The inverses of tails of the Riemann zeta function for some natural numbers and real numbers in critical strip and related topics (with WonTae Hwang, with Donggyun Kim)

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Introduction

Riemann zeta function

Among various types of zeta functions, one of the most famous and important is the Riemann zeta function. One of the Millennium Prize Problems is the Riemann Hypothesis, which is related to the non-trivial zeros of the Riemann zeta function on the critical line.

Definition

1. (Critical strip) The region $0 < \sigma < 1$, where σ is defined by the real part of a complex number $s = \sigma + it$.
2. (Critical line) The line $\sigma = \frac{1}{2}$ in the complex plane on which Riemann Hypothesis asserts that all nontrivial (complex) Riemann zeta function zeros lie.

Riemann zeta function

The *Riemann zeta function* is defined, respectively, by for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

and for $0 < \mathrm{Re}(s) < 1$,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}.$$

Riemann zeta function

In 1900, Max Planck gave a formula for the black-body spectrum, i.e., the probability that the energy emitted by a black body lies in the narrow range of frequencies between f and $f + df$. This is proportional to $B(x)dx$, with $x = hf/(kT)$ and

$$B(x) = \frac{2\pi x^3}{e^x - 1}.$$

where
h: Planck's constant($\approx 6.626 \times 10^{-34} J \cdot s$)
f: Frequency(unit: $1/s$)
k: Boltzmann's constant($\approx 1.38 \times 10^{-23} J/K$)
T: Thermodynamic temperature (absolute temperature) (unit: $1/K$)

Riemann zeta function

The Stefan-Boltzmann law: The total energy radiated per unit surface area of a black body across all wavelengths per unit time(j^*) is directly proportional to the fourth power of the black body's thermodynamic temperature (absolute temperature) T, i.e.,

$$j^* = \sigma T^4,$$

where

$$\sigma = \int_0^\infty B(x)dx = \int_0^\infty \frac{2\pi x^3}{e^x - 1}dx$$

is called the Stefan-Boltzmann constant, and

$$\int_0^\infty \frac{2\pi x^3}{e^x - 1}dx = 12\pi\zeta(4) = \frac{2\pi^5}{15}.$$

Riemann zeta function

The black-body function may be written as

$$B(x) = \frac{2\pi x^3}{e^x - 1} = 2\pi x^3 \sum_{n=1}^{\infty} e^{-nx},$$

and we use the change of variables

$$\begin{aligned}\int_0^\infty B(x)dx &= \int_0^\infty \left(\sum_{n=1}^{\infty} 2\pi (y/n)^3 e^{-y} (1/n) dy \right) \\ &= \int_0^\infty 2\pi y^3 e^{-y} dy \sum_{n=1}^{\infty} \frac{1}{n^4} \\ &= 2\pi \Gamma(4) \zeta(4).\end{aligned}$$

Riemann zeta function

Note: $\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy$. We know that $\Gamma(4) = 3! = 6$. Hence, we get

$$\sigma = \int_0^\infty B(x)dx = 12\pi\zeta(4).$$

Similarly, the whole of the Universe has a background photon density

$$\frac{\text{cosmic photons}}{\text{volume of Universe}} = 16\pi\zeta(3)\left(\frac{kT_0}{hc}\right)^3 \approx 4.12 \times 10^8 m^{-3},$$

where

h: Planck's constant($\approx 6.626 \times 10^{-34} J \cdot s$)

k: Boltzmann's constant($\approx 1.38 \times 10^{-23} J/K$)

T_0: Temperature of the universe($\approx 2.728K$)

Some generalizations

Definition

(The Barnes zeta function) The Barnes zeta function $\zeta_N(s, w|a_1, \cdots, a_N)$ is defined by the formula

$$\zeta_N(s, w|a_1, \cdots, a_N) = \sum_{n_1, \cdots, n_N \geq 0}^{\infty} \frac{1}{(w + n_1 a_1 + \cdots + n_N a_N)^s}$$

for $\mathrm{Re}(s) > N$. Note that

$$\zeta(s, 1|1) = \zeta(s).$$

Some generalizations

Definition

(The Multiple zeta function) The Multiple zeta functions are generalizations of the Riemann zeta function, defined by

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{{n_1}^{s_1} \cdots {n_k}^{s_k}} = \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{i=1}^{k} \frac{1}{{n_i}^{s_i}}$$

for $\mathrm{Re}(s_1) + \cdots + \mathrm{Re}(s_i) > i$ for all i. Note that there are some interesting identities related to the Multiple zeta functions such as

$$\zeta(2,1) = \zeta(3),$$
$$\zeta(a,b) + \zeta(b,a) = \zeta(a)\zeta(b) - \zeta(a+b) \text{ for } a, b > 1$$

which are famous identities of Euler.

Some generalizations

Theorem

(The sum formula) For positive integers k and n,

$$\sum_{i_1+\cdots+i_k=n, i_1>1} \zeta(i_1, \cdots, i_k) = \zeta(n),$$

where the sum is extended over k-tuples $(i_1, \cdots, i_k)$ of positive integers with $i_1 > 1$.

It has beed proved by A. Granville and D. Zagier independently.

A tail of the Riemann zeta functions

A *tail of the Riemann zeta function from n* for an integer $n \geq 1$ are defined, respectively, by for $\mathrm{Re}(s) > 1$,

$$\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s}$$

and for $0 < \mathrm{Re}(s) < 1$,

$$\zeta_n(s) = \frac{1}{1 - 2^{1-s}} \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s}.$$

Some reciprocal sums of the sequences

The reciprocal sums of the Fibonacci and related sequences

Definition

The classical Fibonacci sequence is defined recursively by

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Theorem

(H. Ohtsuka, S. Nakamura; 2008/2009)

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right] = \begin{cases} F_{n-2}, & \text{if } n \geq 2 \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{{F_k}^2} \right)^{-1} \right] = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ is even;} \\ F_nF_{n-1}, & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

The inverse of the tails of the Riemann zeta function when $s = 2, 3, 4$ and 5

The integer part of the inverse of a tail of the Riemann zeta function when $s = 2$ is as follows.

Theorem

(L. Xin; 2016)

$$\left[\zeta_n(2)^{-1}\right] = \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right)^{-1}\right] = n - 1.$$

The inverse of the tails of the Riemann zeta function when $s = 2, 3, 4$ and 5

Proof.

$$\frac{1}{n} = \sum_{k=n}^{\infty} \frac{1}{k(k+1)} < \sum_{k=n}^{\infty} \frac{1}{k^2} < \sum_{k=n}^{\infty} \frac{1}{k(k-1)} = \frac{1}{n-1}.$$

Hence, we have the inequality

$$n - 1 < \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} < n.$$

□

The inverse of the tails of the Riemann zeta function when $s = 2, 3, 4$ and 5

Through far more calculations than previous one, we can get these following results.

Theorem

(L. Xin; 2016)

$$\left[\zeta_n(3)^{-1}\right] = \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^3}\right)^{-1}\right] = 2n(n-1).$$

The inverse of the tails of the Riemann zeta function when $s = 2, 3, 4$ and 5

Theorem

(H. Xu; 2016, L. Xin, L. Xiaoxue; 2017)

$$
\begin{aligned}
&\left[\zeta_n(4)^{-1}\right] \\
&= \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^4}\right)^{-1}\right] \\
&= \begin{cases} 24m^3 - 18m^2 + \left[\frac{3(5m-1)}{2}\right], & \text{if } n = 2m; \\ 24m^3 - 54m^2 + \left[\frac{3(58m-17)}{4}\right], & \text{if } n = 2m-1. \end{cases}
\end{aligned}
$$

The inverse of the tails of the Riemann zeta function when $s = 2, 3, 4$ and 5

Theorem

(H. Xu; 2016) For all integer $n \geq 4$, we have

$$\left[\zeta_n(5)^{-1}\right] = \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^5}\right)^{-1}\right] = 4n^4 - 8n^3 + 9n^2 - 5n + \left[\frac{(n+1)(n-2)}{3}\right].$$

Our results

Introduction of our results

At this point, we naturally have the following questions.

- A bound of the inverse of the tails of the Riemann zeta function when $s = 6$ or greater integer.
- A bound of the inverse of the tails of the Riemann zeta function when s is in critical strip.

The inverse of the tails of the Riemann zeta function when $s = 6$

- We expected that the degree of polynomial-like expression of $\left[\zeta_n(6)^{-1}\right] = \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^6}\right)^{-1}\right]$ is 5.
- We obtained the polynomial part of the expected polynomial-like expression:
$f(n) = 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n+$ (the remainder terms).

The inverse of the tails of the Riemann zeta function when $s = 6$

Let (the remainder terms) $= g(n)$ and for any sufficiently large n, we have

$$g(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{48}; \\ -\frac{23}{48}, & \text{if } n \equiv 1 \pmod{48}; \\ & \vdots \\ -\frac{13}{48}, & \text{if } n \equiv 47 \pmod{48}. \end{cases}$$

From this pattern, we obtained the remainder terms and hence we got the expected polynomial-like expression $f(n)$ and proved it is true. The result is as follows: for an integer n, let n_{48} be the remainder when n is divided by 48. Then, we have the following Theorem.

The inverse of the tails of the Riemann zeta function when $s = 6$

Theorem

(HS; 2017) For each integer $n \geq 829$, we have

$$\left[\zeta_n(6)^{-1}\right] = \left[\left(\sum_{k=n}^{\infty} \frac{1}{k^6}\right)^{-1}\right] = \begin{cases} 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}}{48} - \left[\frac{35-5n_{48}}{48}\right], & \text{if } n \text{ is even;} \\ 5n^5 - \frac{25}{2}n^4 + \frac{75}{4}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}+18}{48} - \left[\frac{17-5n_{48}}{48}\right], & \text{if } n \text{ is odd.} \end{cases}$$

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

Instead of studying

$$\zeta_n(s) = \frac{1}{1-2^{1-s}} \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s}$$

for $0 < s < 1$, we observed the function

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^s}$$

which is relatively easy to treat.

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

For convenience, we define

Definition

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^s} = \begin{cases} A_{n,s}, & \text{if } n \text{ is even;} \\ B_{n,s}, & \text{if } n \text{ is odd.} \end{cases}$$

Immediately,

$$\zeta_n(s) = \begin{cases} -\frac{1}{1-2^{1-s}} A_{n,s}, & \text{if } n \text{ is even;} \\ -\frac{1}{1-2^{1-s}} B_{n,s}, & \text{if } n \text{ is odd.} \end{cases}$$

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

We obtained the following bounds related to $\zeta_n(s)$.

Theorem

(KS; 2018) Let s be a real number with $0 < s < 1$. Then for any positive even number n,

$$2\left(n-\frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n-\frac{1}{4}\right)^s$$

and for any $\epsilon > 0$,

$$2\left(n-\frac{1}{2}\right)^s < A_{n,s}^{-1} < 2\left(n-\frac{1}{2}+\epsilon\right)^s$$

for any sufficiently large even number n.

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

Theorem

(KS; 2018) Let s be a real number with $0 < s < 1$. Then for any positive odd number n,

$$-2\left(n-\frac{1}{4}\right)^s < B_{n,s}^{-1} < -2\left(n-\frac{1}{2}\right)^s$$

and for any $\epsilon > 0$,

$$-2\left(n-\frac{1}{2}+\epsilon\right)^s < B_{n,s}^{-1} < -2\left(n-\frac{1}{2}\right)^s$$

for any sufficiently large odd number n.

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

Corollary

(KS; 2018) Let s be a real number with $0 < s < 1$. Then for any positive even number n,

$$-2\left(1-2^{1-s}\right)\left(n-\frac{1}{4}\right)^{s} < \zeta_n(s)^{-1} < -2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^{s}$$

and for any positive odd number n,

$$2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^{s} < \zeta_n(s)^{-1} < 2\left(1-2^{1-s}\right)\left(n-\frac{1}{4}\right)^{s}.$$

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

Corollary

(KS; 2018) For any positive number ϵ and any real number s with $0 < s < 1$,

$$-2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}+\epsilon\right)^{s} < \zeta_n(s)^{-1} < -2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^{s}$$

for any sufficiently large even number n and

$$2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}\right)^{s} < \zeta_n(s)^{-1} < 2\left(1-2^{1-s}\right)\left(n-\frac{1}{2}+\epsilon\right)^{s}$$

for any sufficiently large odd number n.

The inverse of the tails of the Riemann zeta function when $0 < s < 1$

Corollary

(KS; 2018) For any positive integer n and $s = \frac{1}{2}, \frac{1}{3}$, *or* $\frac{1}{4}$,

$$\left[\frac{1}{1-2^{1-s}}\zeta_n(s)^{-1}\right] = \left[(-1)^{n+1}2\left(n-\frac{1}{2}\right)^s\right].$$

Questions

- Bounds of $\zeta_n(s)^{-1}$ for any integer $s \geq 7$
- Integer parts of $\zeta_n(s)^{-1}$ for any $0 < s < 1$
- Bounds of $||\zeta_n(s)^{-1}||$ for $s = \sigma + it$ where $0 < \sigma < 1$ and $t \neq 0$
- Identities related to $\zeta_n(s)$

Further research on the reciprocal of the Riemann zeta function tails

Further research on the reciprocal of the Riemann zeta function tails

Lemma

(S; 2019) For any positive even integer n, we have

$$2\left(n-\frac{1}{2}\right)^{2/3}=2\left(n^2-n+\frac{1}{4}\right)^{1/3}<A_{n,2/3}^{-1}<2\left(n^2-n+\frac{3}{4}\right)^{1/3}$$

and for any positive odd integer n, we have

$$-2\left(n^2-n+\frac{3}{4}\right)^{1/3}<B_{n,2/3}^{-1}<-2\left(n^2-n+\frac{1}{4}\right)^{1/3}.$$

Hence, if there is no positive integer solution (x, y) for $x^3 = 8y^2 - 8y + 3, 8y^2 - 8y + 4, 8y^2 - 8y + 5$, we can make a formula consisted of the gauss function.

Further research on the reciprocal of the Riemann zeta function tails

Theorem

(J.H. Silverman) For each integer $m \neq 0$, let

$$N(m) = \#\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y^2 = x^3 + m\}.$$

Then $N(m)$ is finite.

Corollary

(S; 2019) For any positive integer n,

$$\left[\frac{1}{1 - 2^{1/3}} \zeta_n \left(\frac{2}{3}\right)^{-1}\right] = \left[(-1)^{n+1} 2 \left(n - \frac{1}{2}\right)^{2/3}\right].$$

Further research on the reciprocal of the Riemann zeta function tails

Lemma

(HS; 2019) For any even integer $n \geq 4$ and $s = 2/d$, where $d \geq 3$ is any positive integer, we have

$$2\left(n-\frac{1}{2}\right)^{s} = 2\left(n^2-n+\frac{1}{4}\right)^{s/2} < A_{n,s}^{-1} < 2\left(n^2-n+\frac{3}{4}\right)^{s/2}$$

and for any odd integer $n \geq 3$, we have

$$-2\left(n^2-n+\frac{3}{4}\right)^{s/2} < B_{n,s}^{-1} < -2\left(n^2-n+\frac{1}{4}\right)^{s/2}.$$

Further research on the reciprocal of the Riemann zeta function tails

Now, we give a result on the finiteness of the integer points of certain affine curves, which will be used later:

Lemma

(HS; 2019) The affine curve $C_m : x^d - 2^d y^2 - 2^d y + 2^{d-2} + m = 0$ *defined over* $\mathbb{Q}$ *has only finitely many integer points for each* $1 \leq m \leq 2^{d-1} - 1$.

Further research on the reciprocal of the Riemann zeta function tails

In the sequel, let C_m denote the affine curve defined in previously for $1 \leq m \leq 2^{d-1} - 1$. Using the above finiteness result on integral points, we have the following

Lemma

(HS; 2019) Let $d \geq 5$ be any integer. Then there exists an integer $n_0 > 0$ with the property that there is no integer between $2\left(n^2 - n + \frac{1}{4}\right)^{\frac{1}{d}}$ and $2\left(n^2 - n + \frac{3}{4}\right)^{\frac{1}{d}}$ for every integer $n \geq n_0$.

Further research on the reciprocal of the Riemann zeta function tails

Corollary

(HS; 2019) Let s be of the form $1/d$ or $2/d$, where $d \geq 3$ is any positive integer. Then there exists an integer $N > 0$ such that we have

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n - \frac{1}{2}\right)^s\right]$$

for every integer $n \geq N$.

Further research on the reciprocal of the Riemann zeta function tails

Question

Let $0 < s < 1$ be any real number. Then, for each s, is there an integer $N > 0$ that satisfies the following equation

$$\left[\frac{1}{1-2^{1-s}} \cdot \zeta_n(s)^{-1}\right] = \left[(-1)^{n+1} \cdot 2\left(n - \frac{1}{2}\right)^s\right]$$

for every integer $n \geq N$?


Thank you

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