Pólya enumeration theorems in algebraic geometry

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Macdonald's formula

Given a topological space X, we define its n-th **symmetric power** as the quotient space

$$\mathrm{Sym}^n(X) := X^n/S_n,$$

where S_n actions on X^n by permuting coordinates.

Examples

1. $\mathrm{Sym}^n(\mathbb{A}^1) \simeq \mathbb{A}^n$.
2. $\mathrm{Sym}^n(\mathbb{P}^1) \simeq \mathbb{P}^n$.

Macdonald's formula

If Y is a topological space with finite singular Betti numbers $h^0(Y), h^1(Y), h^2(Y), \dots$, then we define the **Poincaré series** of Y as:

$$\chi_u(Y) := \sum_{i=0}^{\infty} (-u)^i h^i(Y),$$

which generalizes the Euler characteristics since $\chi_1(Y) = \chi(Y)$, as long as $h^i(Y) = 0$ for $i \gg 0$.

Theorem (Macdonald)

Let X be a compact complex manifold of (complex) dimension d. Then

$$\sum_{n=0}^{\infty} \chi_u(\mathrm{Sym}^n(X)) t^n = \frac{(1-ut)^{h^1(X)} \cdots (1-u^{2d-1}t)^{h^{2d-1}(X)}}{(1-t)^{h^0(X)} \cdots (1-u^{2d}t)^{h^{2d}(X)}}.$$

Grothendieck's formula

Given a variety X over a finite field $\mathbb{F}_q$, the **zeta series** of X is

$$Z_X(t) := \exp\left(\sum_{r=1}^{\infty} \frac{|X(\mathbb{F}_{q^r})|t^r}{r}\right) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}.$$

Theorem (Grothendieck)

Let X be a projective variety of dimension d. Then

$$Z_X(t) = \frac{\det(\mathrm{id}_{H^1(X)} - \mathrm{Frob}_{q,1}^* t) \cdots \det(\mathrm{id}_{H^{2d-1}(X)} - \mathrm{Frob}_{q,2d-1}^* t)}{\det(\mathrm{id}_{H^0(X)} - \mathrm{Frob}_{q,0}^* t) \cdots \det(\mathrm{id}_{H^{2d}(X)} - \mathrm{Frob}_{q,2d}^* t)}.$$

The notation Frob_q (called the **Frobenius**) means an automorphism on X, which gives the q-th power on the points over $\overline{\mathbb{F}_q}$. Here, we wrote

- $H^i(X)$ to mean an l-adic étale cohomology of $X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and
- $\mathrm{Frob}^*_{q,i}$ the $\mathbb{Q}_l$-linear map on $H^i(X)$ induced by Frob_q.

Why is this relevant to Macdonald's formula?

Observation (Kapranov)

For any quasi-projective variety X over $\mathbb{F}_q$, we have

$$Z_X(t) = \sum_{n=0}^{\infty} |\mathrm{Sym}^n(X)(\mathbb{F}_q)| t^n.$$

Grothendieck's formula

Example

We have

$$Z_{\mathbb{A}^1_{\mathbb{F}_q}}(t) = \prod_{\substack{P \in \mathbb{F}_q[t], \\ \text{monic irred}}} \frac{1}{1 - t^{\deg(P)}} = \prod_{\substack{P \in \mathbb{F}_q[t], \\ \text{monic irred}}} (1 + t^{\deg(P)} + t^{\deg(P^2)} + \cdots)$$

$$= \sum_{\substack{f \in \mathbb{F}_q[t], \\ \text{monic}}} t^{\deg(f)} = \sum_{n=0}^{\infty} q^n t^n,$$

and

$$|\mathrm{Sym}^n(\mathbb{A}^1)(\mathbb{F}_q)| = |\mathbb{A}^n(\mathbb{F}_q)| = q^n.$$

Grothendieck's formula

Example

Using the Euler product, one can observe

$$Z_{\mathbb{P}^1_{\mathbb{F}_q}}(t) = \left(\frac{1}{1-t}\right) Z_{\mathbb{A}^1_{\mathbb{F}_q}}(t) = (1 + t + t^2 + \cdots)(1 + qt + q^2t^2 + \cdots)$$
$$= \sum_{n=0}^{\infty} (1 + q + q^2 + \cdots + q^n)t^n.$$

We may also check

$$|\mathrm{Sym}^n(\mathbb{P}^1)(\mathbb{F}_q)| = |\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \cdots + q^n.$$

Comparing the two formulas

We now compare the two formulas:

1. (Macdonald)

$$\sum_{n=0}^{\infty} \chi_u(\mathrm{Sym}^n(X))t^n = \frac{(1-ut)^{h^1(X)}\cdots(1-u^{2d-1})^{h^{2d-1}(X)}}{(1-t)^{h^0(X)}\cdots(1-u^{2d})^{h^{2d}(X)}}.$$

2. (Grothendieck)

$$\sum_{n=0}^{\infty} |\mathrm{Sym}^n(X)(\mathbb{F}_q)|t^n$$
$$= \frac{\det(\mathrm{id}_{H^1(X)} - \mathrm{Frob}^*_{q,1}t)\cdots\det(\mathrm{id}_{H^{2d-1}(X)} - \mathrm{Frob}^*_{q,2d-1}t)}{\det(\mathrm{id}_{H^0(X)} - \mathrm{Frob}^*_{q,0}t)\cdots\det(\mathrm{id}_{H^{2d}(X)} - \mathrm{Frob}^*_{q,2d}t)}.$$

Comparing the two formulas

It turns out

$$\begin{aligned}
|\mathrm{Sym}^n(X)(\mathbb{F}_q)| &= \left|(\mathrm{Sym}^n(X)(\overline{\mathbb{F}_q}))^{\mathrm{Frob}_q}\right| \\
&= \sum_{i=0}^{\infty}(-1)^i \mathrm{Tr}(\mathrm{Frob}_{q,i}^* \circlearrowleft H^i(\mathrm{Sym}^n(X), \mathbb{Q}_l)) \\
&=: L_1(\mathrm{Frob}_q^*),
\end{aligned}$$

the Lefschetz number of Frob_q^*, where l does not divide q.

Definition

If $\phi = \oplus_{i=0}^{\infty}\phi_i$ is a graded linear endomorphism on a graded vector space $V = \bigoplus_{i=0}^{\infty} V_i$, then the **Lefschetz series** is

$$L_u(\phi) := \sum_{i=0}^{\infty}(-u)^i \mathrm{Tr}(\phi_i).$$

Comparing the two formulas

Theorem (C.)

Let X be either

1. a compact complex manifold of dimension d (singular $H^\bullet$) or
2. a projective variety of dimension d over a finite field $\mathbb{F}_q$ (l-adic $H^\bullet$).

Then for any endomorphism F on X, we have

$$\sum_{n=0}^{\infty} L_u(\mathrm{Sym}^n(F)^*)t^n = \frac{\det(\mathrm{id}_{H^1(X)} - F_1^* ut) \cdots \det(\mathrm{id}_{H^{2d-1}(X)} - F_{2d-1}^* u^{2d-1} t)}{\det(\mathrm{id}_{H^0(X)} - F_0^* t) \cdots \det(\mathrm{id}_{H^{2d}(X)} - F_{2d}^* u^{2d} t)}.$$

Comparing the two formulas

1. Taking $F = \mathrm{id}_X$ in the singular setting, Theorem restores Macdonald's formula.
2. Taking $F = \mathrm{Frob}_q$ and $u = 1$ in the l-adic setting, Theorem restores Grothendieck's formula.

Technical limitations for the l-adic setting: for each n, the prime l needs to be taken with $l > n$.

Pólya enumeration theorems: algebraic geometry

Theorem (C. – "Pólya enumeration in AG")

We keep the same hypotheses as in the previous theorem. Let G be a subgroup of S_n and consider the G-action on X^n by permuting coordinates. Then

$$L_u((F^n/G)^* \circlearrowleft H^\bullet(X^n/G)) = Z_G(L_u(F^*), L_{u^2}((F^*)^2), \dots, L_{u^n}((F^*)^n)),$$

where writing $m_i(g)$ to mean the number of i-cycles of g in S_n, we wrote

$$Z_G(x_1, \dots, x_n) := \frac{1}{|G|} \sum_{g \in G} x_1^{m_1(g)} \cdots x_n^{m_n(g)}.$$

Pólya enumeration theorems: algebraic geometry

How does this imply the rationality results? An important identity from combinatorics gives:

$$\sum_{n=0}^{\infty} Z_{S_n}(x_1, \ldots, x_n) t^n = \exp\left(\sum_{r=1}^{\infty} \frac{x_r t^r}{r}\right),$$

so taking $x_r = L_{u^r}((F^*)^r) = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}((uF_i^*)^r)$ and applying Theorem, we have

$$\sum_{n=0}^{\infty} L_u(\mathrm{Sym}^n(F)^*) t^n = \prod_{i=0}^{2d} \exp\left(\sum_{r=1}^{\infty} \frac{\mathrm{Tr}((uF_i^*)^r) t^r}{r}\right)^{(-1)^i}$$
$$= \prod_{i=0}^{2d} \det(\mathrm{id}_{H^i(X)} - F_i^* t)^{(-1)^{i+1}}.$$

Pólya enumeration theorems: algebraic geometry

We have used a well-known identity

$$\exp\left(\sum_{r=1}^{\infty} \frac{\mathrm{Tr}(A^r)t^r}{r}\right) = \frac{1}{\det(\mathrm{id} - tA)},$$

which holds for any square matrix A over a field.

Pólya enumeration theorems: combinatorics

The polynomial $Z_G(x_1, \dots, x_n)$ is called the **Pólya cycle index** in combinatorics. Taking X to be a finite set of points with the discrete topology, a special case of the above theorem in the singular setting gives

$$\begin{aligned} |X^n/G| &= \chi(X^n/G) \\ &= Z_G(\chi(X), \dots, \chi(X)) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(X)^{m(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} |X|^{m(g)}, \end{aligned}$$

where $m(g) = m_1(g) + \cdots + m_n(g)$ is the number of all cycles in the cycle decomposition of g in S_n.

Pólya enumeration theorems: combinatorics

Example

Take $n = 7$ and $g = (1\ 2\ 3)(4\ 5)(6\ 7)$. Then we must have

- $x_1 = x_2 = x_3$, $x_4 = x_5$, and $x_6 = x_7$, so
- $(X^7)^g = |X|^1|X|^2 = |X|^3$.

Here, we had

$$m(g) = m_1(g) + m_2(g) + m_3(g) = 0 + 2 + 1 = 3.$$

The example generalizes to give $|(X^n)^g| = |X|^{m(g)}$. Thus, by Burnside's Lemma, we have

$$|X^n/G| = \frac{1}{|G|}\sum_{g\in G}|(X^n)^g| = \frac{1}{|G|}\sum_{g\in G}|X|^{m(g)}.$$

The identity

$$|X^n/G| = \frac{1}{|G|}\sum_{g\in G}|X|^{m(g)}$$

is called the (non-weighted) **Pólya enumeration theorem** in combinatorics. One may view

$$X^n = \mathrm{Hom}_{\mathbf{Set}}([n], X),$$

where $[n] = \{1, 2, \ldots, n\}$. We may think of

- $[n]$ as the set of n vertices on a graph,
- G as the group of symmetries of the graph,
- X as the set of colors to be put on these vertices,
- elements of X^n, or maps $[n] \to X$, as colorings to the vertices, and
- X^n/G the set of colorings on the graph up to the symmetries of the graph.

Ideas behind the proof

We again consider the case where X is a finite set with the discrete topology and dig deeper into what's going on when we are using Burnside's Lemma. The key is the isomorphism

$$\mathbb{Q}X^n/G \simeq ((\mathbb{Q}X)^{\otimes n})^G,$$

given by $[x_1, \ldots, x_n] \mapsto x_1 \otimes \cdots \otimes x_n$, where G acts on $(\mathbb{Q}X)^{\otimes n}$ by

$$g \cdot (x_1 \otimes \cdots \otimes x_n) := x_{g^{-1}(1)} \otimes \cdots \otimes x_{g^{-1}(n)}.$$

Therefore, we have

$$|X^n/G| = \dim_{\mathbb{Q}}((\mathbb{Q}X)^{\otimes n})^G.$$

Ideas behind the proof

Consider the $\mathbb{Q}$-linear averaging map

$$e_G := \frac{1}{|G|}\sum_{g\in G} : (\mathbb{Q}X)^{\otimes n} \to (\mathbb{Q}X)^{\otimes n}.$$

We have $e_G((\mathbb{Q}X)^{\otimes n}) = ((\mathbb{Q}X)^{\otimes n})^G$, so any G-invariant tensor can be written as $e_G(v)$ for some $v \in ((\mathbb{Q}X)^{\otimes n})^G$. For any linear map ϕ on $\mathbb{Q}X$, we have

$$\phi^{\otimes n}(e_G(v)) = \phi^{\otimes n}\left(\frac{1}{|G|}\sum_{g\in G} gv\right) = \frac{1}{|G|}\sum_{g\in G} g\phi^{\otimes n}(v).$$

Ideas behind the proof

Thus, we have

$$\phi^{\otimes n} \circ e_G = \frac{1}{|G|} \sum_{g \in G} g\phi^{\otimes n},$$

and since $e_G : (\mathbb{Q}X)^{\otimes n} \twoheadrightarrow ((\mathbb{Q}X)^{\otimes n})^G \hookrightarrow (\mathbb{Q}X)^{\otimes n}$, we have

$$\mathrm{Tr}(\phi^{\otimes n}|_{((\mathbb{Q}X)^{\otimes n})^G}) = \mathrm{Tr}(\phi^{\otimes n} \circ e_G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g\phi^{\otimes n}).$$

Hence, it is enough to show that

$$\mathrm{Tr}(g\phi^{\otimes n}) = \mathrm{Tr}(\phi)^{m_1(g)} \mathrm{Tr}(\phi^2)^{m_2(g)} \cdots \mathrm{Tr}(\phi^n)^{m_n(g)}.$$

Both sides are independent of extending field, so we may work over $\mathbb{C}$. Since both sides are continuous in $\phi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}X, \mathbb{C}X) \simeq \mathbb{C}^{|X|^2}$, and since diagonalizable matrices are dense in $\mathbb{C}^{|X|^2}$, we may assume

$$\phi = \mathrm{diag}(\alpha_1, \ldots, \alpha_{|X|}).$$

Ideas behind the proof

We address the rest of the proof in an example where $n = 5$ and

$$g = (1\ 2)(4\ 5),$$

while $X = \{R, B\}$ and $\phi = \mathrm{diag}(\alpha_R, \alpha_B)$. Our goal is to show that

$$\mathrm{Tr}(g\phi^{\otimes 5}) = (\alpha_R + \alpha_B)(\alpha_R^2 + \alpha_B^2)^2.$$

Given a pure tensor

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \in (\mathbb{C}X)^{\otimes 5},$$

where each x_i is R or B, we have

$$\begin{aligned}(g\phi^{\otimes 5})(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) &= \phi(x_{g(1)}) \otimes \cdots \otimes \phi(x_{g(n)}) \\ &= \phi(x_2) \otimes \phi(x_1) \otimes \phi(x_3) \otimes \phi(x_5) \otimes \phi(x_4) \\ &= \alpha_{x_1}\alpha_{x_2}\alpha_{x_3}\alpha_{x_4}\alpha_{x_5} \cdot x_2 \otimes x_1 \otimes x_3 \otimes x_5 \otimes x_4.\end{aligned}$$

Ideas behind the proof

Thus, to compute $\mathrm{Tr}(g\phi^{\otimes 5})$, we only need to consider such pure tensor with the condition

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 = x_2 \otimes x_1 \otimes x_3 \otimes x_5 \otimes x_4,$$

or in other words: $x_1 = x_2$ and $x_4 = x_5$. Such pure tensors are precisely as follows

1. $R \otimes R \otimes R \otimes R \otimes R \rightsquigarrow \alpha_R^5$,
2. $R \otimes R \otimes R \otimes B \otimes B \rightsquigarrow \alpha_R^3 \alpha_B^2$,
3. $R \otimes R \otimes B \otimes R \otimes R \rightsquigarrow \alpha_R^4 \alpha_B$,
4. $R \otimes R \otimes B \otimes B \otimes B \rightsquigarrow \alpha_R^2 \alpha_B^3$,
5. $B \otimes B \otimes R \otimes R \otimes R \rightsquigarrow \alpha_R^3 \alpha_B^2$,
6. $B \otimes B \otimes R \otimes B \otimes B \rightsquigarrow \alpha_R \alpha_B^4$,
7. $B \otimes B \otimes B \otimes R \otimes R \rightsquigarrow \alpha_R^2 \alpha_B^3$,
8. $B \otimes B \otimes B \otimes B \otimes B \rightsquigarrow \alpha_B^5$.

Therefore, we have

$$\begin{aligned}\mathrm{Tr}(g\phi^{\otimes 5}) &= \alpha_R^5 + \alpha_R^4\alpha_B + 2\alpha_R^3\alpha_B^2 + 2\alpha_R^2\alpha_B^3 + \alpha_R\alpha_B^4 + \alpha_B^5 \\ &= (\alpha_R + \alpha_B)(\alpha_R^2 + \alpha_B^2)^2 \\ &= \mathrm{Tr}(\phi)^{m_1(g)}\mathrm{Tr}(\phi^2)^{m_2(g)} \\ &= \mathrm{Tr}(\phi)^{m_1(g)}\mathrm{Tr}(\phi^2)^{m_2(g)}\mathrm{Tr}(\phi^3)^{m_3(g)}\mathrm{Tr}(\phi^4)^{m_4(g)}\mathrm{Tr}(\phi^5)^{m_5(g)},\end{aligned}$$

because $m_1(g) = 1$ and $m_2(g) = 2$, while $m_3(g) = m_4(g) = m_5(g) = 0$, recalling that $g = (1\ 2)(4\ 5)$. This finishes the sketch of the proof for a special case of the main theorem. □

How to generalize the proof

In the singular setting, what we have presented (which is purportedly due to Schur) is the dimension 0 case. The key was

$$H^0(X^n/G) = \mathbb{Q}X^n/G \simeq ((\mathbb{Q}X)^{\otimes n})^G = (H^0(X)^{\otimes n})^G.$$

For higher dimensional X, it turns out that

$$H^\bullet(X^n/G) \simeq H^\bullet(X^n)^G \simeq (H^\bullet(X)^{\otimes n})^G,$$

where the G-action on $H^\bullet(X^n)$ is given by

$$g \cdot (p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)) = p_{g(1)}^*(v_1) \cup \cdots \cup p_{g(n)}^*(v_n).$$

Since $p_1^*(v_1) \cup \cdots \cup p_n^*(v_n)$ corresponds to $v_1 \otimes \cdots \otimes v_n \in H^\bullet(X)^{\otimes n}$, this G-action may introduce extra signs for v_j with odd degrees.

How to generalize the proof

But Macdonald already wrote an explicit formula for this signed G-representation :)

- Mixing the Macdonald's work with Schur's proof (which I learned from Stembridge), we can win the game.
- We have only worked with the Lefchetz numbers, but the argument works for the Lefchetz series.
- The l-adic setting works the same way.
- There is a "weighted version" for the whole story, and this generalizes a similar looking formula for the Hodge numbers for X^n/G, due to Cheah.

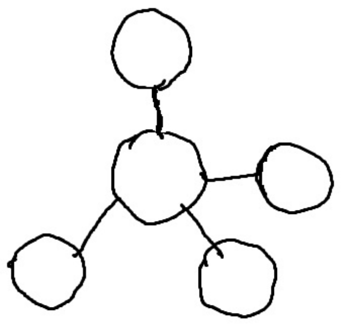
Thank you!

Pólya enumeration theorems: weighted version

Example

Take $X = \{C, H\}$ with the following graph (with $n = 5$). We have six different **weights** of colorings based on how many colors we use:

$$C^5, C^4H, C^3H^2, C^2H^3, CH^4, H^5.$$



Pólya enumeration theorems: weighted version

Question

How many colorings are there in the previous examples with weight CH^4?

Pólya enumeration theorems: weighted version

The answer is 2, not 5, because we need to take into account of the symmetries of the graph. One can figure out that the following polynomial records the numbers of colorings with various weights:

$$C^5 + 2C^4H + 2C^3H^2 + 2C^2H^3 + 2CH^4 + H^5.$$

If the graph gets more complicated, then computing this **weight polynomial** (non-standard terminology) can be more difficult.

Pólya enumeration theorems: weighted version

Theorem (Redfield, Pólya – "Pólya enumeration in CO")

Let $X = \{x_1, \ldots, x_r\}$ be a finite set of colors and $G \leqslant S_n$ the group of symmetries of a graph with n vertices. Set

- $N_{(e_1,\ldots,e_r)}$ to be the number of colorings $[x_{i_1}, \ldots, x_{i_n}] \in X^n/G$ in which x_i occurs e_i times for $1 \leqslant i \leqslant r$, and
- $P_{X^n/G}(x_1, \ldots, x_r) := \sum_{k_1+\cdots+k_r=n} N_{(e_1,\ldots,e_r)} x_1^{e_1} \cdots x_r^{e_r}$, the weight polynomial of X^n/G.

Then

$$P_{X^n/G}(x_1, \ldots, x_r) = Z_G(\boldsymbol{x}, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n),$$

where $\boldsymbol{x}^j := x_1^j + \cdots + x_r^j$.

Remark

The above theorem is often called the **weighted Pólya enumeration theorem** in combinatorics. We have already sketched a proof of it.

Pólya enumeration theorems: weighted version

In the previous example, we have 12 rotational symmetries for the graph, and one may compute that

$$Z_G(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{12}(t_1^4 + 8t_1 t_3 + 3t_2^2),$$

so the theorem says

$$\begin{aligned} P_{X^5/G}(C, H) &= \frac{(C+H)}{12}((C+H)^4 + 8(C+H)(C^3+H^3) + 3(C^2+H^2)^2) \\ &= C^5 + 2C^4H + 2C^3H^2 + 2C^2H^3 + 2CH^4 + H^5. \end{aligned}$$

Pólya enumeration theorems: weighted version

Question

What is the cohomological version of the weighted Pólya enumeration?

Again, the key is that $H^\bullet(X^n/G) \simeq ((H^\bullet(X))^{\otimes n})^G$. In other words, we have $H^\bullet(X^n/G) \hookrightarrow H^\bullet(X^n)$, and each class is of the form

$$p_1^*(\alpha_1) \cup \cdots \cup p_n^*(\alpha_n)$$

with $\alpha_j \in H^\bullet(X)$ such that

$$p_{g(1)}^*(\alpha_1) \cup \cdots \cup p_{g(n)}^*(\alpha_n) = p_1^*(\alpha_1) \cup \cdots \cup p_n^*(\alpha_n)$$

for all $g \in G$. Given any graded endomorphism $\phi = \oplus_{i=0}^\infty \phi_i$ on $H^\bullet(X) = \bigoplus_{i=0}^\infty H^i(X)$, we have an induced map $\phi_{X^n/G}$ on $H^\bullet(X^n/G)$ by

$$p_1^*(\alpha_1) \cup \cdots \cup p_n^*(\alpha_n) \mapsto p_1^*(\phi(\alpha_1)) \cup \cdots \cup p_n^*(\phi(\alpha_n)).$$

Pólya enumeration theorems: weighted version

Theorem (C.)

Keeping the same notations as in "Pólya enumeration in AG", we have

$$L_u(\phi_{X^n/G}) = Z_G(L_u(\phi), L_{u^2}(\phi^2), \ldots, L_{u^n}(\phi^n)).$$

Remark

Taking $X = \{x_1, \ldots, x_r\}$ in the discrete topology in the singular setting and taking $\phi = \mathrm{diag}(x_1, \ldots, x_r)$ on $H^\bullet(X) = H^0(X) \simeq \mathbb{Q}^r$, the above theorem recovers the weighted Pólya enumeration theorem in combinatorics.

Pólya enumeration theorems: weighted version

When X is a smooth projective complex variety of dimension d, taking

$$\phi = \bigoplus_{i=0}^{2d} \bigoplus_{p+q=i} x^p y^q \mathrm{id}_{H^{p,q}(X)},$$

we recover:

Corollary (Cheah)

Keeping the above notations, we have

$$\chi_{u,x,y}(X^n/G) = Z_G(\chi_{u,x,y}(X), \chi_{u^2,x^2,y^2}(X), \dots, \chi_{u^n,x^n,y^n}(X)),$$

where for any complex variety Y, we write

$$\chi_{u,x,y}(Y) := \sum_{i=0}^{\infty} \sum_{p+q=i} h^{p,q}(H^i(Y)) x^p y^q (-u)^i.$$

Remark

The paper contains more stories: https://arxiv.org/abs/2003.04825.

For example, we may obtain the S_n-rationality results for the weighted version, and there are also rationality results for alternating groups A_n. There are some works in progress

1. with Yifeng Huang on replacing X^n with $X^n \backslash$big diagonal and
2. with Y. Nancy Wang on replacing the singular cohomology with the Dolbeault cohomology.

Thank you, again!