Mirror Symmetry of Pairings

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An identity

Consider following formal power series:

$$\phi(q) = \sum_{k\in\mathbb{Z}} (-1)^{k+1}(k+\frac{1}{2})q^{(6k+3)^2},$$

$$\psi(q) = \sum_{k\in\mathbb{Z}} (-1)^{k+1}(6k+1)q^{(6k+1)^2},$$

$$c = \sum_{k\geq 0} (-1)^k \left(q^{(6k+1)^2} - q^{(6k+5)^2}\right).$$

Theorem (Cho-Kim-L.-Shin)

$$c^2\left(\phi\cdot q\frac{\partial\psi}{\partial q} - \psi\cdot q\frac{\partial\phi}{\partial q}\right) = 8\phi(27\phi^3 - \psi^3).$$

Proof & Why

$c^2\left(\phi \cdot q\frac{\partial \psi}{\partial q} - \psi \cdot q\frac{\partial \phi}{\partial q}\right)$ and $8\phi(27\phi^3 - \psi^3)$ are modular forms with same level and weight, so it suffices to check the identity up to certain order (576 in this case).

Question

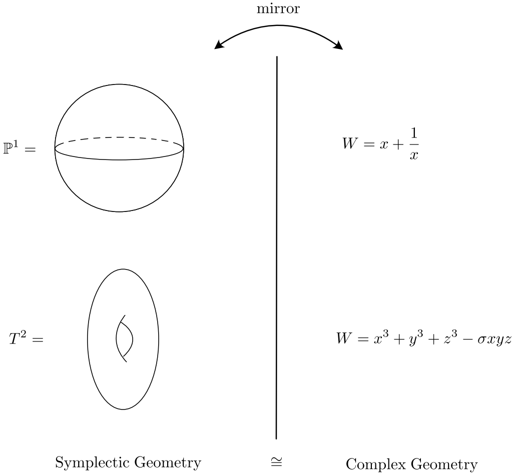
Where on earth do they come from?

Answer

They arose in the study of pairings of closed string algebras*.*

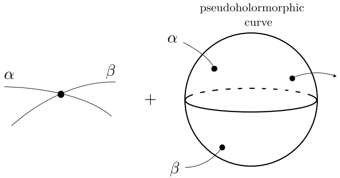
Mirror symmetry

- Mirror symmetry means an equivalence of two string theories.
- Mathematically, it is a duality between symplectic geometry and complex geometry.



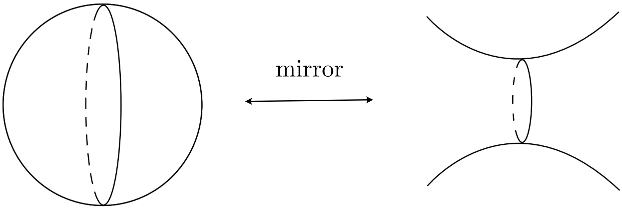
Closed string mirror symmetry

mirror

	Quantum cohomology	Jacobian ring
multiplication		$\left(\frac{\mathbb{C}[x,x^{-1}]}{\partial_x W}, \cdot\right)$
inner product	$\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta$	$\langle f, g \rangle = \oint_{\{\lvert\partial_x W\rvert = \epsilon\}} \frac{fg}{\partial_x W} dx$

Example: $\mathbb{P}^1$

Construction of a mirror pair is due to T-duality, by Strominger-Yau-Zaslow.



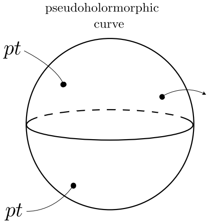
$\mathbb{P}^1$

$$W = x + \frac{1}{x}$$

Mirror function is a disc counting on $\mathbb{P}^1$.
(cf. Cho-Oh, Fukaya-Oh-Ohta-Ono etc)

Example: $\mathbb{P}^1$

mirror



$pt \cup_Q pt$
$= 1 \cdot \mathbb{P}^1$

$$W = x + \frac{1}{x}$$

$$\left(\frac{\mathbb{C}[x, x^{-1}]}{\partial_x W}, \ \cdot \ \right), \qquad x^2 = 1$$

$$\langle pt, \mathbb{P}^1 \rangle = 1$$

$$\langle x, 1 \rangle = \oint_{\{|\partial_x W| = \epsilon\}} \frac{x}{\partial_x W} dx = 1$$

Quantum cohomology $\cong$ Jacobian ring

History

- There have been several works on equivalences of ring structures (cf. Batyrev, Givental, Iritani, Fukaya-Oh-Ohta-Ono etc).
- There have been only few works on equivalences of pairings (cf. Fukaya-Oh-Ohta-Ono '10, for toric manifolds with Morse mirror functions).
- Idea (Cho-L.-Shin): modify the ring isomorphism suitably by the *quantum volume* c.

An elliptic orbifold sphere $\mathbb{P}^1_{3,3,3}$

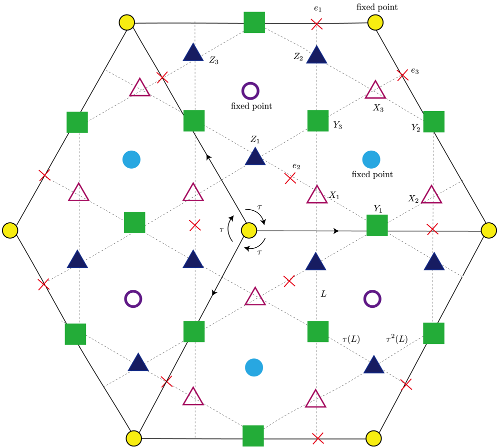
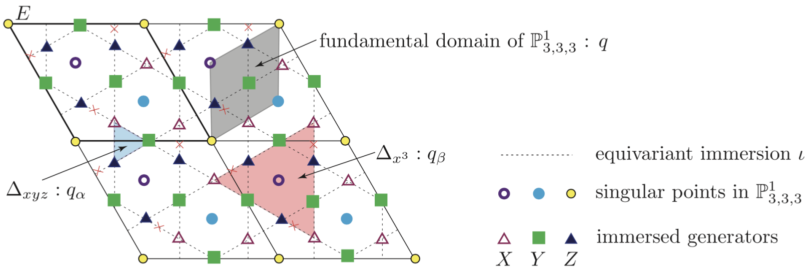


Figure: $\mathbb{Z}/3$-group action on T^2

An elliptic orbifold sphere $\mathbb{P}^1_{3,3,3}$



Mirror function is given by counting suitable discs whose boundaries are on the dotted lines.

$$W = \phi(q)(x^3 + y^3 + z^3) - \psi(q)xyz.$$

$\phi(q)$ and $\psi(q)$ are areas of discs.

Closed string mirror symmetry of $\mathbb{P}^1_{3,3,3}$

Theorem (Amorim-Cho-Hong-Lau)

There is a ring isomorphism

$$\mathfrak{ks} : QH(\mathbb{P}^1_{3,3,3}) \to Jac(W),$$

and in particular,

$$\mathfrak{ks}(pt) = \frac{1}{8} q \frac{\partial W}{\partial q}.$$

Question

Does $\mathfrak{ks}$ also preserve pairings?

What to check

If $\mathfrak{ks}$ preserves pairings, then the following should commute:

$$\begin{tikzcd} QH(\mathbb{P}^1_{3,3,3}) \arrow[r, "tr_A"] \arrow[d, "\mathfrak{ks}"'] & k \arrow[d, "id"] \\ Jac(W) \arrow[r, "tr_B"] & k, \end{tikzcd}$$

where

$$tr_A(\alpha) = \langle \alpha, 1 \rangle = \int_{\mathbb{P}^1_{3,3,3}} \alpha,$$

$$tr_B(f) = \langle f, 1 \rangle = Res\left(\frac{f}{\partial_x W \cdot \partial_y W \cdot \partial_z W} \right).$$

What to check

If $\mathfrak{ks}$ preserves pairings, then the following should commute:

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QH(\mathbb{P}^1_{3,3,3}) \arrow[r, "tr_A"] \arrow[d, "\mathfrak{ks}"'] & k \arrow[d, "id"] \\
Jac(W) \arrow[r, "tr_B"] & k,
\end{tikzcd}$$

hence

$$tr_A(pt) = \int_{\mathbb{P}^1_{3,3,3}} pt = 1,$$

$$tr_B(\mathfrak{ks}(pt)) = Res \left(\frac{\frac{1}{8} q \frac{\partial W}{\partial q}}{\partial_x W \cdot \partial_y W \cdot \partial_z W} \right) = 1.$$

Computation of residue pairing

$$tr_B(\mathfrak{ks}(pt)) = Res\left(\begin{array}{c} \frac{1}{8}q\frac{\partial W}{\partial q} \\ \partial_x W \cdot \partial_y W \cdot \partial_z W \end{array}\right) = Res\left(\begin{array}{c} \frac{1}{8}q\frac{\partial W}{\partial q} \cdot \det C \\ x^l y^l z^l \end{array}\right),$$

where

$$C \cdot \begin{pmatrix} \partial_x W \\ \partial_y W \\ \partial_z W \end{pmatrix} = \begin{pmatrix} x^l \\ y^l \\ z^l \end{pmatrix}.$$

Define

$$C = \frac{1}{3\phi(27\phi^3 - \psi^3)} \begin{pmatrix} (27\phi^3 - \psi^3)x^2 - 9\phi^2\psi yz & 3\phi\psi^2 z^2 & \psi^3 xz \\ -\psi^3 xy & (27\phi^3 - \psi^3)y^2 + 9\phi^2\psi xz & 3\phi\psi^2 x^2 \\ 3\phi\psi^2 y^2 & -\psi^3 yz & (27\phi^3 - \psi^3)z^2 - 9\phi^2\psi xy \end{pmatrix},$$

then

$$C \cdot \begin{pmatrix} \partial_x W \\ \partial_y W \\ \partial_z W \end{pmatrix} = \begin{pmatrix} x^4 \\ y^4 \\ z^4 \end{pmatrix}.$$

$\mathfrak{ks}$ does NOT preserve pairings

We come up with

$$tr_B(\mathfrak{ks}(pt)) = Res_{\{x=y=z=0\}}\left(\begin{array}{c} \frac{1}{8} q \frac{\partial W}{\partial q} \cdot \det C \\ x^4 y^4 z^4 \end{array}\right) = 1$$

$$\Leftrightarrow \phi \cdot q \frac{\partial \psi}{\partial q} - \psi \cdot q \frac{\partial \phi}{\partial q} = 8\phi(27\phi^3 - \psi^3),$$

which is not true. Indeed, recalling the theorem,

$$c^2\left(\phi \cdot q \frac{\partial \psi}{\partial q} - \psi \cdot q \frac{\partial \phi}{\partial q}\right) = 8\phi(27\phi^3 - \psi^3)$$

for some power series c.

Question

What is c?

Homological Mirror Symmetry

The right framework to investigate pairings is provided by Homological Mirror Symmetry.

$$
\begin{tikzcd}
Fuk(L,L) \arrow[r, "OC"] \arrow[d, "\mathcal{F}"'] \arrow[dr, "Z"] & QH(X) \arrow[d, "\mathfrak{ks}"] \\
MF(\mathcal{F}L, \mathcal{F}L) \arrow[r, "OC"] & Jac(W)
\end{tikzcd}
$$

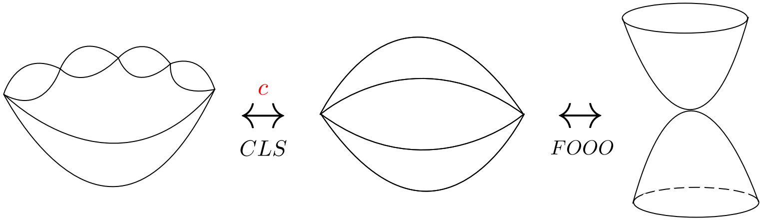
Theorem (Fukaya-Oh-Ohta-Ono)

The upper diagram commutes.

Theorem (Cho-L.-Shin)

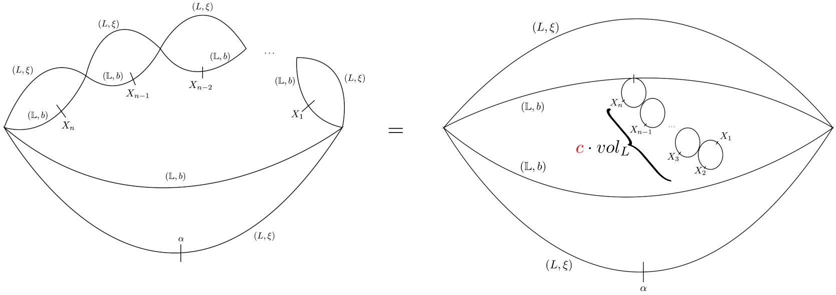
The lower diagram commutes with scaling factor c.

Proofs by figures



$OC \circ \mathcal{F}$ $\quad$ Z $\quad$ $\mathfrak{ks} \circ OC$

c as a quantum volume



c is given by *Lagrangian quantum products* of degree 1 elements.

Modification of $\mathfrak{ks}$

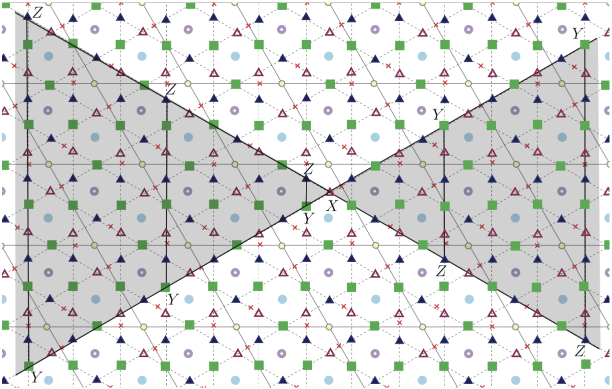
Combining two theorems above, we are led to the modified diagram:

$$\begin{tikzcd}
Fuk(L,L) \arrow[r, "OC"] \arrow[d, "\mathcal{F}"'] & QH(X) \arrow[d, "c\cdot\mathfrak{ks}"] \\
MF(\mathcal{F}L,\mathcal{F}L) \arrow[r, "OC"] & Jac(W).
\end{tikzcd}$$

Goal

Show that $c \cdot \mathfrak{ks}$ preserves pairings.

Computation of c in $\mathbb{P}^1_{3,3,3}$



Counting of triangles gives rise to

$$c = \sum_{k \geq 0} (-1)^k \left(q^{(6k+1)^2} - q^{(6k+5)^2} \right).$$

Mirror symmetry of pairings for $\mathbb{P}^1_{3,3,3}$

$c \cdot \mathfrak{ks} : QH(\mathbb{P}^1_{3,3,3}) \to Jac(W)$ preserves pairings

$$\Leftrightarrow c^2 tr_B(\mathfrak{ks}(pt)) = c^2 Res \left(\begin{array}{c} \frac{1}{8} q \frac{\partial W}{\partial q} \cdot \det C \\ x^4 y^4 z^4 \end{array} \right) = 1$$

$$\Leftrightarrow c^2 \left(\phi \cdot q \frac{\partial \psi}{\partial q} - \psi \cdot q \frac{\partial \phi}{\partial q} \right) = 8\phi(27\phi^3 - \psi^3).$$

Now, we figured out the meaning of the mysterious identity.

Generalization of Fukaya-Oh-Ohta-Ono's result

For compact toric manifolds, the following was known.

Theorem (Fukaya-Oh-Ohta-Ono '10)

$\mathfrak{ks}$ *preserves pairings up to some* constant, *when the toric manifold has Morse mirror function.*

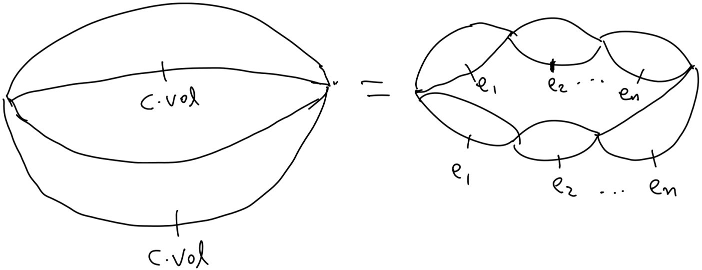
- We found that the constant is c.
- By considering

$$\begin{tikzcd} Fuk(L,L) \arrow[r, "OC"] \arrow[d, "\mathcal{F}"'] & QH(X) \arrow[d, "c\cdot\mathfrak{ks}"] \\ MF(\mathcal{F}L,\mathcal{F}L) \arrow[r, "OC"] & Jac(W), \end{tikzcd}$$

  we can generalize FOOO's theorem.

Idea of the generalization

By disassmbling holomorphic annulus, we have



Theorem (Polishchuk-Vaintrob, Amorim-Cho-L.)

The right hand side is $\det(Hess(W))$.

Now we can use a property of residues (see Hartshorne's *"Residue and duality"* for example):

$$Res\left(\frac{\det(Hess(W))}{\partial_{x_1} W \cdots \partial_{x_n} W}\right) = \mu \text{ where } \mu = \dim Jac(W).$$

What's next?

- Deal with manifolds before taking global quotients.
- Consider Mukai pairings on Hochschild homologies, and investigate the relation with previous pairings.

Thank you very much!