $\Phi(\overline{f}) = \Phi(f)^*$,
3. Field equation: $\Phi((\Box - m^2 - \xi R)f) = 0$,
4. Commutation relations: $[\Phi(f), \Phi(g)] = -\mathrm{i}E(f, g)\mathbb{1}$,

where $E = E^+ - E^-$ causal propagator of the Klein-Gordon operator $(\Box - m^2 - \xi R)$.

Mathematical structure of QFT

What is a state for KG theory in CS?

We call a linear map $\omega : \mathcal{A}(M) \to \mathbb{C}$ an algebraic state if

1. $\omega(\mathbb{1}) = 1$
2. $\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}(M)$.

ω prescribes correlation functions, but notice that we have commited to no Hilbert space representation. E.g. the Wightman two-point function of the theory is

$$\omega(\Phi(f)\Phi(g)) = G^+(f,g). \tag{3}$$

- If ω is quasi-free, all n-point functions follow from Wightman function.
- Renormalisation requires Hadamard property.

$$G^+(x,y) - \left[\frac{1}{2\pi^2} \frac{\Delta^{1/2}(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln(\sigma_\epsilon(x,y)) \right] \in C^\infty(M \times M)$$

Mathematical structure of QFT

How to recover Hilbert space representation?

The GNS construction theorem allows one to obtain a Hilbert space representation, such that

- Input: algebra $\mathcal{A}(M)$ and state $\omega : \mathcal{A}(M) \to \mathbb{C}$.
- Output: GNS representation $(\mathcal{H}_\omega, \pi_\omega, \mathcal{D}_\omega, \Omega_\omega)$

Such that π_ω is faithful, $\mathcal{D}_\omega \subset \mathcal{H}_\omega$ is dense, Ω_ω is cyclic (the "vacuum" state) and

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle. \tag{4}$$

In particular, $\pi(\Phi(f)) = \hat{\Phi}(f)$ is an operator-valued distribution in Hilbert space – our textbook quantum field and

$$\omega(\Phi(f)\Phi(g)) = \langle \Omega_\omega | \hat{\Phi}(f) \hat{\Phi}(g) \Omega_\omega \rangle \tag{5}$$

Final remarks

- QFT is a theory of fields, while the notion of particle is secundary.
- No distinguished representation for QFT in CS: Algebraic programme. The KG construction here presented can be generalise to other (non-linear) field theories. Modern formulation in terms of category theory.
- Experimental challenges: Unruh & Hawking effects $\Rightarrow$ Nobel.
- Theoretical challenges: Black hole evaporation, measurement problem, semiclassical gravity.
- Mathematical challenges: Mathematical structure of QFT, perturbative problems, renormalisation, semiclassical gravity.
- Technological applications? (Exploiting Unruh, Casimir effects...?)

Some literature:

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Thanks!