# Bordered Heegaard Floer homology and knot complement

Jaepil Lee

Yonsei University

May 13, 2019

# Smooth structure of manifolds

**Theorem (Topological Poincaré Conjecture)**

*If a smooth m-manifold $\Sigma^m$ is homotopy equivalent to $S^m$ and $m \geq 5$, then $\Sigma^m$ and $S^m$ must be homeomorphic.*

**Sketch of proof.**

Apply the topological *h*-cobordism theorem. □

# Smooth structure of manifolds

**Theorem (Topological Poincaré Conjecture)**

*If a smooth m-manifold* $\Sigma^m$ *is homotopy equivalent to* $S^m$ *and* $m \geq 5$*, then* $\Sigma^m$ *and* $S^m$ *must be homeomorphic.*

**Sketch of proof.**

Apply the topological *h*-cobordism theorem. □

But $\Sigma^m$ is not necessarily diffeomorphic to $S^m$. In fact,

**Theorem (Milnor)**

*In dimension 7, there is an exotic sphere, which is homeomorphic but not diffeomorphic to the standard Euclidean 7-sphere.*

# Exoticness of four dimension

In four dimension,

**Theorem**

*There is a smooth manifold $M$ which is homeomorphic but not diffeomorphic to $\mathbb{R}^4$.*

**Theorem**

*Every 3-manifold $\Sigma^3$ with the same homology as the 3-sphere $S^3$ must bound a contractible topological 4-manifold.*

**Conjecture (Smooth 4-dimensional Poincaré conjecture)**

*If a smooth 4-manifold $\Sigma^4$ is homotopy equivalent to $S^4$, then $\Sigma^4$ and $S^4$ must be diffeomorphic.*

# Advertising the Heegaard Floer homology

Heegaard Floer theory assigns;

- to a closed 3-manifold $Y^3$, chain complexes $CF^*(Y)$, $* \in \{\wedge, \pm, \infty\}$ well defined up to homotopy
- to a 4-manifold $W$ with $\partial W = Y_1 \cup -Y_2$, chain maps $F_W^* : CF^*(Y_1) \to CF^*(Y_2)$ well defined up to chain homotopy

# Advertising the Heegaard Floer homology

Heegaard Floer theory is able to;

- detects smooth structure of 4-manifold
- detects genus of knots / Thurston norm
- detects fiberedness of knots and 3-manifolds
- find the minimal genus representatives of homology class in 4-manifold

and the list goes on.

# Construction of 3-manifold

## Handle decomposition of manifold

### Definition

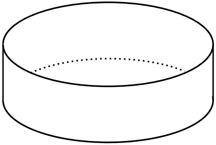
For $0 \leq k \leq 3$, an *3-dimesional k-handle h* is an 3-ball $D^k \times D^{3-k}$. It is attached to the boundary of an 3-manifold $X$ along the *attaching region* $\partial D^k \times D^{3-k}$ by an embedding $\phi : \partial D^k \times D^{3-k} \to \partial X$. Also $\partial D^k \times \{0\}$ is called the *core*.

# Construction of 3-manifold

## Handle decomposition of manifold

### Definition

For $0 \leq k \leq 3$, an *3-dimesional $k$-handle $h$* is an 3-ball $D^k \times D^{3-k}$. It is attached to the boundary of an 3-manifold $X$ along the *attaching region* $\partial D^k \times D^{3-k}$ by an embedding $\phi : \partial D^k \times D^{3-k} \to \partial X$. Also $\partial D^k \times \{0\}$ is called the *core*.

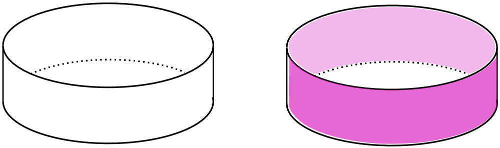


# Construction of 3-manifold

## Handle decomposition of manifold

### Definition

For $0 \leq k \leq 3$, an *3-dimesional $k$-handle $h$* is an 3-ball $D^k \times D^{3-k}$. It is attached to the boundary of an 3-manifold $X$ along the *attaching region* $\partial D^k \times D^{3-k}$ by an embedding $\phi : \partial D^k \times D^{3-k} \to \partial X$. Also $\partial D^k \times \{0\}$ is called the *core*.

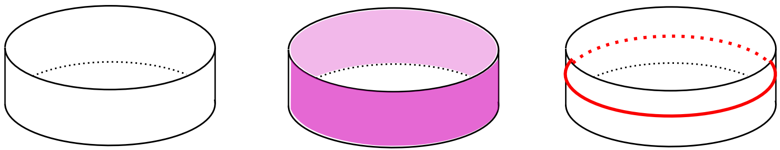


# Construction of 3-manifold

## Handle decomposition of manifold

### Definition

For $0 \leq k \leq 3$, an *3-dimesional $k$-handle $h$* is an 3-ball $D^k \times D^{3-k}$. It is attached to the boundary of an 3-manifold $X$ along the *attaching region* $\partial D^k \times D^{3-k}$ by an embedding $\phi : \partial D^k \times D^{3-k} \to \partial X$. Also $\partial D^k \times \{0\}$ is called the *core*.

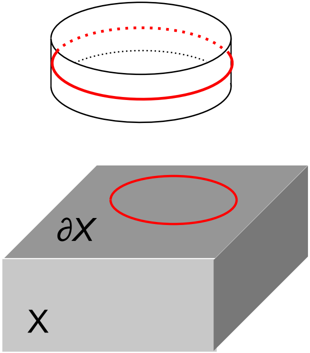


- The resulting manifold $X \cup_\phi h$ is a smooth manifold.
- The (isotopy type of) embedding $\phi$ is determined by the image of $\phi|_{\partial D^k \times \{0\}}$, called an *attaching sphere*.

# Construction of 3-manifold

## Handle decomposition of manifold

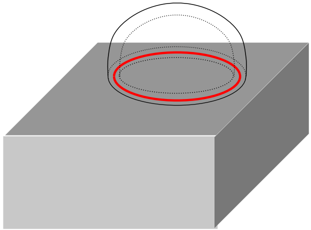
Example : 2-handle



# Construction of 3-manifold

## Handle decomposition of manifold

Example : 2-handle



# Construction of 3-manifold

## Heegaard diagram

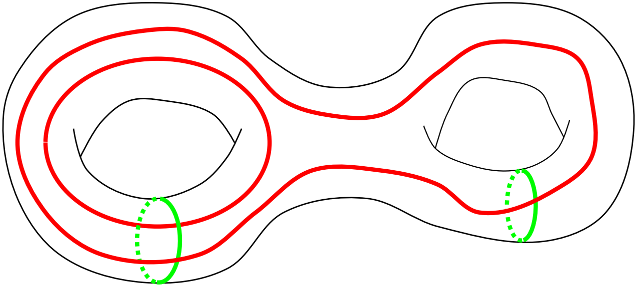
A Heegaard diagram consists of the following data:

- a genus $g$ surface $\Sigma$,
- a collection of disjoint closed curves $\boldsymbol{\alpha} := \{\alpha_1, \cdots, \alpha_g\}$ such that $\Sigma \backslash \boldsymbol{\alpha}$ is connected,
- another set of curves $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$ satisfying the same condition (but $\boldsymbol{\alpha}-$ and $\boldsymbol{\beta}$- curves may intersect)

# Construction of 3-manifold

## Heegaard diagram

- thicken $\Sigma$ to $\Sigma \times [0,1]$
- attach 3-dimensional 2-handles along $\boldsymbol{\alpha}$ on $\Sigma \times \{1\}$
- attach 3-dimensional 2-handles along $\boldsymbol{\beta}$ on $\Sigma \times \{0\}$
- attach two closed 3-balls in obvious way



# Construction of 3-manifold

## Heegaard diagram

**Theorem**

*Every closed, oriented 3-manifold $Y$ admits a Heegaard diagram representing $Y$.*

# Construction of 3-manifold

## Heegaard diagram

For a technical reason, we choose a point $z \in \Sigma$ away from the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves. Then,

**Definition**

The quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is called a *Heegaard quadruple*.

# Heegaard Floer chain complex

In 2003, Ozsváth and Szabó defined a Lagrangian Floer intersection homology on the symmetric product of the Heegaard diagram.

# Heegaard Floer chain complex

In 2003, Ozsváth and Szabó defined a Lagrangian Floer intersection homology on the symmetric product of the Heegaard diagram.

The Lagrangian Floer intersection homology requires the following.

- a symplectic manifold $(M^{2n}, \omega)$
- two Lagrangian submanifolds $(L_1, L_2)$

# Heegaard Floer chain complex

In 2003, Ozsváth and Szabó defined a Lagrangian Floer intersection homology on the symmetric product of the Heegaard diagram.

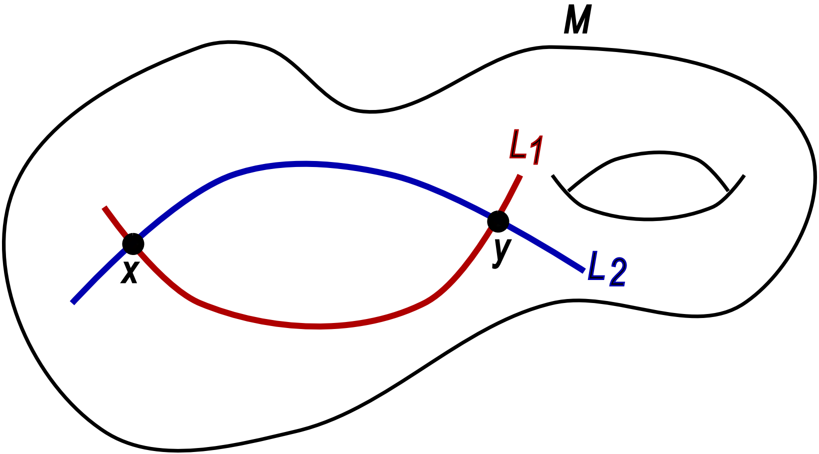
The Lagrangian Floer intersection homology requires the following.

- a symplectic manifold $(M^{2n}, \omega)$
- two Lagrangian submanifolds $(L_1, L_2)$

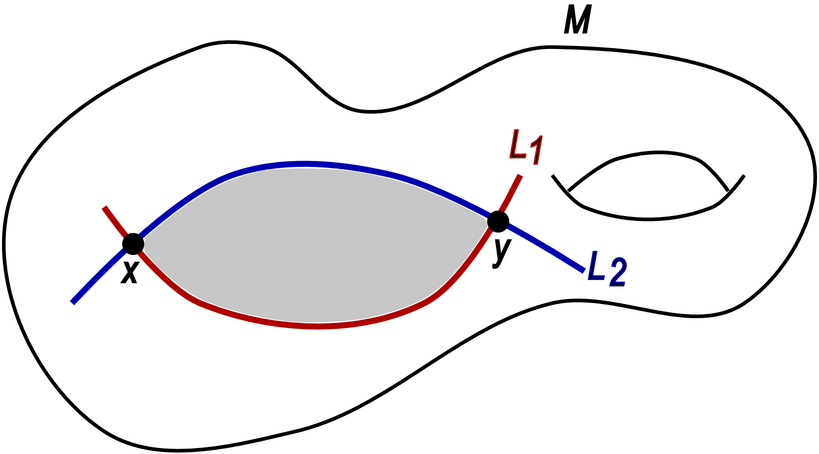
Then,

- The chain complex is generated by intersection points between $L_1$ and $L_2$.
- A differential between two points is determined by the $J$-holomorphic disk whose boundary lies in $L_1$ and $L_2$.

# Heegaard Floer chain complex



# Heegaard Floer chain complex



# Heegaard Floer chain complex

In Lagrangian intersection Floer theory, computation of differential is usually very hard, involving solving highly non-linear PDEs.

- For given two generators, make sure there is a topological disk between them.
- Even if there is a topological disk, need to ensure existence of a holomorphic representative of the disk.

# Heegaard Floer chain complex

A $k$-fold *symmetric product* of $X$ $\mathrm{Sym}^k(X)$ is

$$\mathrm{Sym}^k(X) := (\underbrace{X \times \cdots \times X}_{k\text{-times}})/S_k.$$

If $X$ is $n$-dimensional, then $\mathrm{Sym}^k(X)$ is $n \cdot k$ dimensional.

# Heegaard Floer chain complex

A $k$-fold *symmetric product* of $X$ $\mathrm{Sym}^k(X)$ is

$$\mathrm{Sym}^k(X) := (\underbrace{X \times \cdots \times X}_{k\text{-times}})/S_k.$$

If $X$ is $n$-dimensional, then $\mathrm{Sym}^k(X)$ is $n \cdot k$ dimensional.

Fix a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where the genus of $\Sigma$ is g. Applying $g$-fold symmetric product, we have

# Heegaard Floer chain complex

A $k$-fold *symmetric product* of $X$ $\mathrm{Sym}^k(X)$ is

$$\mathrm{Sym}^k(X) := (\underbrace{X \times \cdots \times X}_{k\text{-times}})/S_k.$$

If $X$ is $n$-dimensional, then $\mathrm{Sym}^k(X)$ is $n \cdot k$ dimensional.

Fix a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where the genus of $\Sigma$ is g. Applying $g$-fold symmetric product, we have

- from genus-$g$ surface $\Sigma$, $2g$-dimensional symplectic manifold $\mathrm{Sym}^g(\Sigma)$
- from $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$, $g$-dimensional Lagrangian submanifold $\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g / S_g$
- from $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$, $g$-dimensional Lagrangian submanifold $\mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g / S_g$

# Heegaard Floer chain complex

We want do Lagrangian Floer intersection theory on

- a symplectic manifold $\mathrm{Sym}^g(\Sigma)$
- two Lagrangian submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$

# Heegaard Floer chain complex

We want do Lagrangian Floer intersection theory on

- a symplectic manifold $\mathrm{Sym}^g(\Sigma)$
- two Lagrangian submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$

**Definition (hard version)**

The *Heegaard Floer homology* of $Y^3$ is the Lagrangian Floer intersection homology of symplectic manifold $\mathrm{Sym}^g(\Sigma)$ and Lagrangian submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, where $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ represents $Y^3$.

# Heegaard Floer chain complex

There is a projection map $\mathrm{Sym}^g(\Sigma) \to \Sigma$. Under the projection,

| in $\mathrm{Sym}^g(\Sigma)$ | on $\Sigma$ |
|---|---|
| topological disk connecting two generators | a (linear combination of) region on $\Sigma$ bounded by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (called *domain*) |
| $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ | $g$ points on $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ satisfying... |

# Heegaard Floer chain complex

There is a projection map $\mathrm{Sym}^g(\Sigma) \to \Sigma$. Under the projection,

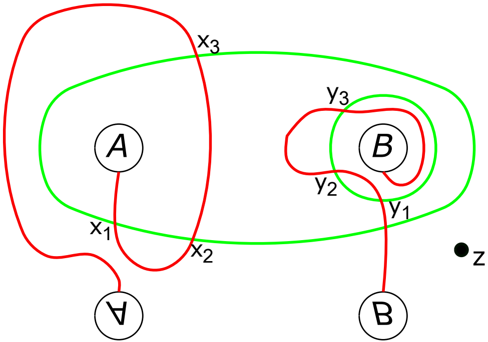
| in $\mathrm{Sym}^g(\Sigma)$ | on $\Sigma$ |
|---|---|
| topological disk connecting two generators | a (linear combination of) region on $\Sigma$ bounded by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (called *domain*) |
| $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ | $g$ points on $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ satisfying... |

...satisfying the following conditions.

- exactly one element on each $\boldsymbol{\alpha}$-curve
- exactly one element on each $\boldsymbol{\beta}$-curve

# Heegaard Floer chain complex

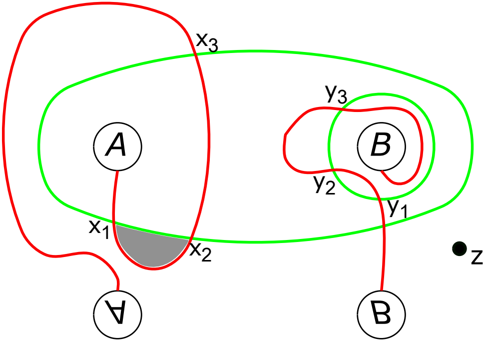
Example : Heegaard diagram of $S^3$




There are 9 generators : $x_i y_j$, where $i, j = 1, 2, 3$

# Heegaard Floer chain complex

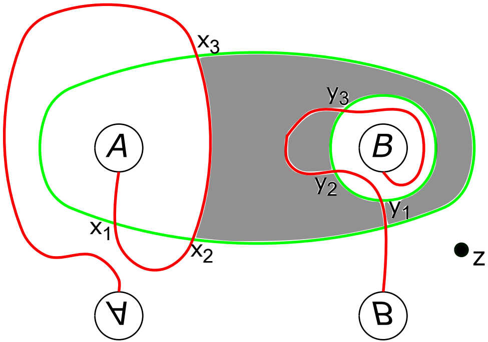
Example : Heegaard diagram of $S^3$




This domain implies differentials $x_1 y_j \mapsto x_2 y_j$

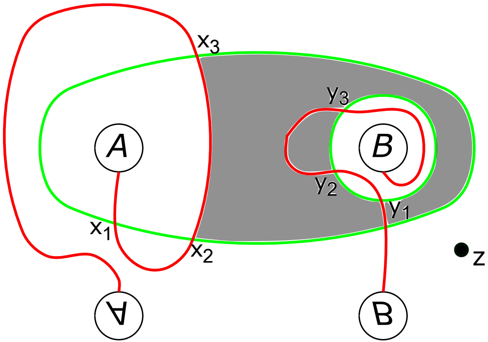
# Heegaard Floer chain complex

Example : Heegaard diagram of $S^3$



# Heegaard Floer chain complex

Example : Heegaard diagram of $S^3$




The existence of differential $x_3 y_j \mapsto x_2 y_j$?

# Heegaard Floer chain complex

- $\pi_2(\mathbf{x}, \mathbf{y}) :=$ space of all domains connecting from a generator $\mathbf{x}$ to a generator $\mathbf{y}$
- $\mathcal{M}(B) :=$ space of holomorphic curves "contained" in domain $B$

**Fact**. If the Maslov index difference between $\mathbf{x}$ and $\mathbf{y}$ is one, then $\mathcal{M}(B)/\mathbb{R}$ is compact zero dimensional manifold(i.e., finite number of points), if it exists.

# Heegaard Floer chain complex

## Definition (hat-version)

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram of 3-manifold $Y$. $\mathcal{S}(\mathcal{H})$ be a set of generators. Then $\widehat{CF}(\mathcal{H})$ is $\mathbf{Z}_2$-vector space generated by $\mathcal{S}(\mathcal{H})$ equipped with a differential

$$\widehat{\partial}\mathbf{x} := \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B) = 1}} \#(\mathcal{M}(B)/\mathbb{R}) \cdot \mathbf{y},$$

where $B$ does not contain the domain with $z$.

# Heegaard Floer chain complex

A priori, the homology $H_*(\widehat{CF}(\mathcal{H}))$ depends on the choice of the Heegaard diagram $\mathcal{H}$. However, thanks to the heroic analysis of Ozsváth and Szabó,

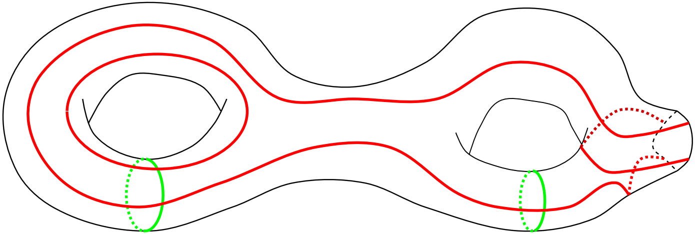
**Theorem**

*Suppose $\mathcal{H}$ represent a 3-manifold $Y$. Then the homotopy type of the homology $H_*(\widehat{CF}(\mathcal{H}))$ is independent from choice of the Heegaard diagram.*

Thus we write $\widehat{HF}(Y) := H_*(\widehat{CF}(\mathcal{H}))$.
(We are omitting the $Spin^c$-structure)

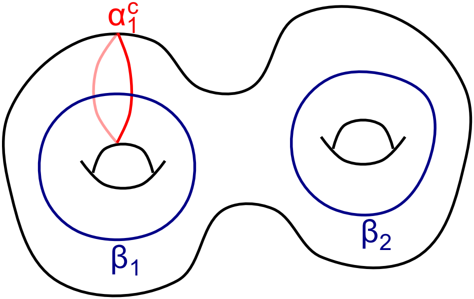
# Bordered Floer homology

Bordered Heegaard Floer homology is invented to compute the Heegaard-Floer type invariant on the sliced Heegaard diagram, and the invariant is intended to be glued to another sliced diagram.



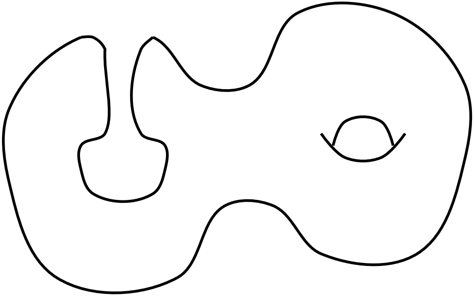
# The bordered Heegaard diagram

- Let $(\overline{\Sigma}^g, \alpha_1^c, \cdots, \alpha_{g-1}^c, \beta_1, \cdots, \beta_g)$ be a Heegaard diagram with one $\alpha$-circle removed.
- The resulting 3-manifold has two boundary components, a sphere and a torus.
- Let $\alpha_1^a, \alpha_2^a$ be circles on $\overline{\Sigma}^g$ intersecting at one point $p$, giving basis of the fundamental group of the torus.



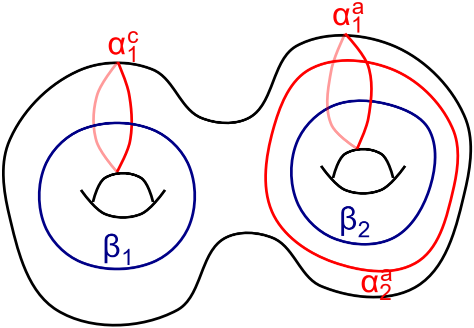
# The bordered Heegaard diagram

- Let $(\overline{\Sigma}^g, \alpha_1^c, \cdots, \alpha_{g-k}^c, \beta_1, \cdots, \beta_g)$ be a Heegaard diagram with one $\alpha$-circle removed.
- The resulting 3-manifold has two boundary components, a sphere and a torus.
- Let $\alpha_1^a, \alpha_2^a$ be circles on $\overline{\Sigma}^g$ intersecting at one point $p$, giving basis of the fundamental group of the torus.



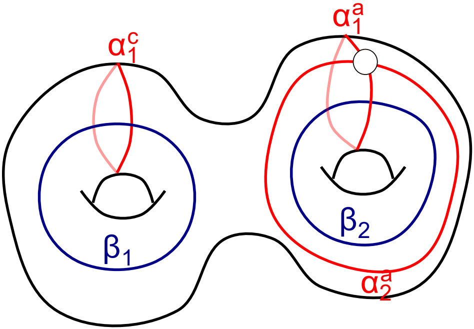
# The bordered Heegaard diagram

- Let $(\overline{\Sigma}^g, \alpha_1^c, \cdots, \alpha_{g-k}^c, \beta_1, \cdots, \beta_g)$ be a Heegaard diagram with one $\alpha$-circle removed.
- The resulting 3-manifold has two boundary components, a sphere and a torus.
- Let $\alpha_1^a, \alpha_2^a$ be circles on $\overline{\Sigma}^g$ intersecting at one point $p$, giving basis of the fundamental group of the torus.



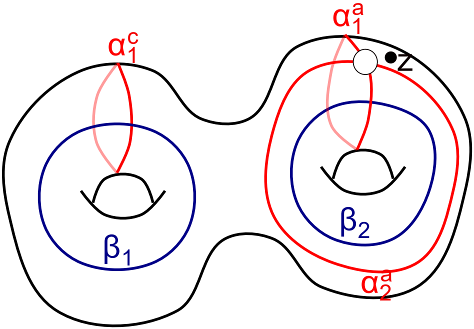
# The bordered Heegaard diagram

- Let $\Sigma^g := \overline{\Sigma}^g - \mathbb{D}_\epsilon(p)$.
- $(\Sigma^g, (\alpha_1^c, \cdots \alpha_{g-1}^c, \alpha_1^a, \alpha_2^a), (\beta_1, \cdots, \beta_g))$ is called a *bordered Heegaard diagram*.
- Fix a point $z \in \Sigma^g$ near $p$.



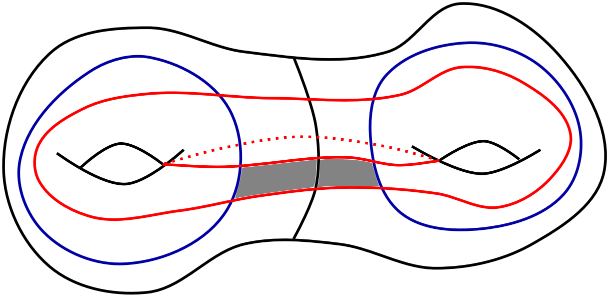
# The bordered Heegaard diagram

- Let $\Sigma^g := \overline{\Sigma}^g - \mathbb{D}_\epsilon(p)$.
- $(\Sigma^g, (\alpha_1^c, \cdots \alpha_{g-1}^c, \alpha_1^a, \alpha_2^a), (\beta_1, \cdots, \beta_g))$ is called a *bordered Heegaard diagram*.
- Fix a point $z \in \Sigma^g$ near $p$.



# Torus algebra

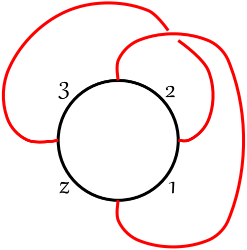
- The bordered Floer homology is intended to compute HF of the union of two Heegaard diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$.
- It turns out the holomorphic curves crossing the boundary $\partial\Sigma_1 = \partial\Sigma_2$ gives an algebra of *Reeb chords* on $(\partial\Sigma_1, \boldsymbol{\alpha})$.



# Torus algebra

The circular puncture is divided into 4 segments by the foot of the arcs.

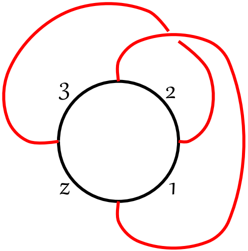
We label each segment, except for the segment adjacent to $z$, from 1 to 3, respecting the induced orientation or reversed induced orientation.

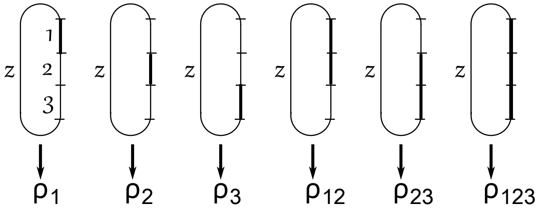


# Torus algebra

- Each segment $I \in \{1, 2, 3, 12, 23, 123\}$ is an oriented Reeb chord relative to $\partial\alpha_i^a$.
- Torus algebra $\mathcal{A}(T)$ is generated by $\rho_I$, which is called an algebra element of Reeb chord $I$.
- Multiplication is the concatenation of segments. e.g., $\rho_1 \cdot \rho_2 = \rho_{12}$ and so on.
- If the concatenation is non-sensible, then the multiplication is zero.

# Torus algebra



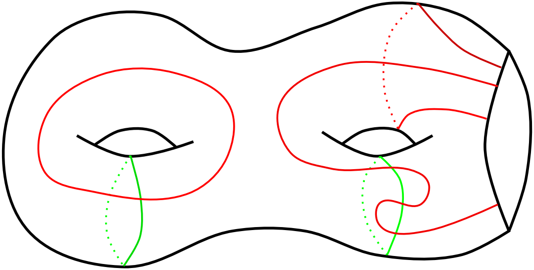
# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The generator of the left module is a set of $g$-points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ such that

- exactly one element on each $\boldsymbol{\beta}$-circle
- exactly one element on each $\boldsymbol{\alpha}$-circle
- no two points of the same $\boldsymbol{\alpha}$-arc

Let $\mathcal{S}(\mathcal{H})$ denote the set of generators.



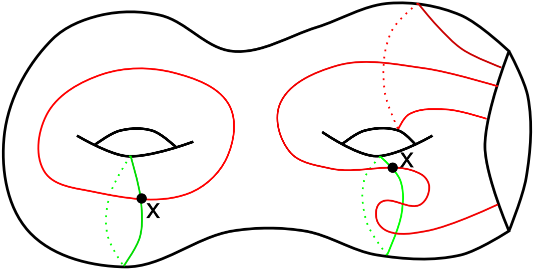
# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The generator of the left module is a set of $g$-points in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ such that

- exactly one element on each $\boldsymbol{\beta}$-circle
- exactly one element on each $\boldsymbol{\alpha}$-circle
- no two points of the set on the same $\boldsymbol{\alpha}$-arc

Let $\mathcal{S}(\mathcal{H})$ denote the set of generators.



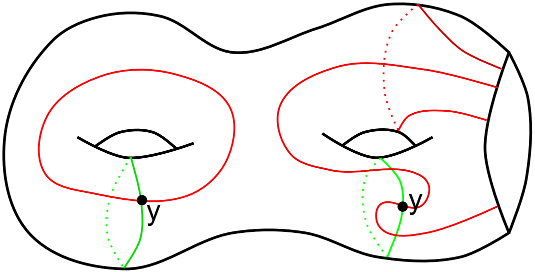
# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The generator of the left module is a set of $g$-points in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ such that

- exactly one element on each $\boldsymbol{\beta}$-circle
- exactly one element on each $\boldsymbol{\alpha}$-circle
- no two points of the set on the same $\boldsymbol{\alpha}$-arc

Let $\mathcal{S}(\mathcal{H})$ denote the set of generators.



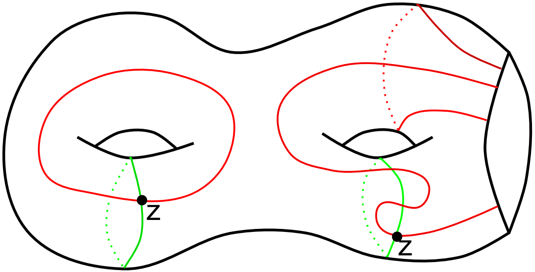
# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The generator of the left module is a set of $g$-points in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ such that

- exactly one element on each $\boldsymbol{\beta}$-circle
- exactly one element on each $\boldsymbol{\alpha}$-circle
- no two points of the set on the same $\boldsymbol{\alpha}$-arc

Let $\mathcal{S}(\mathcal{H})$ denote the set of generators.



# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The differential of the left module is

$$d(\mathbf{x}) = \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mathrm{ind}(B) = 1}} \#(\mathcal{M}(B)/\mathbb{R}) \cdot \overrightarrow{\rho}\, \mathbf{y},$$

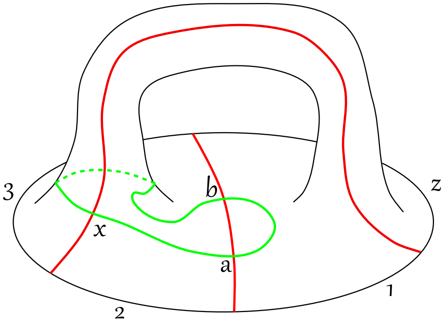
where

- $\overrightarrow{\rho}$ is the multiplication of all algebra element $\rho_I$ appear in $B$.
- the domain $B$ does not contain $z$.

We will see an example.

# Bordered Floer homology

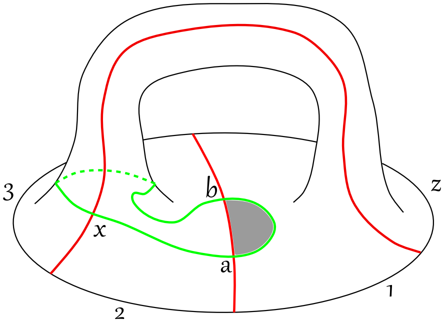
## left $\mathcal{A}(T)$-module $\widehat{CFD}$




$$\begin{aligned} d(b) &= a + \rho_3 x \\ d(x) &= \rho_2 a \\ d(a) &= 0 \end{aligned}$$

# Bordered Floer homology

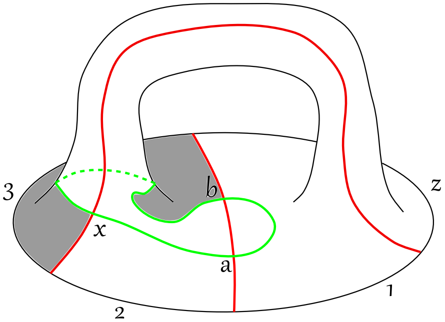
## left $\mathcal{A}(T)$-module $\widehat{CFD}$




$$\begin{aligned} d(b) &= \mathbf{a} + \rho_3 x \\ d(x) &= \rho_2 a \\ d(a) &= 0 \end{aligned}$$

# Bordered Floer homology

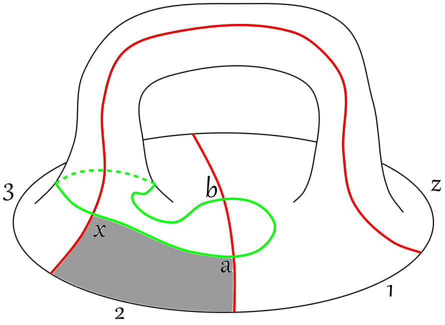
left $\mathcal{A}(T)$-module $\widehat{CFD}$




$$
\begin{aligned}
d(b) &= a + \boldsymbol{\rho_3}\mathbf{x} \\
d(x) &= \rho_2 a \\
d(a) &= 0
\end{aligned}
$$

# Bordered Floer homology

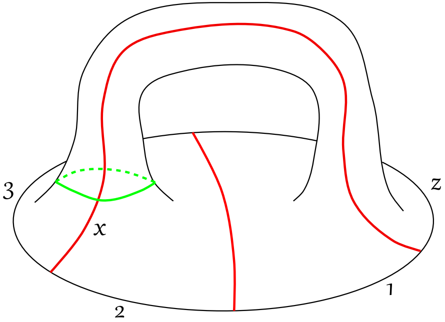
left $\mathcal{A}(T)$-module $\widehat{CFD}$




$$\begin{aligned} d(b) &= a + \rho_3 x \\ d(x) &= \boldsymbol{\rho_2 a} \\ d(a) &= 0 \end{aligned}$$

# Bordered Floer homology

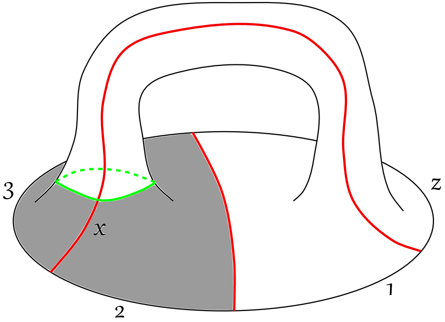
left $\mathcal{A}(T)$-module $\widehat{CFD}$




$$d(x) = \rho_{23}x$$

# Bordered Floer homology

left $\mathcal{A}(T)$-module $\widehat{CFD}$

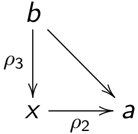



$$d(x) = \boldsymbol{\rho_2 \rho_3 x} = \rho_{23} x$$

# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The first chain complex:

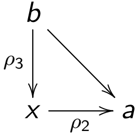



The second chain complex:

# Bordered Floer homology

## left $\mathcal{A}(T)$-module $\widehat{CFD}$

The first chain complex:




The second chain complex:


In fact, they are homotopy equivalent.

**Theorem**

*If the two punctured diagrams $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ and $\mathcal{H}' = (\Sigma', \alpha', \beta', z')$ represent the same 3-manifold $Y$, then $\widehat{CFD}(\mathcal{H})$ and $\widehat{CFD}(\mathcal{H}')$ are homotopy equivalent.*

# Bordered Floer homology

## $\widehat{CFA}$

By dualizing $\widehat{CFD}(\mathcal{H})$, we obtain a right $\mathcal{A}(F)$-module $\widehat{CFA}(\mathcal{H})$. It is...

- a right $\mathcal{A}(T)$-module.

# Bordered Floer homology

## $\widehat{CFA}$

By dualizing $\widehat{CFD}(\mathcal{H})$, we obtain a right $\mathcal{A}(F)$-module $\widehat{CFA}(\mathcal{H})$. It is...

- a right $\mathcal{A}(T)$-module.
- There is maps

$$m_{1+k} : \widehat{CFA} \otimes \underbrace{\mathcal{A}(T) \otimes \cdots \otimes \mathcal{A}(T)}_{\mathrm{k}-times} \to \widehat{CFA}$$

satisfying $\mathcal{A}_\infty$-relations, by analysing degenerated curves.

# Bordered Floer homology

## $\widehat{CFA}$

By dualizing $\widehat{CFD}(\mathcal{H})$, we obtain a right $\mathcal{A}(F)$-module $\widehat{CFA}(\mathcal{H})$. It is...

- a right $\mathcal{A}(T)$-module.
- There is maps

$$m_{1+k} : \widehat{CFA} \otimes \underbrace{\mathcal{A}(T) \otimes \cdots \otimes \mathcal{A}(T)}_{\text{k}-times} \to \widehat{CFA}$$

  satisfying $\mathcal{A}_\infty$-relations, by analysing degenerated curves.
- Usually $\widehat{CFA}(\mathcal{H})$ is harder to compute than $\widehat{CFD}(\mathcal{H})$, but there is an explicit dualizing algorithm in the case of torus boundary (Hedden, Levine).

# Bordered Floer homology

## $\widehat{CFA}$

By dualizing $\widehat{CFD}(\mathcal{H})$, we obtain a right $\mathcal{A}(F)$-module $\widehat{CFA}(\mathcal{H})$. It is...

- a right $\mathcal{A}(T)$-module.
- There is maps

$$m_{1+k} : \widehat{CFA} \otimes \underbrace{\mathcal{A}(T) \otimes \cdots \otimes \mathcal{A}(T)}_{\mathrm{k}-times} \to \widehat{CFA}$$

  satisfying $\mathcal{A}_\infty$-relations, by analysing degenerated curves.
- Usually $\widehat{CFA}(\mathcal{H})$ is harder to compute than $\widehat{CFD}(\mathcal{H})$, but there is an explicit dualizing algorithm in the case of torus boundary (Hedden, Levine).
- Taking $\mathcal{A}_\infty$-tensor product with $\widehat{CFD}$, it gives $\widehat{CF}$ of closed 3-manifold.

# Bordered Floer homology

## $\widehat{CFA}$

By dualizing $\widehat{CFD}(\mathcal{H})$, we obtain a right $\mathcal{A}(F)$-module $\widehat{CFA}(\mathcal{H})$. It is...

- a right $\mathcal{A}(T)$-module.
- There is maps

$$m_{1+k} : \widehat{CFA} \otimes \underbrace{\mathcal{A}(T) \otimes \cdots \otimes \mathcal{A}(T)}_{\text{k}-times} \to \widehat{CFA}$$

  satisfying $\mathcal{A}_\infty$-relations, by analysing degenerated curves.
- Usually $\widehat{CFA}(\mathcal{H})$ is harder to compute than $\widehat{CFD}(\mathcal{H})$, but there is an explicit dualizing algorithm in the case of torus boundary (Hedden, Levine).
- Taking $\mathcal{A}_\infty$-tensor product with $\widehat{CFD}$, it gives $\widehat{CF}$ of closed 3-manifold.
- $\mathcal{A}_\infty$ structure sometimes gives the signed count of moduli space for hard domains.

# Knot complement

- There is a Heegaard Floer chain complex of knot.

# Knot complement

- There is a Heegaard Floer chain complex of knot.
- It is endowed with $\mathbb{Z} \otimes \mathbb{Z}$-filtration.

# Knot complement

- There is a Heegaard Floer chain complex of knot.
- It is endowed with $\mathbb{Z} \otimes \mathbb{Z}$-filtration.
- The differential respects the filtration.

# Knot complement

- There is a Heegaard Floer chain complex of knot.
- It is endowed with $\mathbb{Z} \otimes \mathbb{Z}$-filtration.
- The differential respects the filtration.
- Filtered chain homotopy type of the chain complex is a knot invariant.

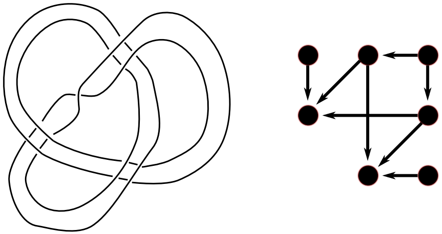
# Knot complement

- There is a Heegaard Floer chain complex of knot.
- It is endowed with $\mathbb{Z} \otimes \mathbb{Z}$-filtration.
- The differential respects the filtration.
- Filtered chain homotopy type of the chain complex is a knot invariant.
- For a classical knot $K$ ($K \subset S^3$), it is computable in general.

# Knot complement

- There is a Heegaard Floer chain complex of knot.
- It is endowed with $\mathbb{Z} \otimes \mathbb{Z}$-filtration.
- The differential respects the filtration.
- Filtered chain homotopy type of the chain complex is a knot invariant.
- For a classical knot $K$ ($K \subset S^3$), it is computable in general.

For example,



# Knot complement

There is an algorithm to compute $\widehat{CFD}$ of a knot $K \subset S^3$ complement.
For example,