NOTES ON HOCHSCHILD COHOMOLOGY

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ABSTRACT. This note is for a talk on deformation of algebra and Hochschild cohomology.

CONTENTS

1. DEFORMATION OF ALGEBRA

Let A be an associative algebra.

Definition 1.1. *A* ***$\hbar$-linear product*** $*$ *on* $A[\![\hbar]\!]$ *is a bilinear operation* $* : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$ *such that*

$$\Big(\sum_{i=0}^{\infty} a_i \hbar^i\Big) * \Big(\sum_{j=0}^{\infty} b_j \hbar^j\Big) = \sum_{i,j=0}^{\infty} (a_i * b_j)\hbar^{i+j}. \tag{1}$$

A ***deformation product*** *of the algebra* A *is an associative* $\hbar$*-linear product* $*$ *on* $A[\![\hbar]\!]$ *such that*

$$a * b|_{\hbar=0} = m_A(a,b) = ab$$

for any $a, b \in A$.

Remark 1.2. *By* $\hbar$*-linearity, a* $\hbar$*-linear product* $*$ *is uniquely determined by the restriction* $*|_{A\times A} : A \times A \to A[\![\hbar]\!]$,

$$*(a,b) = \sum_{k=0}^{\infty} B_k(a,b)\hbar^i = B_0(a,b) + B_1(a,b)\hbar + B_2(a,b)\hbar^2 + \cdots ,$$

where $a, b \in A$, $B_k : A \times A \to A$ *are bilinear maps. If* $*$ *is a deformation product, then* $B_0 = m_A$.

Furthermore, any sequence $B_k : A \times A \to A$ of bilinear maps induces a $\hbar$-linear product on $A[[\hbar]]$ by (1)*:*

$$\begin{aligned}\Big(\sum_{i=0}^{\infty} a_i \hbar^i\Big) * \Big(\sum_{j=0}^{\infty} b_j \hbar^j\Big) &= \sum_{i,j=0}^{\infty} (a_i * b_j) \hbar^{i+j} \\ &= \sum_{i,j=0}^{\infty} \Big(\sum_{k=0}^{\infty} B_k(a_i, b_j) \hbar^k\Big) \hbar^{i+j} \\ &= \sum_{N=0}^{\infty} \Big(\sum_{\substack{i,j,k\geq 0 \\ i+j+k=N}} B_k(a_i, b_j)\Big) \hbar^N.\end{aligned}$$

However, the induced $\hbar$-linear product $*$ may NOT be associative.

Main problem today: What sequence $(B_k)_{k=1}^{\infty}$ induces a deformation product?

Let us compare the two elements

$$\Big(\big(\sum_{i=0}^{\infty} a_i \hbar^i\big) * \big(\sum_{j=0}^{\infty} b_j \hbar^j\big)\Big) * \big(\sum_{k=0}^{\infty} c_k \hbar^k\big)$$

and

$$\big(\sum_{i=0}^{\infty} a_i \hbar^i\big) * \Big(\big(\sum_{j=0}^{\infty} b_j \hbar^j\big) * \big(\sum_{k=0}^{\infty} c_k \hbar^k\big)\Big)$$

in terms of the coefficient maps $(B_l)_{l=1}^{\infty}$:

$$\begin{aligned}\Big(\big(\sum_{i=0}^{\infty} a_i \hbar^i\big) * \big(\sum_{j=0}^{\infty} b_j \hbar^j\big)\Big) * \big(\sum_{k=0}^{\infty} c_k \hbar^k\big) &= \sum_{N=0}^{\infty} \Big(\sum_{\substack{i,j,p\geq 0 \\ i+j+p=N}} B_p(a_i, b_j)\, \hbar^N\Big) * \big(\sum_{k=0}^{\infty} c_k \hbar^k\big) \\ &= \sum_{N=0}^{\infty} \sum_{\substack{i,j,p\geq 0 \\ i+j+p=N}} \sum_{q=0}^{\infty} B_q(B_p(a_i, b_j), c_k)\, \hbar^{N+k+q} \\ &= \sum_{N=0}^{\infty} \sum_{\substack{i,j,k,p,q\geq 0 \\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k)\, \hbar^N.\end{aligned}$$

Furthermore, since $B_0 = m_A$,

$$\begin{aligned}\sum_{\substack{i,j,k,p,q\geq 0 \\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k) = &\sum_{\substack{i,j,k\geq 0 \\ i+j+k=N}} m_A(m_A(a_i, b_j), c_k) + \sum_{\substack{i,j,k\geq 0, p\geq 1 \\ i+j+k+p=N}} m_A(B_p(a_i, b_j), c_k) \\ &+ \sum_{\substack{i,j,k\geq 0, q\geq 1 \\ i+j+k+q=N}} B_q(m_A(a_i, b_j), c_k) + \sum_{\substack{i,j,k\geq 0, p,q\geq 1 \\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k).\end{aligned}$$

Similarly,

$$\big(\sum_{i=0}^{\infty} a_i \hbar^i\big) * \Big(\big(\sum_{j=0}^{\infty} b_j \hbar^j\big) * \big(\sum_{k=0}^{\infty} c_k \hbar^k\big)\Big) = \sum_{N=0}^{\infty} \sum_{\substack{i,j,k,p,q\geq 0 \\ i+j+k+p+q=N}} B_q(a_i, B_p(b_j, c_k))\, \hbar^N,$$

and

$$\sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(a_i, B_p(b_j, c_k)) = \sum_{\substack{i,j,k\geq 0\\ i+j+k=N}} m_A(a_i, m_A(b_j, c_k)) + \sum_{\substack{i,j,k\geq 0, p\geq 1\\ i+j+k+p=N}} m_A(a_i, B_p(b_j, c_k)) + \sum_{\substack{i,j,k\geq 0, q\geq 1\\ i+j+k+q=N}} B_q(a_i, m_A(b_j, c_k)) + \sum_{\substack{i,j,k\geq 0, p,q\geq 1\\ i+j+k+p+q=N}} B_q(a_i, B_p(b_j, c_k)).$$

Thus, the induced product $*$ is associative iff

$$\begin{aligned}
&\sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k) - \sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(a_i, B_p(b_j, c_k))\\
&= \sum_{\substack{i,j,k\geq 0, p\geq 1\\ i+j+k+p=N}} m_A(B_p(a_i, b_j), c_k) + \sum_{\substack{i,j,k\geq 0, q\geq 1\\ i+j+k+q=N}} B_q(m_A(a_i, b_j), c_k) + \sum_{\substack{i,j,k\geq 0, p,q\geq 1\\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k)\\
&- \sum_{\substack{i,j,k\geq 0, p\geq 1\\ i+j+k+p=N}} m_A(a_i, B_p(b_j, c_k)) - \sum_{\substack{i,j,k\geq 0, q\geq 1\\ i+j+k+q=N}} B_q(a_i, m_A(b_j, c_k)) - \sum_{\substack{i,j,k\geq 0, p,q\geq 1\\ i+j+k+p+q=N}} B_q(a_i, B_p(b_j, c_k))\\
&= 0. \quad (2)
\end{aligned}$$

Example 1.3. *The sequence $B_0 = m_A$, $B_1 = B_2 = \cdots = 0$ induces the natural product on $A[\![\hbar]\!]$. (Associative.)*

Example 1.4. *The sequence: $B_0 = B_1 = m_A$, $B_2 = B_3 = \cdots = 0$ induces the product*

$$\Big(\sum_{i=0}^{\infty} a_i \hbar^i\Big) * \Big(\sum_{j=0}^{\infty} b_j \hbar^j\Big) = \sum_{N=0}^{\infty} \Big(\sum_{i+j=N} a_i b_j + \sum_{i+j=N-1} a_i b_j\Big)\hbar^N$$

which is associative.

Example 1.5. *The sequence: $B_0 = m_A$, $B_1 = m_A \circ \mathrm{tw} : a \otimes b \mapsto ba$, $B_2 = B_3 = \cdots = 0$ induces the product*

$$\Big(\sum_{i=0}^{\infty} a_i \hbar^i\Big) * \Big(\sum_{j=0}^{\infty} b_j \hbar^j\Big) = \sum_{N=0}^{\infty} \Big(\sum_{i+j=N} a_i b_j + \sum_{i+j=N-1} b_j a_i\Big)\hbar^N$$

which is NOT associative. (Provided A is non-commutative.)

2. Hochschild cochains of algebra

Definition 2.1. *A **Hochschild n-cochain** of A is a $(n+1)$-linear map $\phi \in \mathrm{Hom}_{\Bbbk}(A^{\otimes(n+1)}, A)$. The space of Hochschild cochains $C^{\bullet}_{\mathcal{H}}(A)$ of A form a graded vector space $C^{\bullet}_{\mathcal{H}}(A) = \bigoplus_{n\geq -1} C^n_{\mathcal{H}}(A)$, where*

$$C^n_{\mathcal{H}}(A) := \mathrm{Hom}_{\Bbbk}(A^{\otimes(n+1)}, A), \qquad n \geq -1.$$

The Hochschild complex is endowed with a natural "pre-Lie system" structure. More precisely, for $\phi \in C^{n_1}_{\mathcal{H}}(A), \psi \in C^{n_2}_{\mathcal{H}}(A)$, we define the Hochschild composition $\phi \bullet \psi \in C^{n_1+n_2}_{\mathcal{H}}(A)$ by

$$\phi \bullet \psi(a_0, \cdots, a_{n_1+n_2}) := \sum_{j=0}^{n_1} (-1)^{jn_2} \phi(a_0, \cdots, a_{j-1}, \psi(a_j, \cdots, a_{j+n_2}), \cdots, a_{n_1+n_2}).$$

Definition 2.2. *The* ***Gerstenhaber bracket*** $[\![\phi,\psi]\!] \in C_{\mathcal{H}}^{n_1+n_2}(A)$ *of* $\phi \in C_{\mathcal{H}}^{n_1}(A), \psi \in C_{\mathcal{H}}^{n_2}(A)$ *is defined by*

$$[\![\phi,\psi]\!] := \phi \bullet \psi - (-1)^{n_1 n_2}\psi \bullet \phi.$$

Let $m_A \in C_{\mathcal{H}}^1(A)$ *be the multiplication of* A*. The* ***Hochschild differential*** *is*

$$d_{\mathcal{H}} := [\![m_A, -]\!] : C_{\mathcal{H}}^{\bullet}(A) \to C_{\mathcal{H}}^{\bullet+1}(A).$$

Proposition 2.3 ([3])**.** *The Hochschild cochains together with Hochschild differential and Gerstenhaber bracket* $(C_{\mathcal{H}}^{\bullet}(A), d_{\mathcal{H}}, [\![-,-]\!])$ *form a* ***dgla****, i.e.,*

- $[\![-,-]\!] : C_{\mathcal{H}}^n(A) \times C_{\mathcal{H}}^m(A) \to C_{\mathcal{H}}^{n+m}(A)$ *is a graded Lie bracket:*
  - *(graded) skew-symmetry*
$$[\![\phi,\psi]\!] = -(-1)^{|\phi||\psi|}[\![\psi,\phi]\!],$$
  - *(graded) Jacobi identity*
$$[\![\phi,[\![\psi,\sigma]\!]]\!] = [\![[\![\phi,\psi]\!],\sigma]\!] + (-1)^{|\phi||\psi|}[\![\psi,[\![\phi,\sigma]\!]]\!];$$
- $d_{\mathcal{H}} : C_{\mathcal{H}}^n(A) \to C_{\mathcal{H}}^{n+1}(A)$ *is a differential compatible with* $[\![-,-]\!]$*:*
  - *homological differential*
$$d_{\mathcal{H}} \circ d_{\mathcal{H}} = 0,$$
  - *compatibility*
$$d_{\mathcal{H}}\big([\![\phi,\psi]\!]\big) = [\![d_{\mathcal{H}}(\phi),\psi]\!] + (-1)^{|\phi|}[\![\phi,d_{\mathcal{H}}(\psi)]\!],$$

for any $\phi \in C_{\mathcal{H}}^{|\phi|}(A)$*,* $\psi \in C_{\mathcal{H}}^{|\psi|}(A)$*,* $\sigma \in C_{\mathcal{H}}^{|\sigma|}(A)$*.*

Example 2.4. *Let* $B \in C_{\mathcal{H}}^1(A) = \mathrm{Hom}_{\Bbbk}(A \otimes A, A)$ *be a binary operation on* A*. Then for* $a_0, a_1, a_2 \in A$*,*

$$\begin{aligned} [\![B,B]\!](a_0,a_1,a_2) &= \Big(B\big(B(a_0,a_1),a_2\big) - B\big(a_0,B(a_1,a_2)\big)\Big) \\ &\qquad - (-1)^{1\cdot 1}\Big(B\big(B(a_0,a_1),a_2\big) - B\big(a_0,B(a_1,a_2)\big)\Big) \\ &= 2\Big(B\big(B(a_0,a_1),a_2\big) - B\big(a_0,B(a_1,a_2)\big)\Big). \end{aligned}$$

Therefore,

$$[\![B,B]\!] = 0 \qquad \textit{iff} \qquad B \textit{ is associative}.$$

In particular, $d_{\mathcal{H}}(m_A) = 0$*.*

Consider the dgla $C_{\mathcal{H}}^{\bullet}(A)[\![\hbar]\!] := C_{\mathcal{H}}^{\bullet}(A) \otimes \Bbbk[\![\hbar]\!]$. More precisely, the dgla structure is given by the formulas

$$d_{\mathcal{H}}\Big(\sum_{i=0}^{\infty}\phi_i\hbar^i\Big) = \sum_{i=0}^{\infty} d_{\mathcal{H}}(\phi_i)\hbar^i,$$

$$[\![\sum_{i=0}^{\infty}\phi_i\hbar^i, \sum_{j=0}^{\infty}\psi_j\hbar^j\ ]\!] = \sum_{N=0}^{\infty}\sum_{\substack{i,j\geq 0\\ i+j=N}} [\![\phi_i,\psi_j]\!]\hbar^N.$$

Recall that a deformation product $* : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$ is determined by the restriction

$$*|_{A\times A} = \sum_{k=0}^{\infty} B_k\,\hbar^k = m_A + B_1\,\hbar + B_2\,\hbar^2 + \cdots\ ,$$

where $B_k \in C^1_{\mathcal{H}}(A)$. The associativity (2) of $*$ is rephrased as follows:

$$\sum_{\substack{p,q\geq 0\\ p+q=N}} B_q(B_p(a,b),c) - B_q(a,B_p(b,c)) = 0, \qquad \forall\, a,b,c \in A,\ N \geq 0$$

$$\Leftrightarrow \sum_{\substack{p,q\geq 0\\ p+q=N}} [\![B_p, B_q]\!] = 0, \qquad \forall\, N \geq 0$$

$$\Leftrightarrow [\![\sum_{p=0}^{\infty} B_p\hbar^p, \sum_{q=0}^{\infty} B_q\hbar^q]\!] = \sum_{N=0}^{\infty} \sum_{\substack{p,q\geq 0\\ p+q=N}} [\![B_p, B_q]\!]\, \hbar^N = 0$$

$$\Leftrightarrow d_{\mathcal{H}}\Big(\sum_{k=1}^{\infty} B_k\hbar^k \Big) + \frac{1}{2}[\![\sum_{k=1}^{\infty} B_k\hbar^k, \sum_{k=1}^{\infty} B_k\hbar^k]\!] = 0 \qquad (\because\ B_0 = m_A)$$

Theorem 2.5. *Let A be an associative algebra. A sequence $B_k \in C^1_{\mathcal{H}}(A)$, $k \geq 1$, induces a deformation product iff the formal sum $\Theta := \sum_{k=1}^{\infty} B_k\, \hbar^k$ is a **Maurer-Cartan element** (or **MC element**) in the dgla $C^{\bullet}_{\mathcal{H}}(A)[\![\hbar]\!]$, i.e., $\Theta \in C^1_{\mathcal{H}}(A)[\![\hbar]\!]$ is of degree one and satisfies the Maurer-Cartan equation*

$$d_{\mathcal{H}}(\Theta) + \frac{1}{2}[\![\Theta,\Theta]\!] = 0.$$

3. Coalgebra interpretation

Recall that a **(graded) coalgebra** is a graded vector space C together with a comultiplication $\Delta : C \to C \otimes C$ of degree zero which satisfy "coassociativity." The iterated comultiplication $\Delta^n : C \to C^{\otimes(n+1)}$ is defined by the iteration formula

$$\Delta^0 := \mathrm{id}_C : C \to C$$
$$\Delta^1 := \Delta : C \to C \otimes C$$
$$\Delta^n := (\mathrm{id}_C \otimes \Delta^{n-1}) \circ \Delta : C \to C^{\otimes(n+1)}.$$

A **coderivation** of degree k is an endomorphism $Q \in \mathrm{End}^k(C)$ of degree k such that

$$\Delta \circ Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) \circ \Delta.$$

The space of coderivations of degree k on C is denoted $\mathrm{coDer}^k(C)$, and $\mathrm{coDer}(C) := \bigoplus_{k\in\mathbb{Z}} \mathrm{coDer}^k(C)$.

Proposition 3.1. *The space of coderivations $\mathrm{coDer}(C)$ with the graded commutator*

$$[Q,Q'] := Q \circ Q' - (-1)^{|Q||Q'|} Q' \circ Q$$

is a graded Lie algebra.

Definition 3.2. *Let C be a coalgebra, and V be a vector space. A linear map*

$$p : C \to V$$

*is called a **cogenerator** if the map*

$$C \to \prod_{n\geq 0} V^{\otimes(n+1)} : c \mapsto \big(p(c), p^{\otimes 2}(\Delta c), p^{\otimes 3}(\Delta^2 c), \cdots\big)$$

is injective.

Lemma 3.3. *If $Q \in \mathrm{coDer}(C)$, then*

$$\Delta^n \circ Q = \Big(\sum_{i=0}^{n} \mathrm{id}^{\otimes i} \otimes Q \otimes \mathrm{id}^{\otimes(n-i)} \Big) \circ \Delta^n.$$

Proposition 3.4. *If $Q, Q' \in \mathrm{coDer}(C)$, then*

$$p \circ Q = p \circ Q' \quad \Rightarrow \quad Q = Q'.$$

Proof. It suffices to prove

$$p^{\otimes(n+1)} \circ \Delta^n \circ Q = p^{\otimes(n+1)} \circ \Delta^n \circ Q'$$

for any $n \geq 0$.

$$\begin{aligned} p^{\otimes(n+1)} \circ \Delta^n \circ Q &= p^{\otimes(n+1)} \circ \Big(\sum_{i=0}^{n} \mathrm{id}^{\otimes i} \otimes Q \otimes \mathrm{id}^{\otimes(n-i)} \Big) \circ \Delta^n \\ &= \Big(\sum_{i=0}^{n} p^{\otimes i} \otimes (pQ) \otimes p^{\otimes(n-i)} \Big) \circ \Delta^n \\ &= \Big(\sum_{i=0}^{n} p^{\otimes i} \otimes (pQ') \otimes p^{\otimes(n-i)} \Big) \circ \Delta^n \\ &= p^{\otimes(n+1)} \circ \Big(\sum_{i=0}^{n} \mathrm{id}^{\otimes i} \otimes Q' \otimes \mathrm{id}^{\otimes(n-i)} \Big) \circ \Delta^n \\ &= p^{\otimes(n+1)} \circ \Delta^n \circ Q'. \end{aligned}$$

□

The following colagebra plays a key role in this section.

Definition 3.5. *Let V be a graded vector space. The* ***(reduced) tensor coalgebra*** *$\bar{T}V$ generated by V is the space*

$$\bar{T}V := \bigoplus_{n \geq 1} V^{\otimes n}$$

together with the comultiplication $\Delta : \bar{T}V \to \bar{T}V \otimes \bar{T}V$ defined by the formula

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n-1} \big(v_1 \otimes \cdots \otimes v_i\big) \otimes \big(v_{i+1} \otimes \cdots \otimes v_n\big)$$

for $v_1, \cdots, v_n \in V$.

Lemma 3.6. *The natural projection $p : \bar{T}V \to V$ is a cogenerator.*

Proof. $p^{\otimes n} \Delta^{n-1}|_{V^{\otimes n}} = \mathrm{id}_{V^{\otimes n}}$. □

Proposition 3.7. *The map*

$$\mathrm{coDer}(\bar{T}V) \to \mathrm{Hom}_{\Bbbk}(\bar{T}V, V) : Q \mapsto p \circ Q$$

is an isomorphism of graded vector spaces with the inverse

$$\mathrm{Hom}_{\Bbbk}(\bar{T}V, V) \to \mathrm{coDer}(\bar{T}V) : \phi \mapsto Q_\phi$$

where

$$Q_\phi(v_0 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n-m} (-1)^{|\phi|(|v_0|+\cdots+|v_{i-1}|)} v_0 \otimes \cdots v_{i-1} \otimes \phi(v_i \otimes \cdots v_{i+m}) \otimes \cdots \otimes v_n$$

for $\phi \in \mathrm{Hom}_{\Bbbk}^{|\phi|}(V^{\otimes(m+1)}, V)$.

Proof. It's clear $p \circ Q_\phi = \phi$. Thus it suffices to show $Q_\phi \in \mathrm{coDer}(\bar{T}V)$:

$$\begin{aligned}\Delta \circ Q_\phi &= \Delta\Big(\sum_{i,j\geq 0} \mathrm{id}^{\otimes i} \otimes \phi \otimes \mathrm{id}^{\otimes j} \Big) \\ &= \sum_{i,j\geq 0} \sum_{k=0}^{i} \big(\mathrm{id}^{\otimes k} \big) \otimes \big(\mathrm{id}^{\otimes i-k} \otimes \phi \otimes \mathrm{id}^{\otimes j} \big) \\ &\quad + \sum_{i,j\geq 0} \sum_{k=0}^{j} \big(\mathrm{id}^{\otimes i} \otimes \phi \otimes \mathrm{id}^{\otimes k} \big) \otimes \big(\mathrm{id}^{\otimes j-k} \big) \\ &= (\mathrm{id} \otimes Q_\phi + Q_\phi \otimes \mathrm{id}) \circ \Delta.\end{aligned}$$

□

Remark 3.8. *An A_∞-**structure** on V is equivalent to a coderivation $Q \in \mathrm{coDer}^1\left(\bar{T}(V[1])\right)$ of degree one on $\bar{T}(V[1])$ such that $Q \circ Q = 0$.*

Let A be an algebra. A Hochschild cochain $\phi \in C_{\mathcal{H}}^{\geq 0}(A)$ cogenerates a coderivation $Q_\phi \in \mathrm{coDer}\left(\bar{T}(A[1])\right)$. Furthermore, the Hochschild composition corresponds to the composition of coderivations.

Proposition 3.9. *Let $\phi \in C_{\mathcal{H}}^m(A)$, $\psi \in C_{\mathcal{H}}^n(A)$, $m, n \geq 0$. Suppose $Q_\phi \in \mathrm{coDer}\left(\bar{T}(A[1])\right)$ and $Q_\psi \in \mathrm{coDer}\left(\bar{T}(A[1])\right)$ are the coderivations cogenerated by ϕ and ψ, respectively. Then*

$$\phi \bullet \psi = p \circ (Q_\phi \circ Q_\psi).$$

Remark 3.10. *The subspace $\mathrm{coDer}\left(\bar{T}(A[1])\right) \subset \mathrm{End}_{\Bbbk}\left(\bar{T}(A[1])\right)$ is NOT closed under composition.*

4. An outlook to deformation quantization

In [1, 2], F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer suggested to study "quantum observables" by $C^\infty(M)[\![\hbar]\!]$ and "star product."

Definition 4.1. *Let M be a manifold. A **star product***

$$* : C^\infty(M)[\![\hbar]\!] \times C^\infty(M)[\![\hbar]\!] \to C^\infty(M)[\![\hbar]\!]$$

on M is a deformation product of the algebra $C^\infty(M)$ (with pointwise multiplication m) whose restriction

$$*|_{C^\infty(M)\times C^\infty(M)} = \sum_{k=0}^{\infty} B_k\, \hbar^k = m + B_1\, \hbar + B_2\, \hbar^2 + \cdots$$

satisfies the properties

- *$B_i : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bidifferential operators*
- *B_1 is a Poisson bracket on M.*
- *$1 * f = f * 1 = f$*

Main question in deformation quantization:
Given a Poisson manifold $(M, \{\ ,\ \})$, can one construct a star product such that its classical limit is $\{\ ,\ \}$? I.e.
$\lim_{\hbar\to 0} \frac{1}{\hbar}(f * g - g * f) = \{f, g\}$?

Definition 4.2. *The dgla $(\mathcal{T}_{\mathrm{poly}}^\bullet(M), 0, [-,-])$ of **polyvector fields** on M is defined by*

$$\mathcal{T}_{\mathrm{poly}}^\bullet(M) = \bigoplus_{i\geq -1} \mathcal{T}_{\mathrm{poly}}^i(M), \qquad \mathcal{T}_{\mathrm{poly}}^i(M) = \Gamma(\Lambda^{i+1} T_M)$$

where the ***Schouten bracket***

$$[-,-]:\mathcal{T}^k_{\text{poly}}(M)\otimes\mathcal{T}^l_{\text{poly}}(M)\to\mathcal{T}^{k+l}_{\text{poly}}(M)$$

is characterized by the properties

$$[X,f]=X(f),\qquad\forall\,X\in\mathcal{T}^0_{\text{poly}}(M),f\in\mathcal{T}^{-1}_{\text{poly}}(M)=C^\infty(M)$$
$$[X,Y]=\textit{Lie bracket of vector fields},\qquad\forall\,X,Y\in\mathcal{T}^0_{\text{poly}}(M)$$
$$[\xi,\eta\wedge\zeta]=[\xi,\eta]\wedge\zeta+(-1)^{|\xi||\eta|}\eta\wedge[\xi,\zeta],\qquad\forall\,\xi\in\mathcal{T}^{|\xi|}_{\text{poly}}(M),\ \eta\in\mathcal{T}^{|\eta|}_{\text{poly}}(M),\ \zeta\in\mathcal{T}^{\bullet}_{\text{poly}}(M).$$

Definition 4.3. *The dgla* $(\mathcal{D}^\bullet_{\text{poly}}(M),d_{\mathcal{H}},[\![-,-]\!])$ *of* ***polydifferential operators*** *on* M *is the subdgla*

$$\mathcal{D}^\bullet_{\text{poly}}(M)=\bigoplus_{i\geqslant-1}\mathcal{D}^i_{\text{poly}}(M),$$
$$\mathcal{D}^i_{\text{poly}}(M)=\underbrace{\mathcal{D}(M)\otimes_R\cdots\otimes_R\mathcal{D}(M)}_{i+1\ \textit{factors}},$$
$$R=C^\infty(M),\ \mathcal{D}(M)=\ \textit{differential operators on}\ M,$$

of the dgla of Hochschild cochains $\big(C^\bullet_{\mathcal{H}}\big(C^\infty(M)\big),d_{\mathcal{H}},[\![-,-]\!]\big)$.

Definition 4.4. *The* ***Hochschild–Kostant–Rosenberg map***

$$\text{hkr}:\mathcal{T}^\bullet_{\text{poly}}(M)\to\mathcal{D}^\bullet_{\text{poly}}(M)$$

is defined by the skew-symmetrization formula

$$\text{hkr}(X_1\wedge\cdots\wedge X_k)=\frac{1}{k!}\sum_{\sigma\in S_k}\text{sgn}(\sigma)\,X_{\sigma(1)}\otimes\cdots\otimes X_{\sigma(k)},\quad\forall X_i\in\mathfrak{X}(M).$$

Theorem 4.5 (Kontsevich formality theorem [4])**.** *There exists an* L_∞ *quasi-isomorphism*

$$\Phi:\mathcal{T}^\bullet_{\text{poly}}(M)\rightsquigarrow\mathcal{D}^\bullet_{\text{poly}}(M)$$

such that its first Taylor coefficient is the natural inclusion $\Phi_1=\text{hkr}$.

Remark 4.6. *A* L_∞ *morphism* $\mathcal{F}:\mathfrak{g}\rightsquigarrow\mathfrak{h}$ *induces a map on MC elements*

$$\mathcal{F}:\text{MC}(\mathfrak{g})\to\text{MC}(\mathfrak{h}):\omega\mapsto\sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{F}_k(\omega^k)$$

provided the series converges.

Proposition 4.7. *The following characterizes MC elements in* $\mathcal{T}^\bullet_{\text{poly}}(M)$ *and* $\mathcal{D}^\bullet_{\text{poly}}(M)$ *respectively:*

- $\pi\in\text{MC}(\mathcal{T}^\bullet_{\text{poly}}(M))\Leftrightarrow[\pi,\pi]=0\Leftrightarrow\pi$ *is a Poisson bivector field*
- $\Theta\in\text{MC}(\hbar\mathcal{D}^\bullet_{\text{poly}}(M)[\![\hbar]\!])\Leftrightarrow m+\Theta$ *generates a deformation product.*

Kontsevich's L_∞ morphism induces a map on Maurer–Cartan elements:

$$\Phi:\text{MC}(\hbar\mathcal{T}^\bullet_{\text{poly}}(M)[\![\hbar]\!])\ni\hbar\cdot\pi\mapsto\Theta\in\text{MC}(\hbar\mathcal{D}^\bullet_{\text{poly}}(M)[\![\hbar]\!])$$

Solution of deformation quantization:
Properties of Kontsevich's Φ + properties of MC (associativity)
$\Rightarrow\Phi(\hbar\cdot\pi)$ induces a star product with $B_1=\frac{1}{2}\{\ ,\ \}_\pi$

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