Medial Axis Transform: Shape, Algorithm, and Stability

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for Computer Science

Contents

Shape

"Mathematical Theory of Medial Axis Transform," with H.I.Choi and H.P.Moon, *Pac. J. Math.*, Vol.181, No.1, 1997

Algorithm

"New Algorithm for Medial Axis Transform of Plane Domain," with H.I.Choi, H.P.Moon and N.-S.Wee, *GMIP*, Vol.59, No.6, 1997

Stability

"Hyperbolic Hausdorff Distance for Medial Axis Transform," with H.-P. Seidel, Research Report, MPI-I-2000-4-003, 2000.

"Stability Analysis of Medial Axis Tranform under Relative Hausdorff Distance," with S.W.Lee, ICPR2000, Barcelona, Sep., 2000.

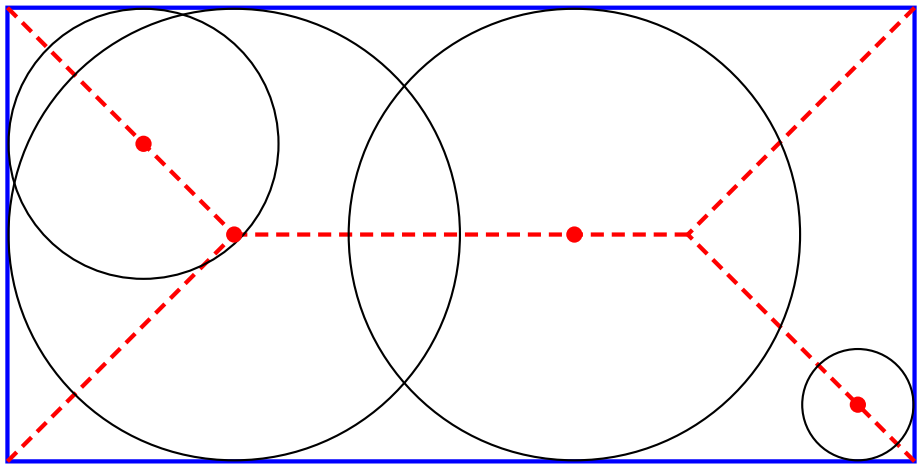
"One-sided Stability of Medial Axis Transform," with H.-P. Seidel, DAGM2001, Munich, Sept., 2001.

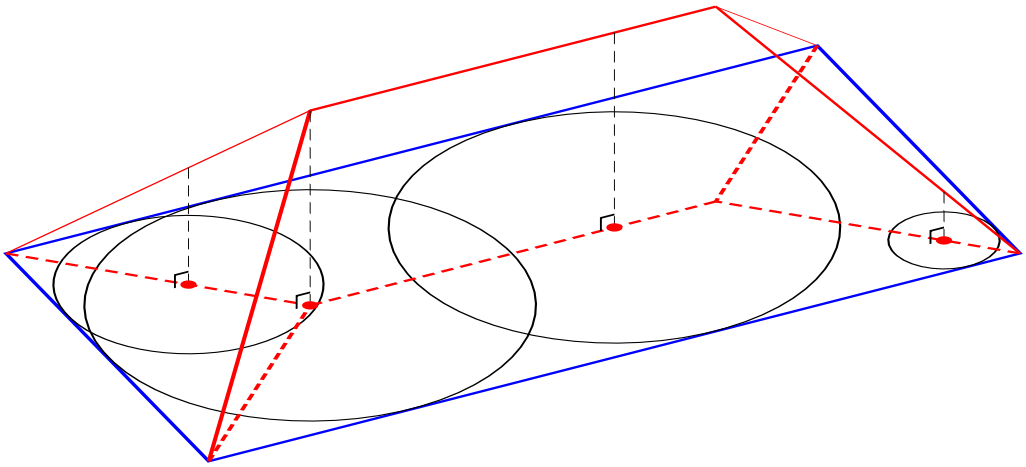
"One-sided Stability of MAT and its Applications," with H.-P. Seidel, Vision, Modeling, and Visualization 2001, Stuttgart, Nov., 2001.

What is Medial Axis Transform?

$$\mathbf{MAT}(\Omega) = \{(p, r) \in \mathbb{R}^2 \times \mathbb{R} \mid B_r(p) \text{ is a maximal ball in } \Omega\}$$

$$\mathbf{MA}(\Omega) = \{p \in \mathbb{R}^2 \mid \exists r \geq 0 \ s.t. \ (p, r) \in \mathbf{MAT}(\Omega)\}$$





Other Definitions

- Grass Fire Model
  - points where wave starting from the boundary meet
- Cut Locus
  - points where the distance to the boundary stop minimizing
- Set of Singularities
  - points where the distance function to the boundary is not differentiable

Merits of Medial Axis Transform

- Preserves topological structures of domains.
- Easy to store and process in computer.– graph structure
- Has natural definitions.
- Has a wide range of applications.

Pitfalls of Medial Axis Transform

- Difficult to compute
- Sensitive to noises
- Not exactly corresponds to human perception

Problems

- What it looks like?
- How to compute it efficiently?
- How to control its sensitivity to noises?

MAT of Normal Domain

Definition

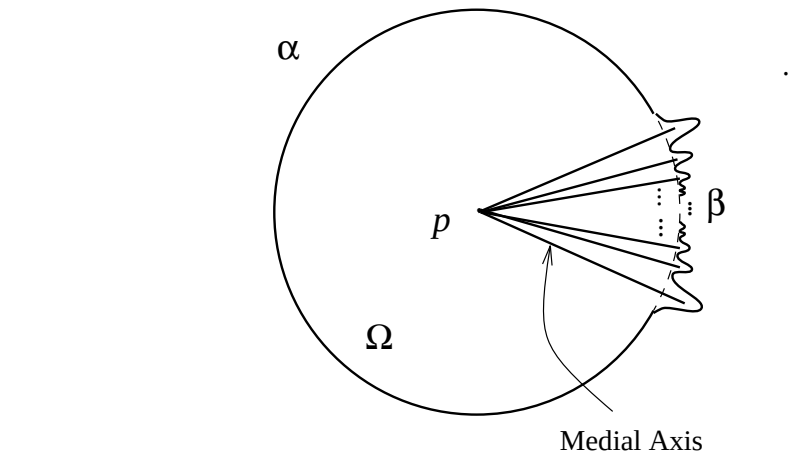
A compact and connected set $\Omega \subset \mathbb{R}^2$ is a *normal domain*, if $\partial\Omega$ is a disjoint union of finite number of simple closed curves, each of which in turn consists of finite number of *real analytic* curve pieces.

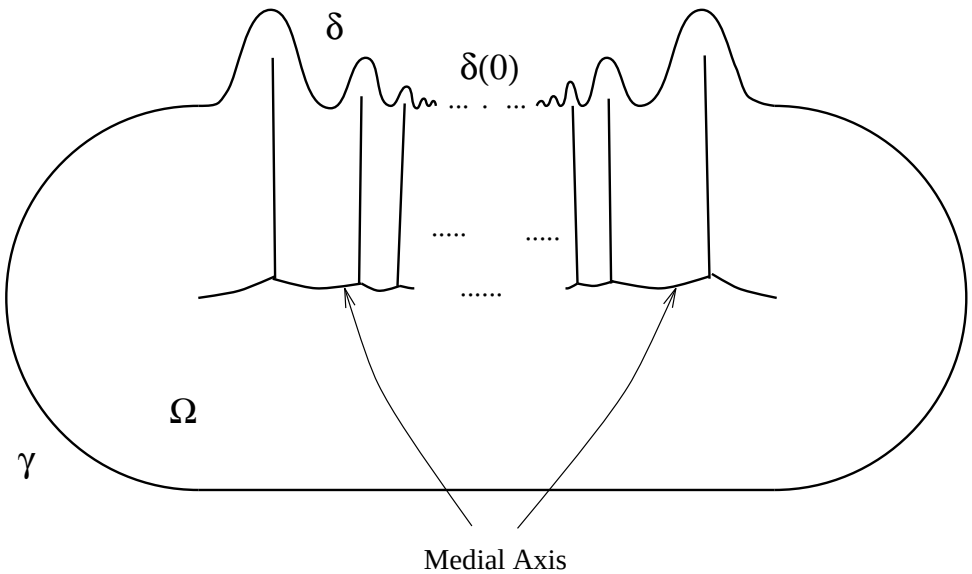
Theorem

If Ω is a normal domain, then $\mathbf{MA}(\Omega)$ and $\mathbf{MAT}(\Omega)$ have *finite graph structures*, and $\mathbf{MA}(\Omega)$ is a *strong deformation retract* of Ω.

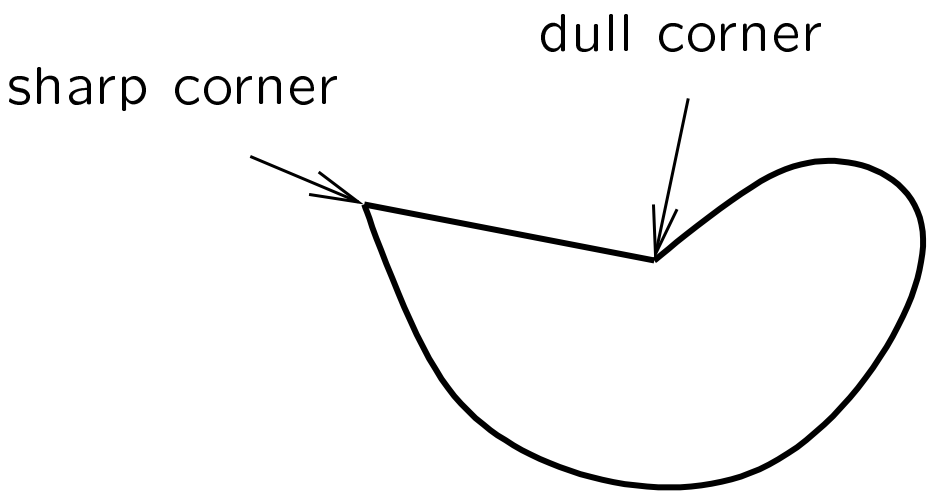
- **MA (MAT)** can exhibit pathological behaviours for non-normal domains.

- Almost every domain in applications is normal.



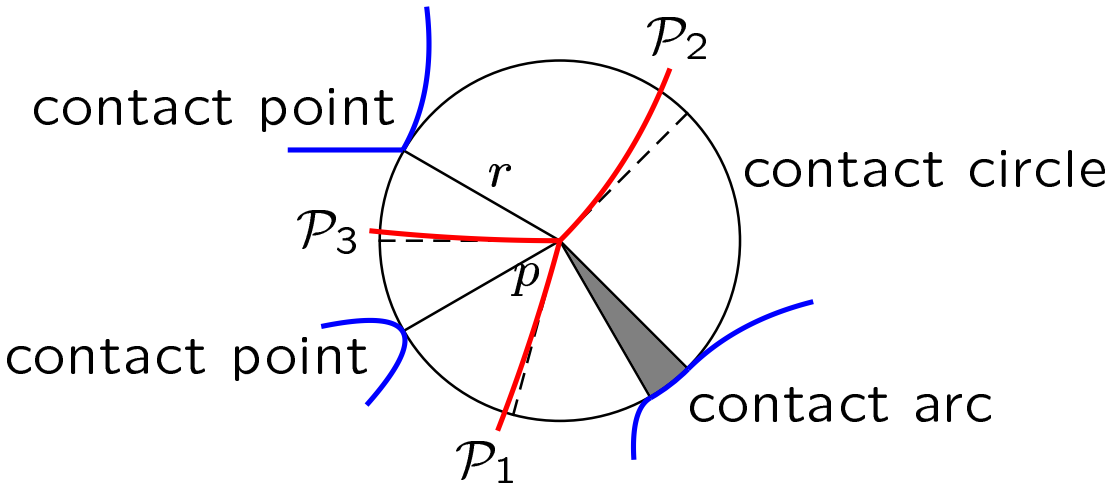


Corners



Sharp corners are always on $\mathbf{MA}$.

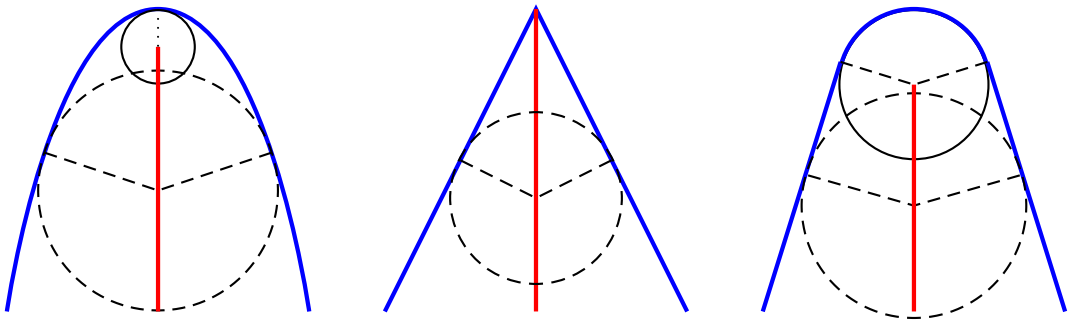
Classification of MAT Points



contact component: contact point or contact arc

- 1-Prong Points : End Points. Finite.
- 2-Prong Points : Generic Points. Correspond to *edges*.
- n-Prong Points ($n \geq 3$) : Bifurcation Points. Finite.

Three Types of 1-Prong Points



Possible places in the boundary for the 1-prong points:

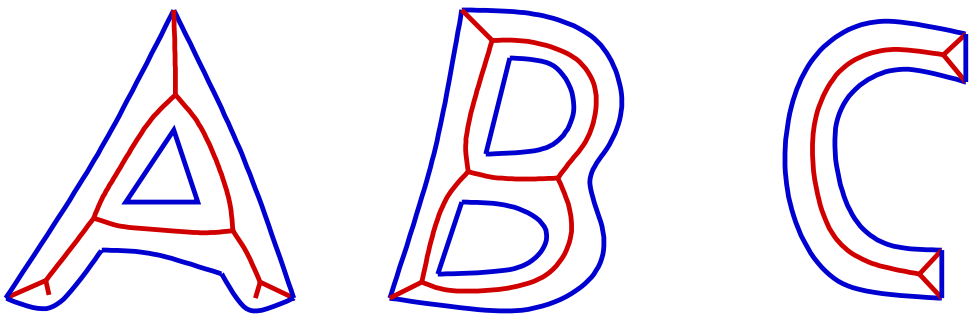
- sharp corner (always)
- point of maximum curvature
- circular arc

1-prong points are found directly from the boundary.

Homotopy Equivalence of Medial Axis

Theorem

If Ω is a normal domain, then $\mathbf{MA}(\Omega)$ is a strong deformation retract of Ω.

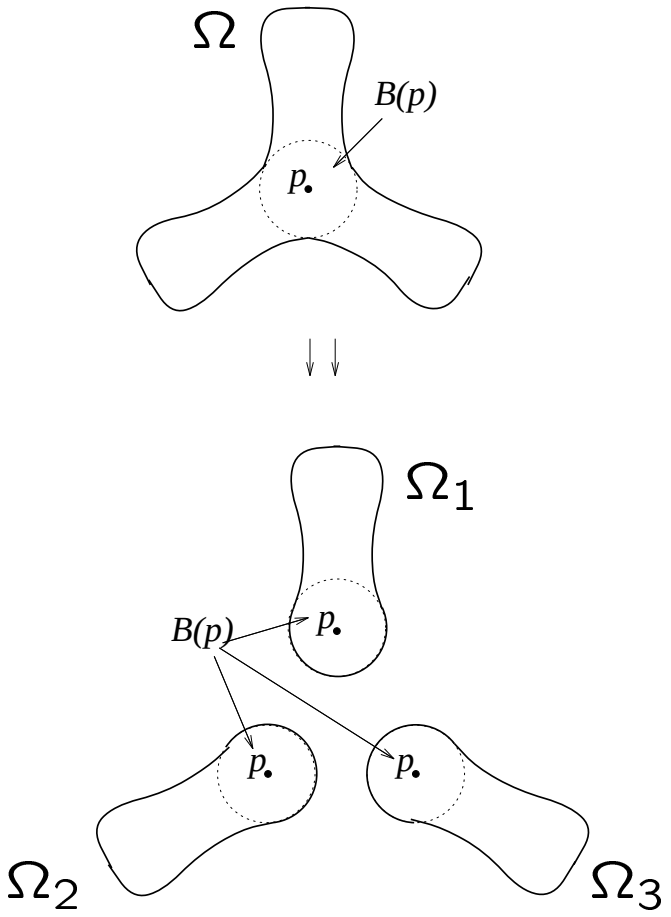


Algorithm

Choi, Choi, Moon, Wee, 1997 : Essentially the first for domains with free-form boundary and non-trivial homology.

- Initialization
  - Killing Homology
  - Circular Arcs
  - Corners
  - Inscribed Osculating circle
- Geometric Algorithm
  - 2-Prong Algorithm
  - Resolution of *Virtual Node*

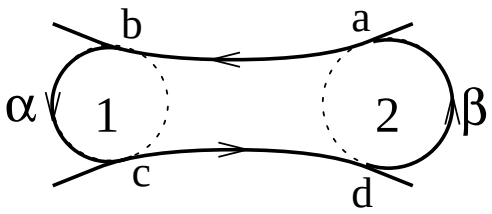
Domain Decomposition



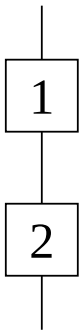
$$\mathbf{MA}(\Omega) = \bigcup_i \mathbf{MA}(\Omega_i)$$

$$\mathbf{MAT}(\Omega) = \bigcup_i \mathbf{MAT}(\Omega_i)$$

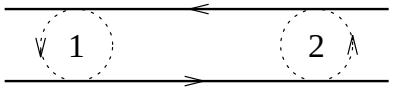
Subdomain and Tree Data Structure



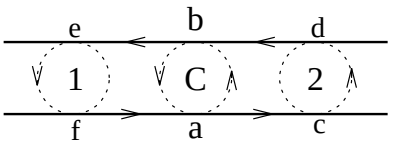
Subdomain



Edge connecting node 1 and 2



Before insertion of C



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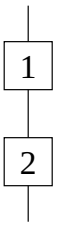

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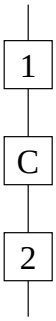
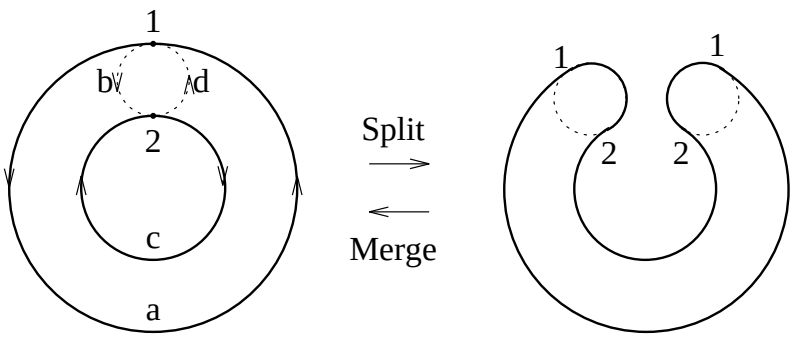
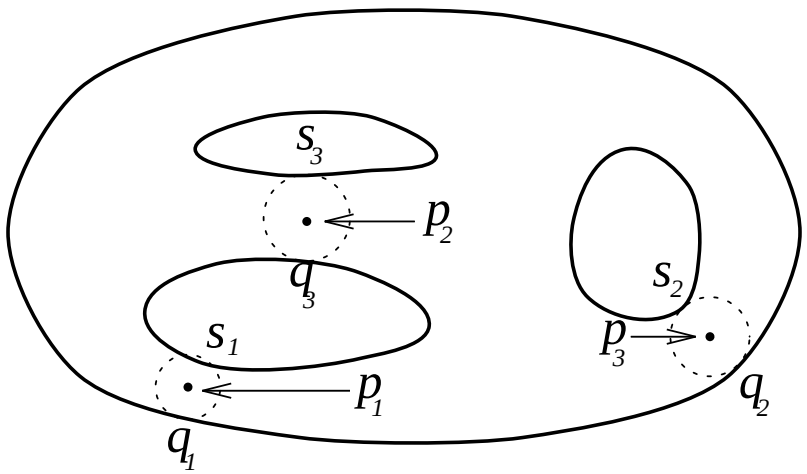


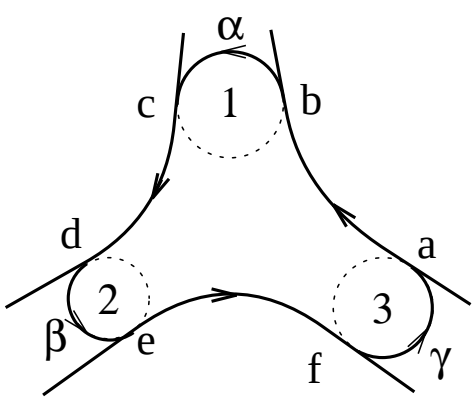
After insertion of C

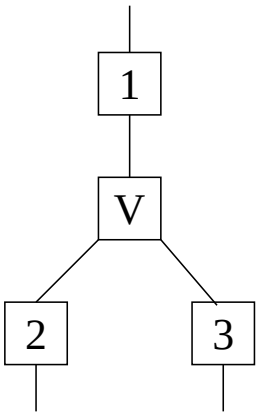
Homology Killing

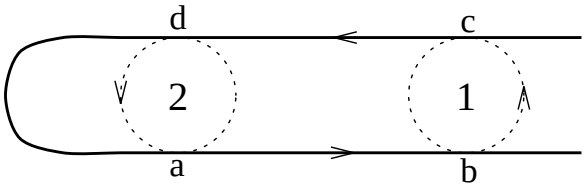


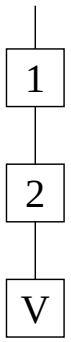


Virtual Node



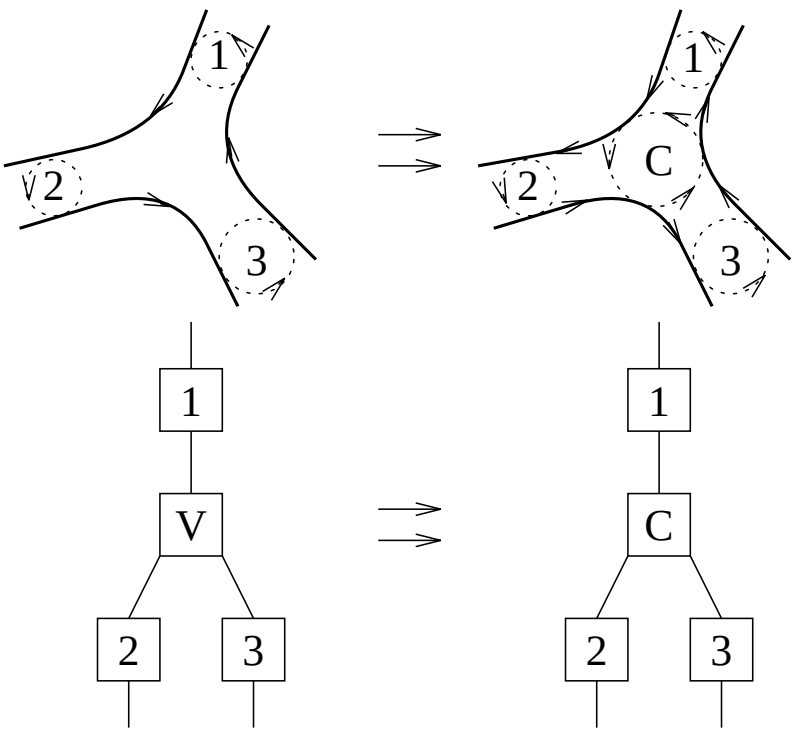




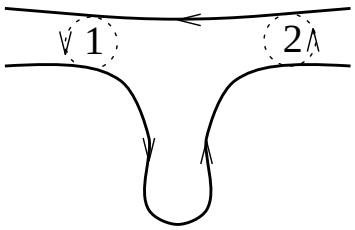


Resolution of Virtual Node

Theoretical guarantee for the existance [Choi, Choi, Moon, 1997].



Operation on The Tree Structure



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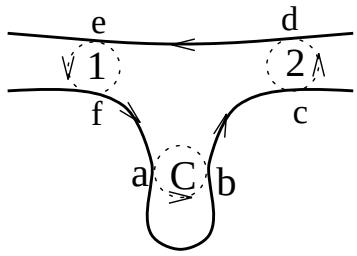
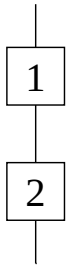
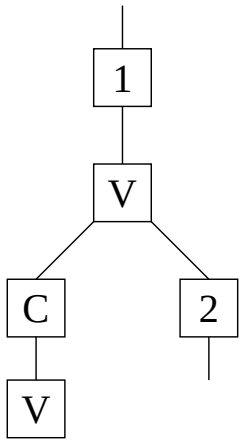


After insertion of C

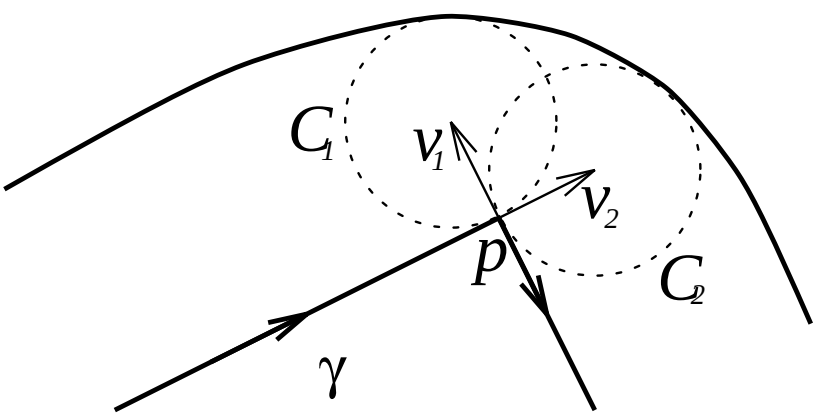


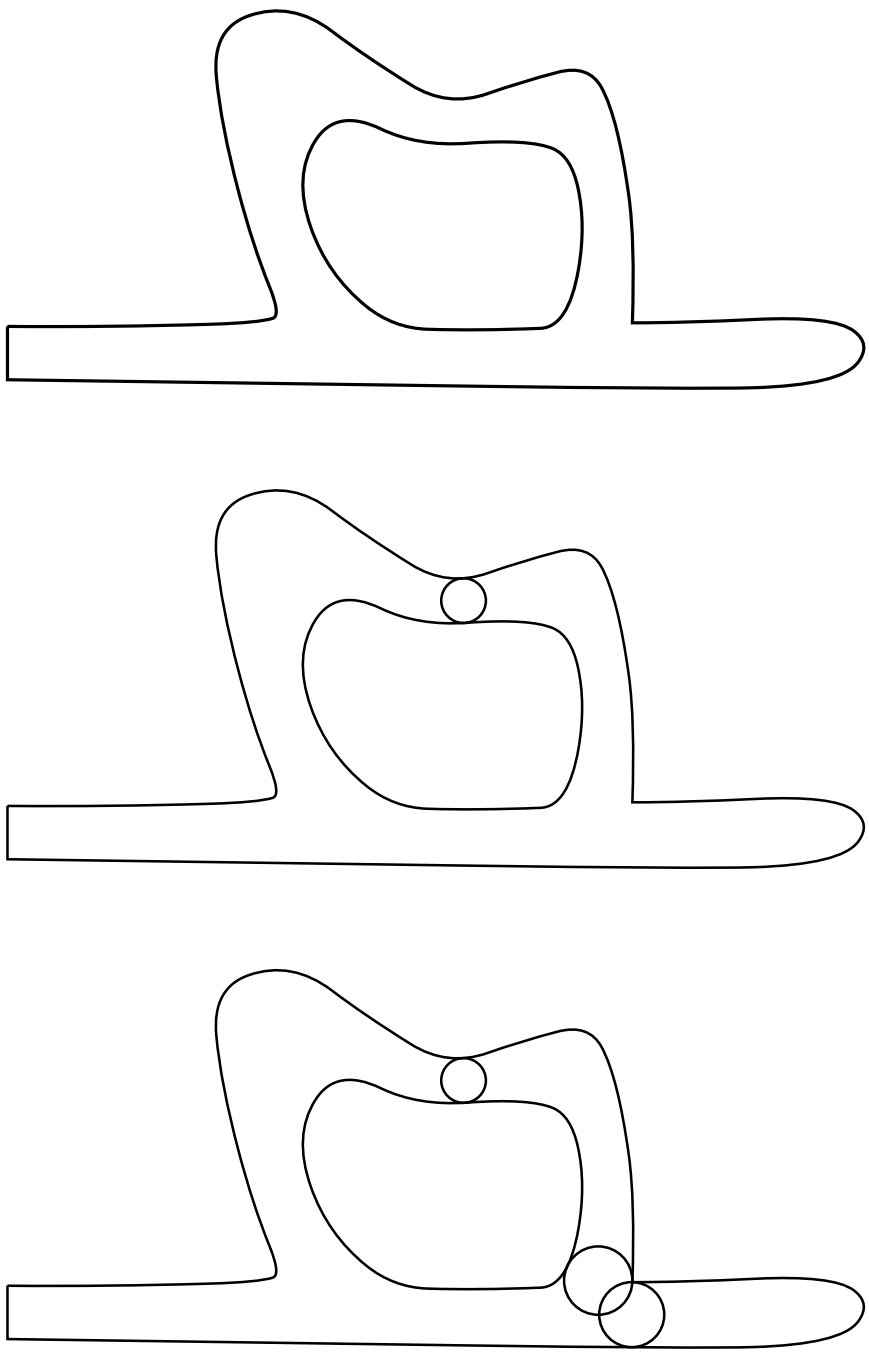
Before insertion of C

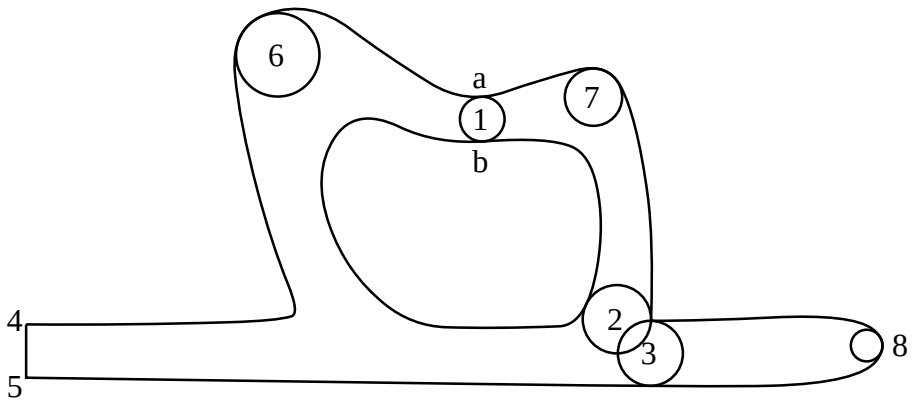


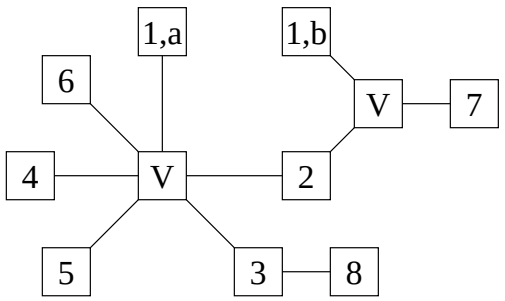
After insertion of C

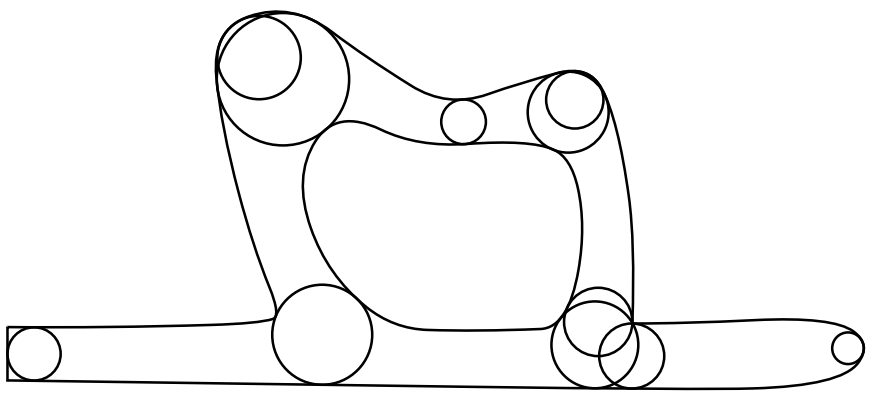
Dull Corner

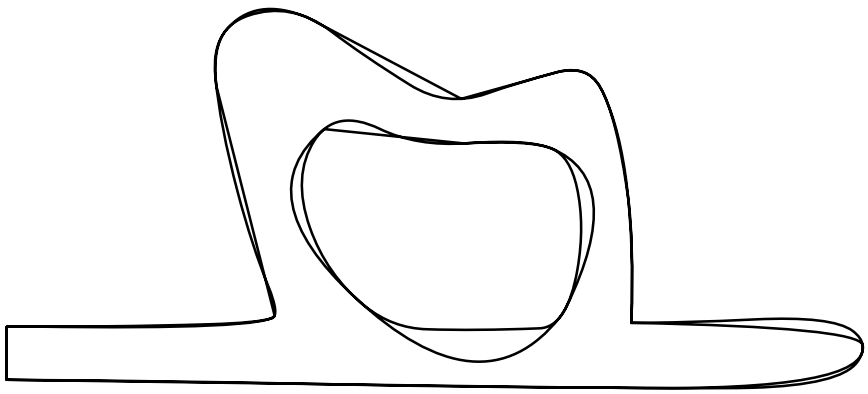


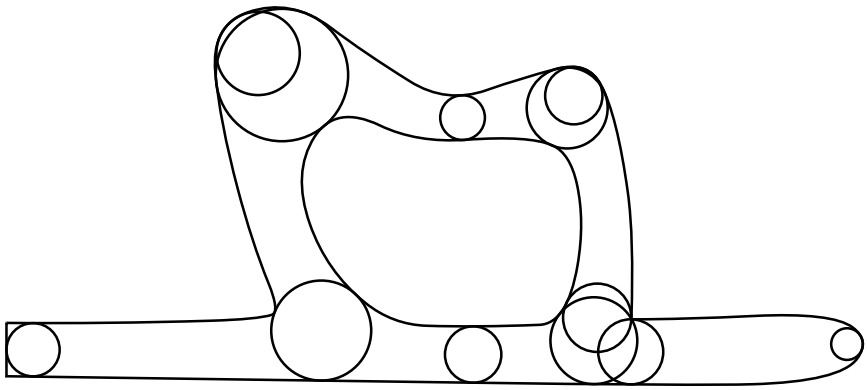


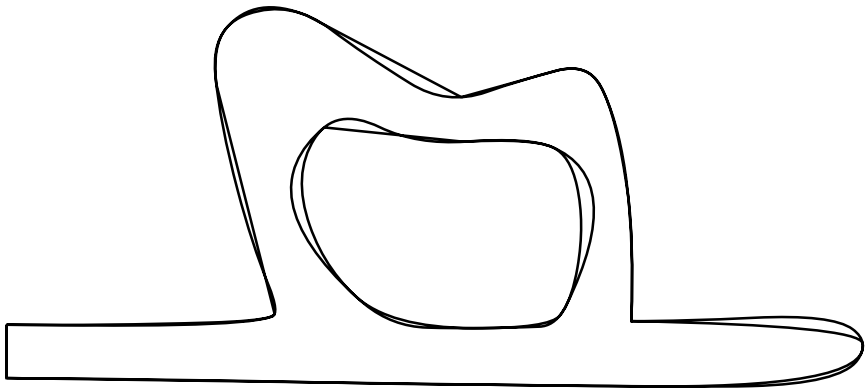

6
a
7
1
b
4
2
3
8
5

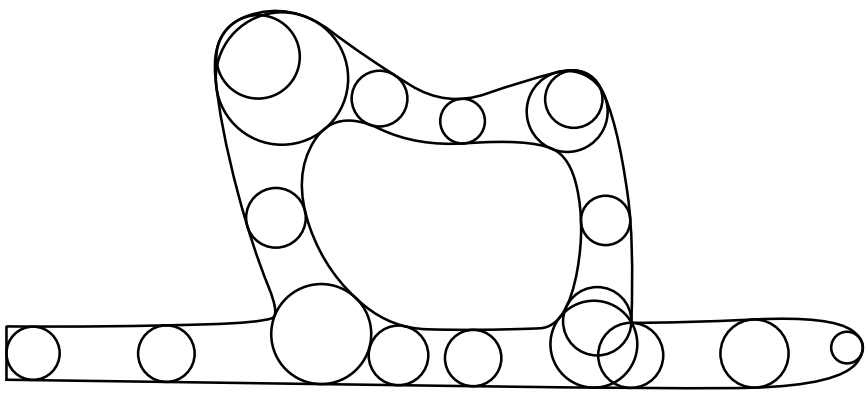

1,a
1,b
6
V
7
4
V
2
5
3
8

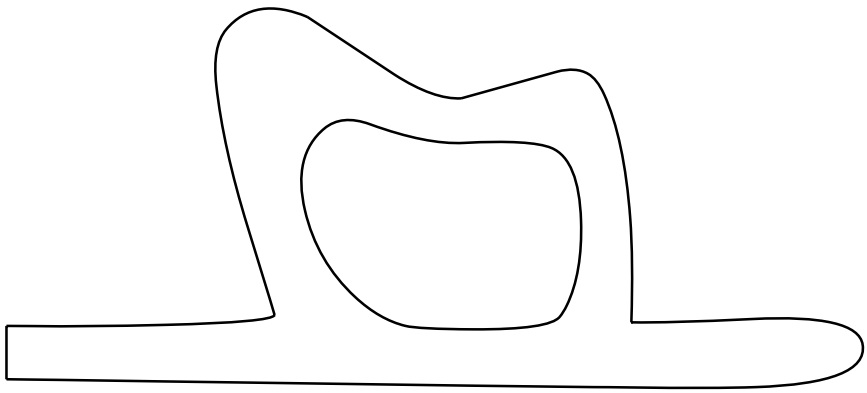


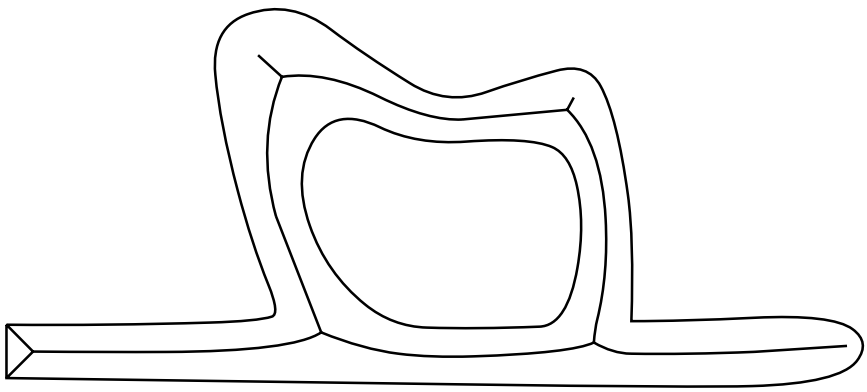












Characteristics of the Algorithm

- Applies to domains with *free-form* boundary and holes

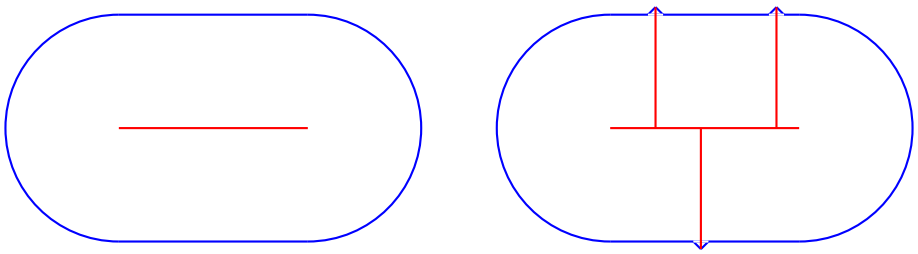
- Guarantees topological correctness from the start

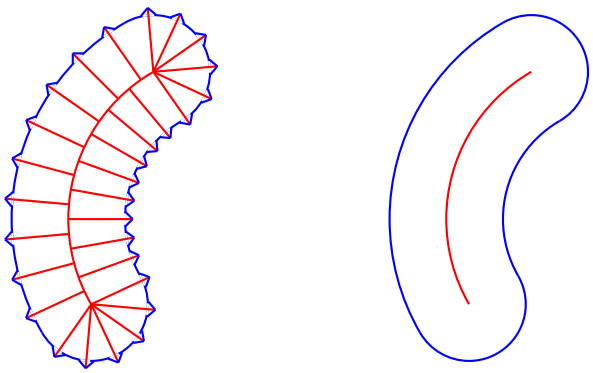
- Approximate nature

- Fast - domain decomposition

Instability of Medial Axis Transform

- Very sensitive to perturbation of the boundary. (not continuous)
- Introduces many *hairy* prongs.
- No explicit and quantitative error analyses before.

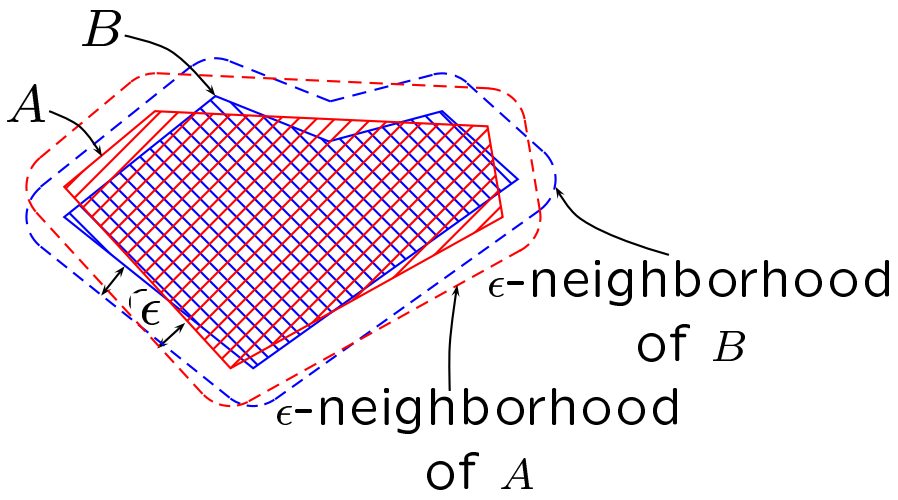




Hausdorff Distance

A, B : compact sets in $\mathbb{R}^n$.

$$\begin{aligned}\mathcal{H}(A,B) &= \max\{\max_{p\in A} d(p,B), \max\{\max_{q\in B} d(q,A)\}\} \\ &= \max\{\mathcal{H}(A|B), \mathcal{H}(B|A)\}.\end{aligned}$$



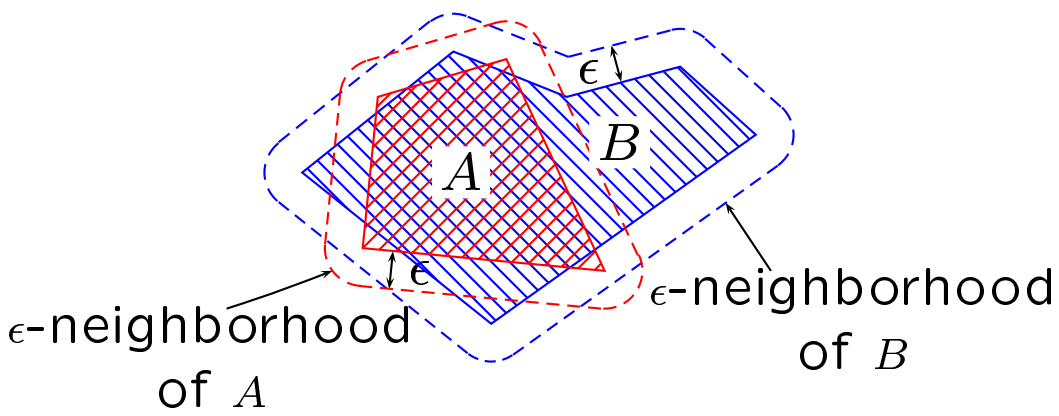
Proposition

Hausdorff distance is a *complete metric* on the set of all (nonempty) compact sets in $\mathbb{R}^n$.

One-sided Hausdorff Distance

A, B: compact sets in $\mathbb{R}^n$

$$\mathcal{H}(A|B) = \max_{p \in A} d(p, B)$$



$\mathcal{H}(A|B)$ measures how much A is contained in B.

Hyperbolic Hausdorff Distance

Definition (Hyperbolic distance)

Let $P_i = (p_i, r_i) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ for $i = 1, 2$. Here, $R_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$. Then the *hyperbolic distance* $d_h(P_1|P_2)$ of P_1 with respect to P_2 is defined by

$$d_h(P_1|P_2) = \max\{d(p_1, p_2) - (r_2 - r_1), 0\}.$$

Definition (Hyperbolic Hausdorff Distance)

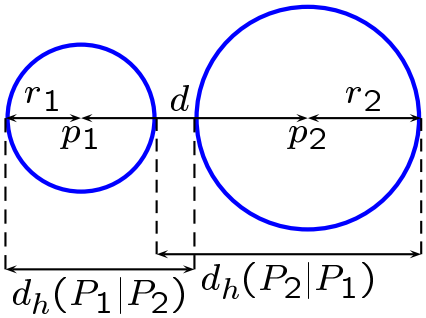
Let M_1, M_2 be two compact sets in $\mathbb{R}_2 \times \mathbb{R}_{\geq 0}$. Then the *hyperbolic Hausdorff distance* $\mathcal{H}_h(M_1, M_2)$ between M_1 and M_2 is defined by

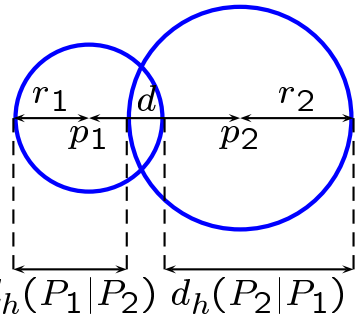
$$\mathcal{H}_h(M_1, M_2) = \max\{\mathcal{H}_h(M_1|M_2), \mathcal{H}_h(M_2|M_1)\},$$

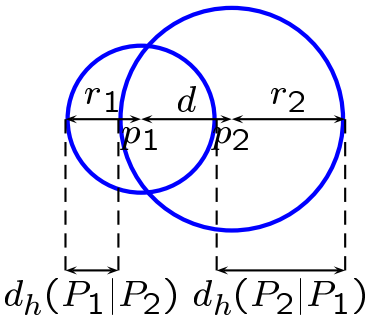
where the *one-sided hyperbolic Hausdorff distance* $\mathcal{H}_h(M_1|M_2)$ of M_1 with respect to M_2 is defined by

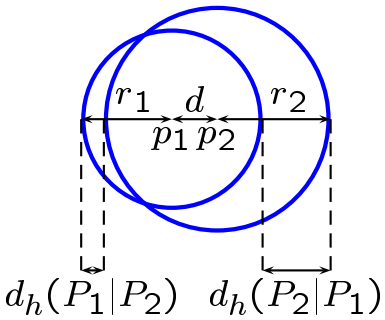
$$\mathcal{H}_h(M_1|M_2) = \max_{P_1 \in M_1} \min_{P_2 \in M_2} d_h(P_1|P_2).$$

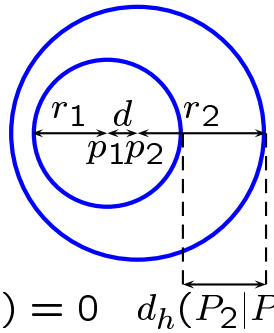
Illustration of Hyperbolic Hausdorff Distance

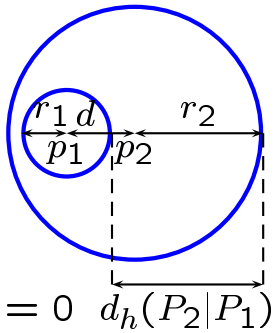












Stability of MAT under Hyperbolic Hausdorff Distance

Let

$$\begin{aligned} \mathcal{C}_n &= \{\Omega \in \mathbb{R}^n \,|\, \Omega, \mathbf{MAT}(\Omega) \text{ are compact}\}, \\ \mathcal{C}_{n,1} &= \{M \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \,|\, M \text{ is compact}\}, \end{aligned}$$

and define

$$\mathcal{H}_\partial(\Omega_1, \Omega_2) = \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}$$

for $\Omega_1, \Omega_2 \in \mathcal{C}_n$.

Theorem

Let Ω_1, Ω_2 be in $\mathcal{C}_n$ for some $n = 1, 2, \cdots$. Then we have

$$\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) \leq 3 \cdot \mathcal{H}_\partial(\Omega_1, \Omega_2),$$

$$\mathcal{H}_\partial(\Omega_1, \Omega_2) \leq \mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)).$$

Thus the transform

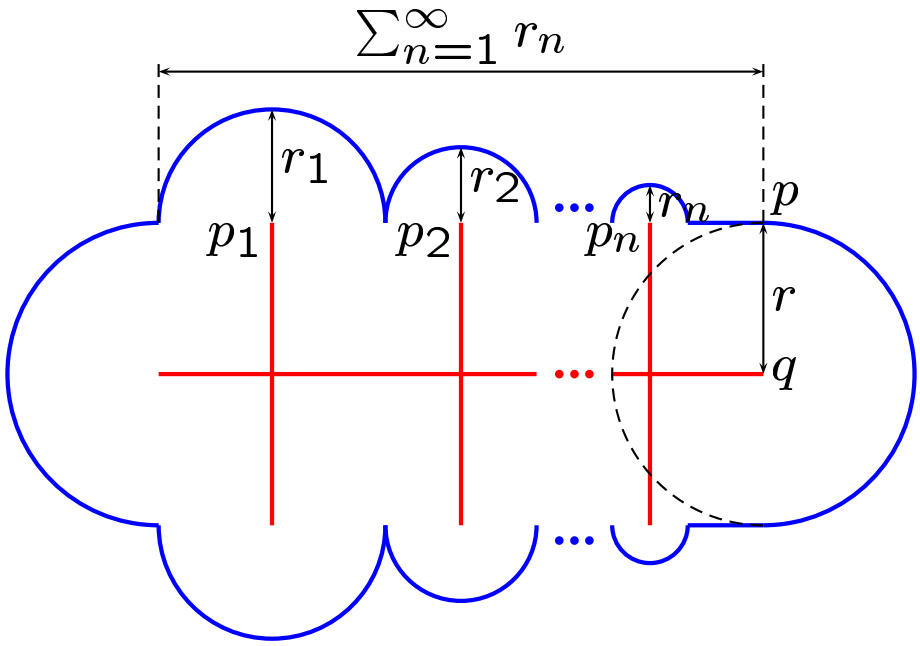
$$\mathbf{MAT} : (\mathcal{C}_n, \mathcal{H}_\partial) \to (\mathcal{C}_{n,1}, \mathcal{H}_h)$$

is continuous (in fact, almost an isometry)

Remark $\mathbf{MAT}(\Omega) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ may not be compact, even if $\Omega \in \mathbb{R}^n$ is compact.

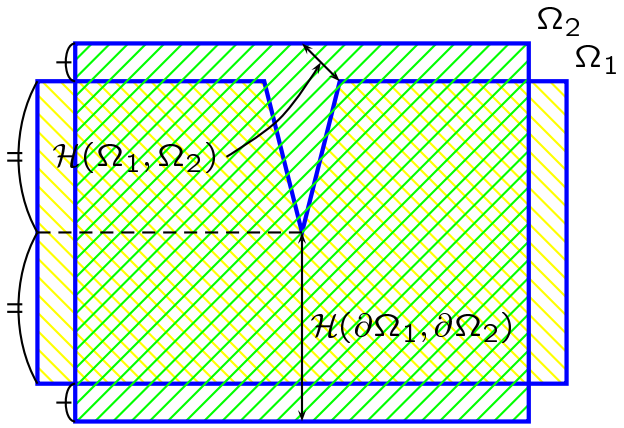
Example

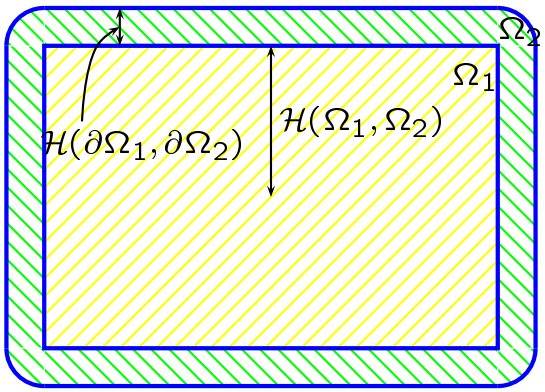
For $n = 1, 2, \cdots$, let Ω_n be the domain in $\mathbb{R}^2$ as depicted below. Here we assume $\sum_{n=1}^{\infty} r_n < \infty$.



Let $\Omega = \lim_{n \to \infty \Omega)n}$ (in the Hausdorff distance). Clearly, Ω is compact, whereas $\mathbf{MAT}(\Omega)$ is not. $((p, 0) \subset (q, r))$

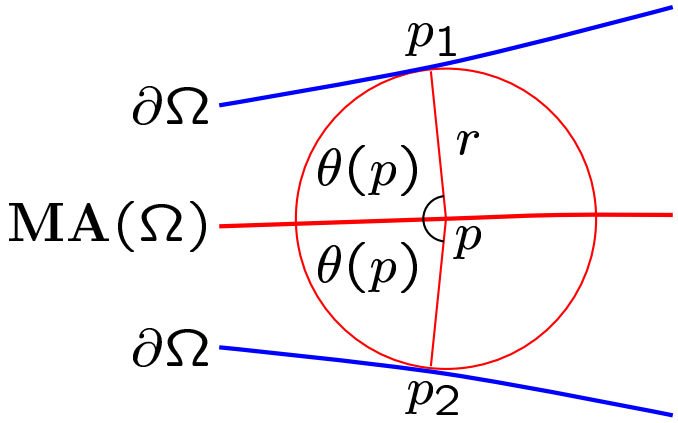
Hausdorff Distance between Domains *vs.* Boundaries





Domain-dependent Constants

Except for some finitely many points, every point in $\mathbf{MA}(\Omega)$ has exactly two contact *points*, the set of which is denoted by $G(\Omega)$.

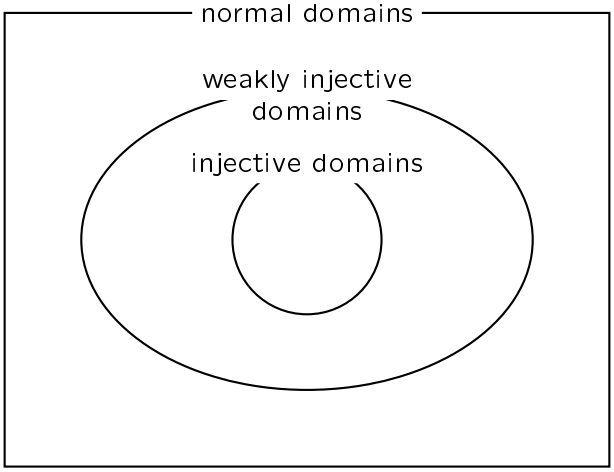


$$\begin{aligned} \rho_\Omega &= \inf\{r : (p,r) \in \mathbf{MAT}(\Omega)\}, \\ \theta_\Omega &= \inf\{\theta(p) : p \in G(\Omega)\}. \end{aligned}$$

$\rho_\Omega = 0 \iff \Omega$ has a sharp corner.

$\theta_\Omega = 0 \iff \mathbf{MA}(\Omega)$ has a 1-prong point which corresponds to a (nontrivial) inscribed osculating circle.

Hierarchy of Domains



Definition A normal domain Ω is:

weakly injective, if $\theta_\Omega > 0$,

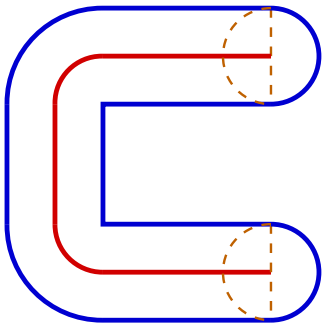
injective, if $\theta_\Omega > 0$ and $\rho_\Omega > 0$.

Ω is injective $\iff$ Every 1-prong points in $\mathbf{MA}(\Omega)$ corresponds to an inscribed circle with the contact arc.

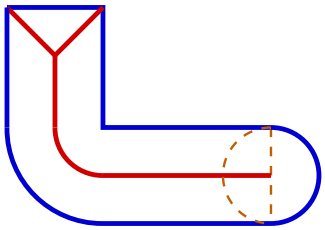
Ω is weakly injective $\iff$ $\mathbf{MA}(\Omega)$ does not have a 1-prong point which corresponds to a (nontrivial) inscribed osculating circle.

Examples

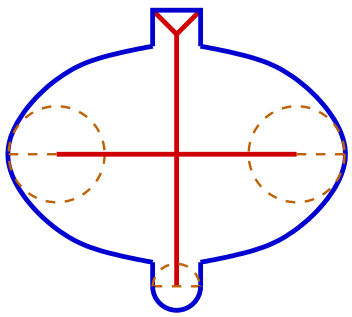
injective:



weakly injective but not injective:



normal but not weakly injective:



One-sided Stability of MAT for Injective Domain

Theorem

Let Ω be an injective domain. Then we have

$$\begin{aligned}
&\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) \\
&\quad \leq \frac{2}{1-\cos\theta_\Omega}\cdot\epsilon + o(\epsilon), \\
&\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) \\
&\quad \leq \frac{\sqrt{4+(3-\cos\theta_\Omega)^2}}{1-\cos\theta_\Omega}\cdot\epsilon + o(\epsilon),
\end{aligned}$$

for every normal domain Ω' such that $\max\{\mathcal{H}(\Omega,\Omega'),\mathcal{H}(\partial\Omega,\partial\Omega')\} \leq \epsilon$.

One-sided Stability of MAT for Weakly Injective Domain

For $\theta \in (0, \pi/2]$, let

$$g(\theta) = 3\left(1 + \frac{2\sqrt{1+\cos^2\theta}}{1-\cos\theta}\right).$$

Theorem

Let Ω be a weakly injective domain. Then we have

$$\begin{aligned} \mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) &\leq g(\theta_\Omega)\cdot\epsilon + o(\epsilon), \\ \mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) &\leq g(\theta_\Omega)\cdot\epsilon + o(\epsilon), \end{aligned}$$

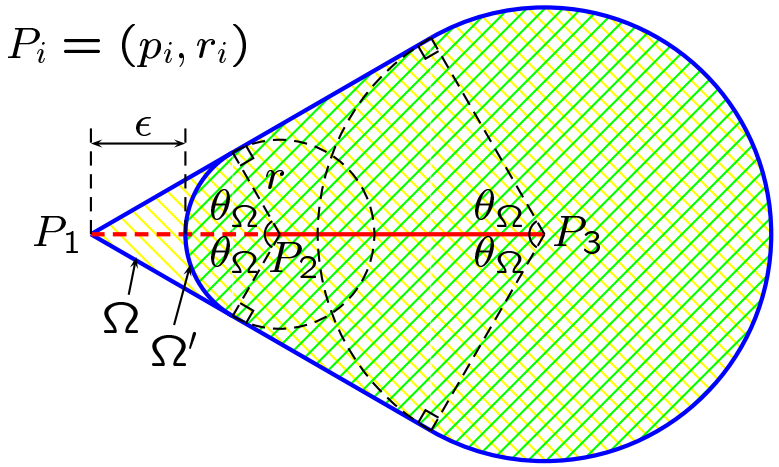
for every normal domain Ω' such that $\max\{\mathcal{H}(\Omega,\Omega'), \mathcal{H}(\partial\Omega,\partial\Omega')\} \leq \epsilon$.

Examples

$$\mathcal{H}(\Omega, \Omega') = \mathcal{H}(\partial\Omega, \partial\Omega') = \epsilon,$$

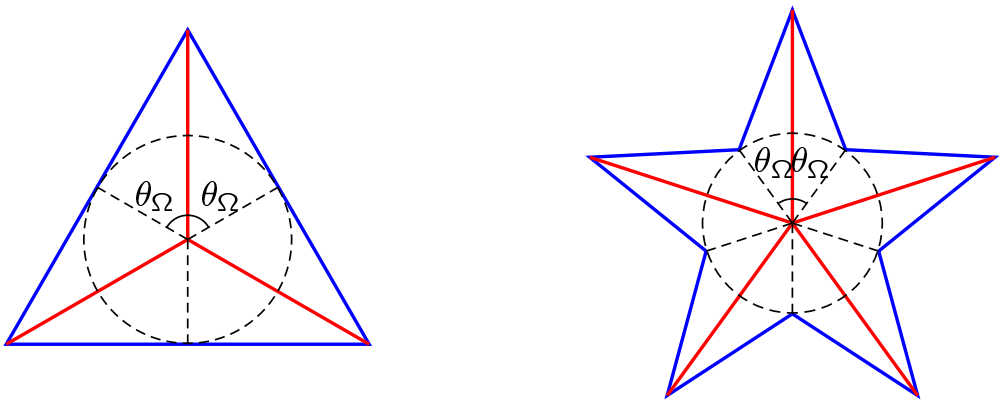
$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) = d(p_1, p_2) = \frac{1}{1-\cos\theta_\Omega} \cdot \epsilon,$$

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) = d(P_1, P_2) = \frac{\sqrt{1+\cos^2\theta_\Omega}}{1-\cos\theta_\Omega} \cdot \epsilon.$$



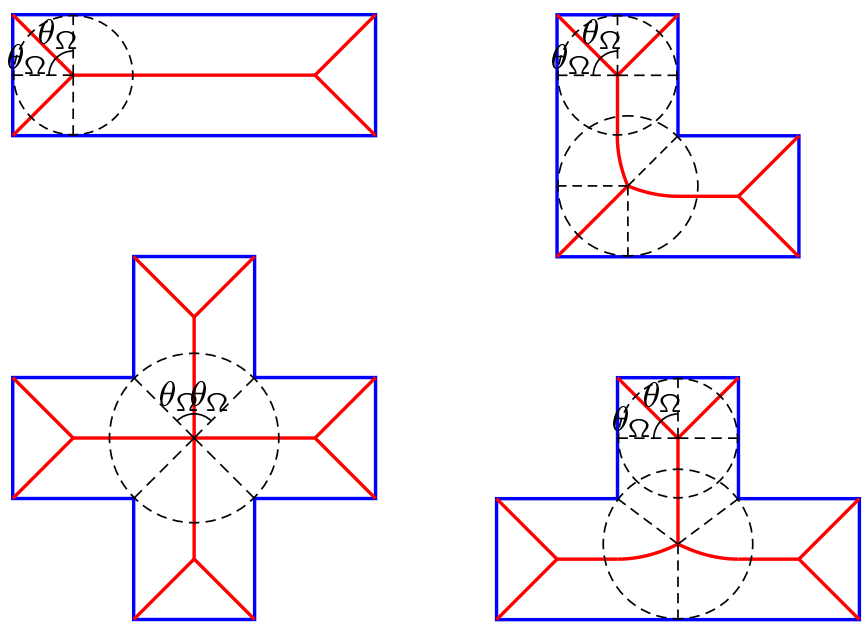
The factor $\frac{1}{1-\cos\theta_\Omega}$ in $g(\theta_\Omega)$, which blows up as $\theta_\Omega \to 0$, is unavoidable. One important consequence is that the class of the weakly injective domains is the largest possible class for which we have a linear bound for the one-sided Hausdorff distance of $\mathbf{MAT}$ (and $\mathbf{MA}$) with respect to the perturbation.

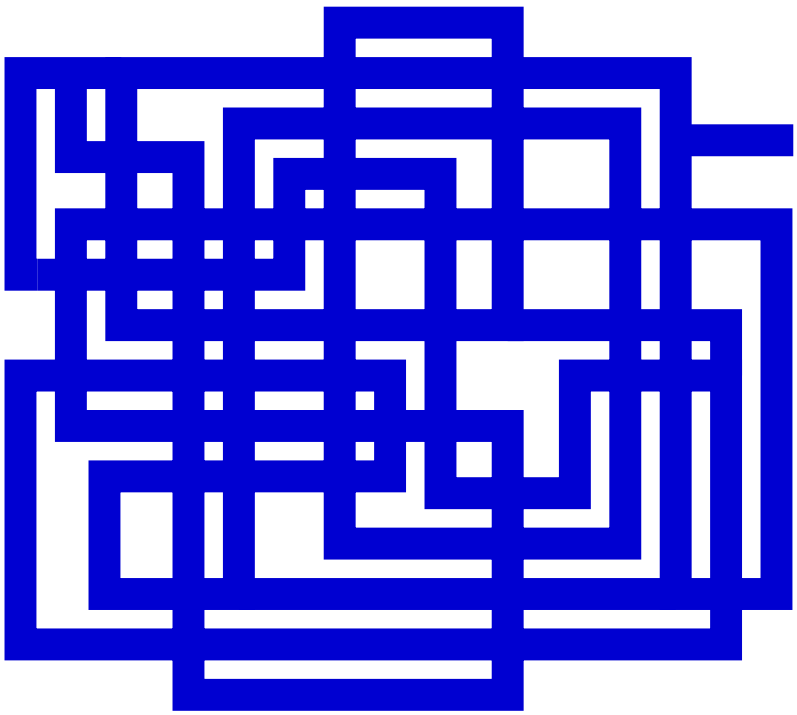
Consider an equilateral triangle and a star-shaped domain depicted respectively as in (a) and (b). Note that $\theta_\Omega = \frac{\pi}{3}$ for (a), and $\theta_\Omega = \frac{\pi}{5}$ for (b). So $g(\theta_\Omega) = 3(1 + 2\sqrt{5}) = 16.416408\cdots$ for (a), and $g(\theta_\Omega) = 43.410203\cdots$ for (b).



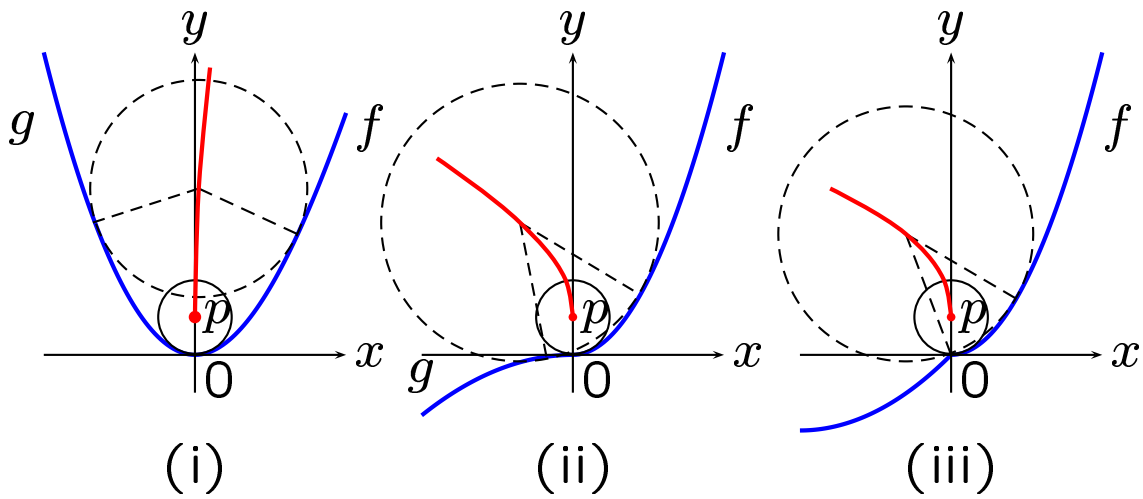
(a) (b)

Consider the rectangular domains with constant widths. Note that $\theta_\Omega = \frac{\pi}{4}$ and $g(\theta_\Omega) = 3\left(1 + 2\sqrt{3}(1+\sqrt{2})\right) = 28.089243\cdots$ for all cases.





Three Types of Non-degenerate 1-Prong Points



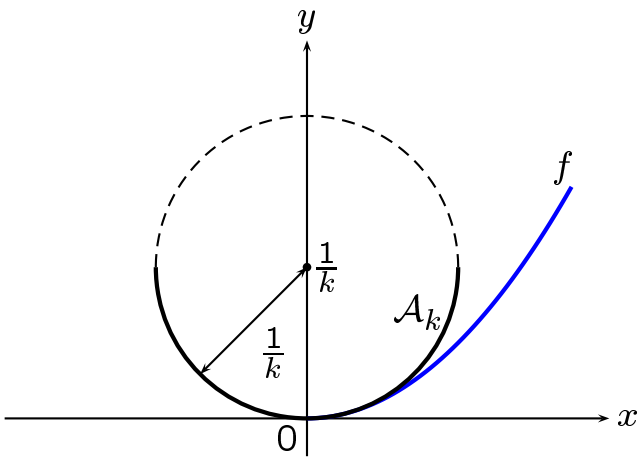
(i) C^2 at 0.

(ii) C^1 but not C^2 at 0.

(iii) not C^1 at 0.

Arc Function

$$
\begin{aligned}
&\mathcal{A}_k(x) \\
&\quad = \quad \frac{1}{k} - \frac{1}{k}\sqrt{1-k^2x^2} \\
&\quad = \quad \frac{1}{2}kx^2 + \frac{1}{8}k^3x^4 + \frac{1}{16}k^5x^6 + \cdots \\
&\quad =: \quad \sum_{n=2}^{\infty} k^{n-1}Q_n x^n .
\end{aligned}
$$



Aymptotic Constants

$P = (p, r) \in \mathbf{MAT}(\Omega)$: non-degenerate 1-prong point

$f(t) = \sum_{n=2} a_n t^n$, $a_2 = \frac{1}{2r} = \frac{k}{2}$.

$A_n = k^{n-1} Q_n - a_n$, $n = 2, 3, \cdots$.

N_P: the largest integer such that $A_2 = A_3 = \cdots = A_{N_P} = 0$.

$$\Longrightarrow N_P \geq 2 \text{ and } A_{N_P+1} > 0.$$

$$N_\Omega = \min \{ N_P : P \text{ is a non-degenerate 1-prong of } \mathbf{MAT}(\Omega) \}.$$

$$K_P = \begin{cases} \dfrac{\sqrt{2} \cdot 12^{\frac{N_P-1}{N_P+1}} (N_P+1)}{(N_P-1)^{\frac{N_P-1}{N_P+1}}} \cdot r^2 A_{N_P+1}^{\frac{2}{N_P+1}}, & P \text{: type (i)} \\ \dfrac{\sqrt{2} \cdot 2^{\frac{3N_P-1}{N_P+1}} \cdot 6^{\frac{N_P-1}{N_P+1}} \cdot N_P^{\frac{2N_P}{N_P+1}}}{(N_P-1)^{\frac{2(N_P-1)}{N_P+1}}} \cdot r^2 A_{N_P+1}^{\frac{2}{N_P+1}}, & P \text{: type (ii), (iii)} \end{cases}$$

$$K_\Omega = \max \{ K_P : P \text{ is a non-degenerate 1-prong point of } \mathbf{MAT}(\Omega) \text{ } s.t. \text{ } N_P = N_\Omega \}.$$

It is important to note that K_Ω can be calculated directly from $\partial\Omega$.

One-sided Stability of MAT for Normal Domain

Theorem

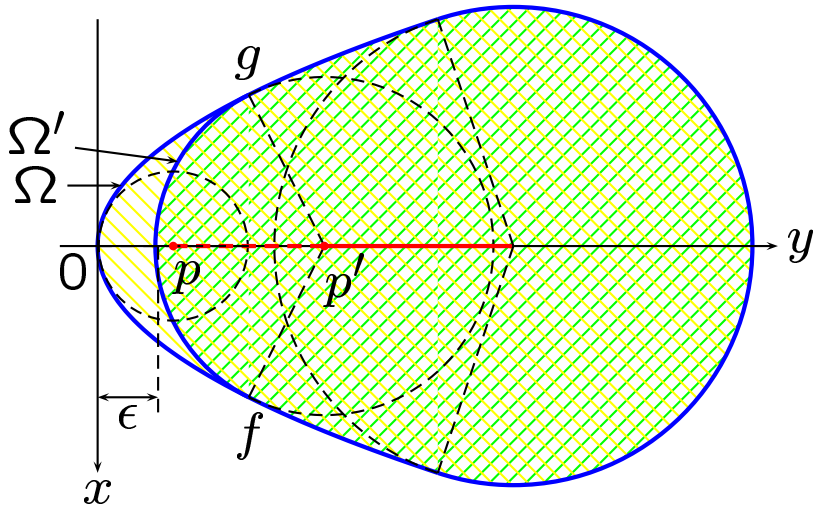
Let Ω be a normal domain which is not weakly injective. Then we have

$$\begin{aligned}
&\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) \\
&\quad \leq \quad K_\Omega \cdot \epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}} + o\left(\epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}}\right), \\
&\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) \\
&\quad \leq \quad K_\Omega \cdot \epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}} + o\left(\epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}}\right),
\end{aligned}$$

for every normal domain Ω' such that $\max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\} \leq \epsilon$.

Examples

Type (i): Quadratic



$f(x) = g(x) = \frac{k}{2}x^2$.

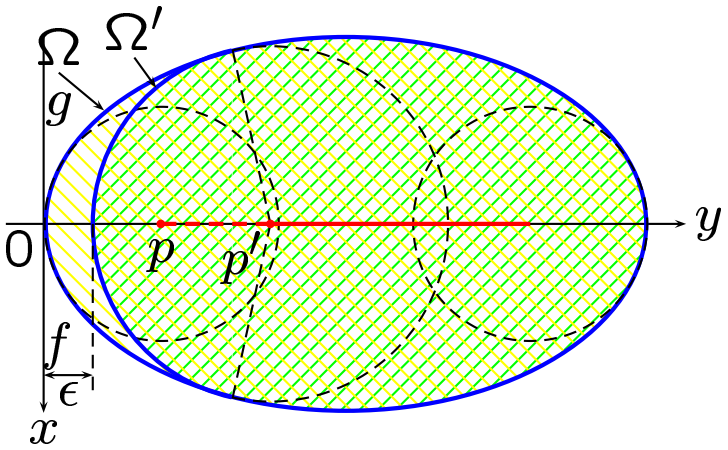
$A_3 = 0$, $A_4 = \frac{k^3}{8}$, and $N_P = 3$.

$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) = \frac{\sqrt{2}}{\sqrt{k}} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) = \frac{2}{\sqrt{k}} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

where $\epsilon = \max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}$.

Type (i): Ellipse



$$\Omega : \frac{x^2}{a^2} + \frac{(y-b)^2}{b^2} \leq 1, \ b > a > 0$$

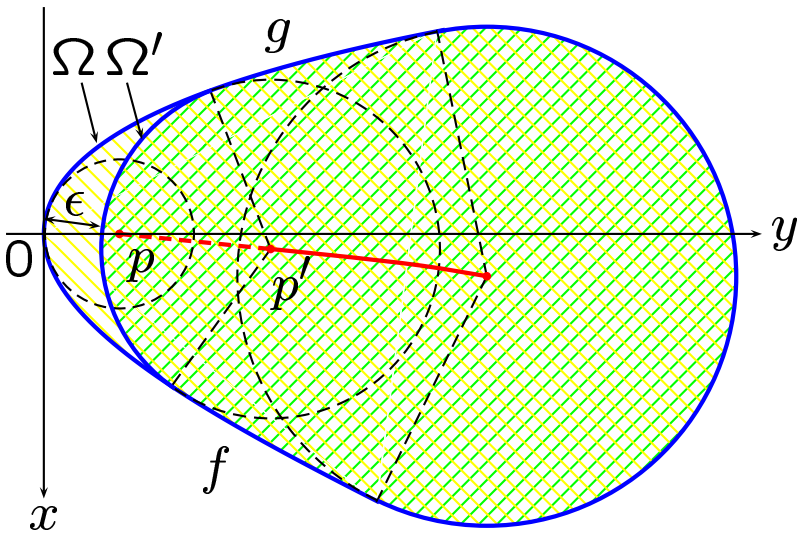
$$\begin{aligned} f(x) &= g(x) = b - b\sqrt{1 - \frac{x^2}{a^2}} \\ &= \frac{b}{2a^2}x^2 + \frac{b}{8a^4}x^4 + \frac{b}{16a^6}x^6 + \cdots, \\ \mathcal{A}_{\frac{b}{a^2}}(x) &= \frac{b}{2a^2}x^2 + \frac{b^3}{8a^6}x^4 + \frac{b^5}{16a^{10}}x^6 + \cdots. \end{aligned}$$

$A_3 = 0$, $A_4 = \frac{b}{8a^6}(b^2 - a^2)$, and $N_P = 3$.

$$\begin{aligned} \mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) &= \frac{\sqrt{2}a}{b\sqrt{b}}\sqrt{b^2 - a^2} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}), \\ \mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) &= \frac{2a}{b\sqrt{b}}\sqrt{b^2 - a^2} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}). \end{aligned}$$

where $\epsilon = \max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}$.

Type (i): Non-symmetric



$f(x) = \frac{1}{2}x^2 - \frac{1}{24}x^3,\ g(x) = \frac{1}{2}x^2 + \frac{1}{32}x^4.$

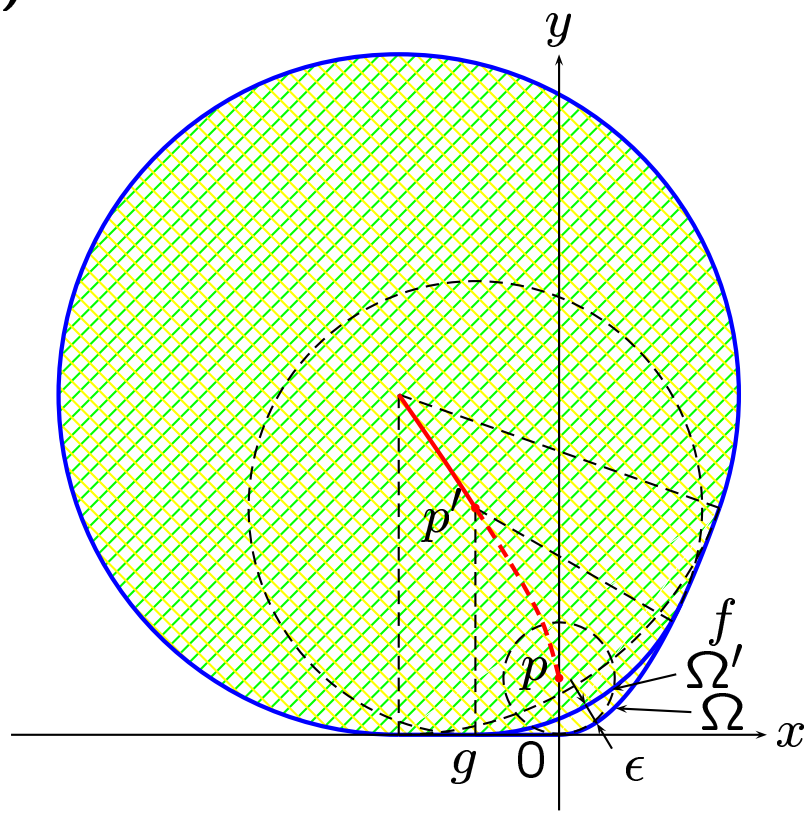
$k = 1,\ A_3 = \frac{1}{24},$ and $N_P = 2.$

$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) = \sqrt[3]{\frac{3}{32}} \cdot \sqrt[3]{\epsilon} + o(\sqrt[3]{\epsilon}),$$

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) = \sqrt{2}\sqrt[3]{\frac{3}{32}} \cdot \sqrt[3]{\epsilon} + o(\sqrt[3]{\epsilon}),$$

where $\epsilon = \max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}.$

Type (ii)



$f(x) = \frac{k}{2}x^2$, $g(x) = 0$.

$A_3 = 0$, $A_4 = \frac{k^3}{8}$, and $N_P = 3$.

$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) = \frac{3\sqrt{3}}{\sqrt{2}\sqrt{k}} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) = \frac{3\sqrt{3}}{\sqrt{k}} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),$$

where $\epsilon = \max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}$.

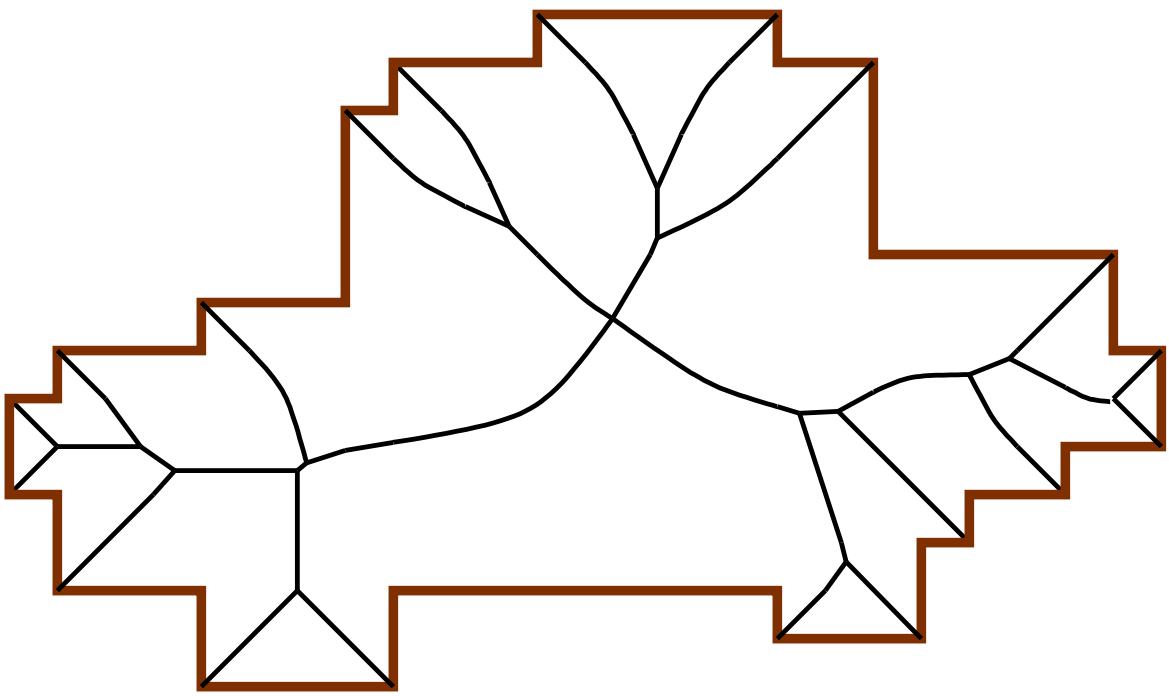
Directions to Applications: Pruning

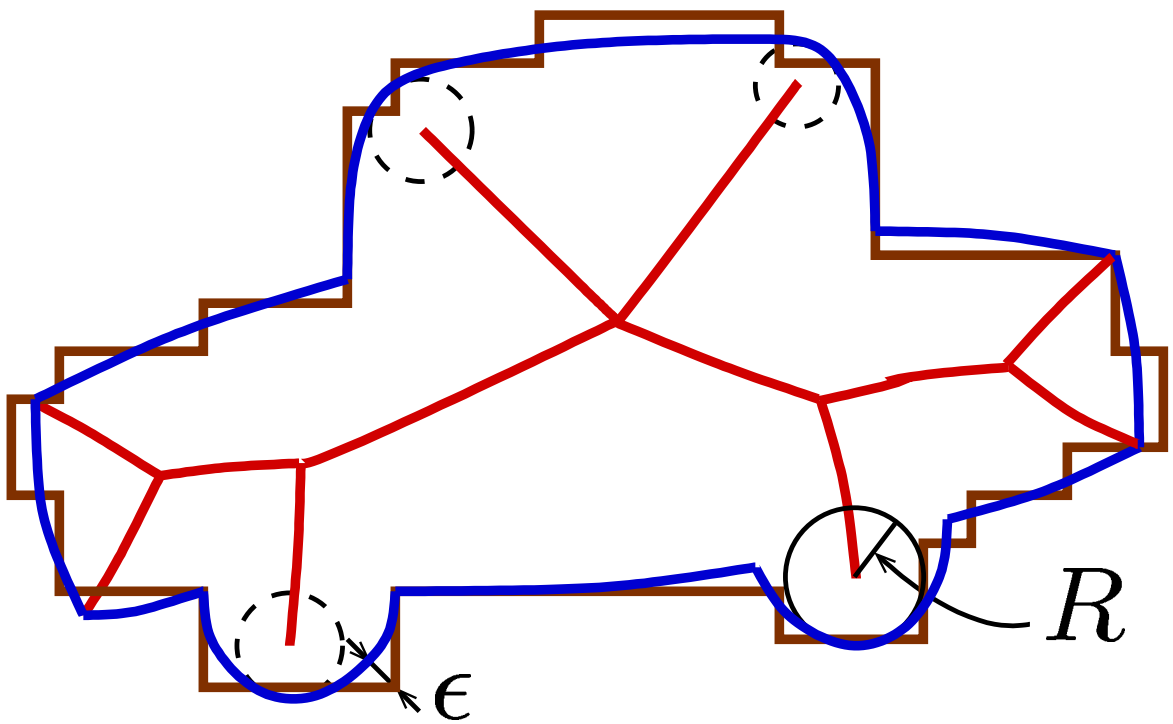
Let Ω be a normal domain which is not weakly injective, and whose boundary consists of *piecewise quadratic splines*. Then we have

$$\begin{aligned} \mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) &\leq 9\sqrt{2}\sqrt{R_\Omega}\cdot\sqrt{\epsilon}+o(\sqrt{\epsilon}), \\ \mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) &\leq 9\sqrt{2}\sqrt{R_\Omega}\cdot\sqrt{\epsilon}+o(\sqrt{\epsilon}), \end{aligned}$$

for any normal domain Ω' with $\min\{\mathcal{H}(\Omega,\Omega'),\mathcal{H}(\partial\Omega,\partial\Omega')\}\leq\epsilon$, where

$$R_\Omega = \max\{r : P=(p,r) \text{ is a non-degenerate 1-prong point of } \mathbf{MAT}(\Omega)\}.$$

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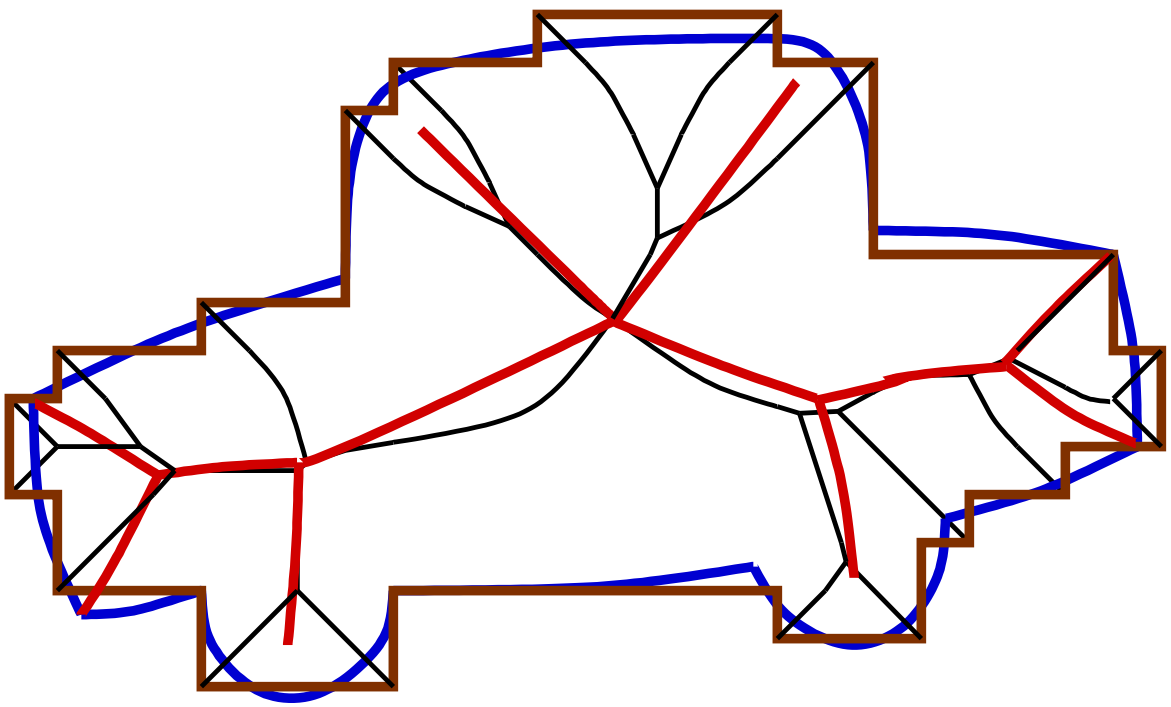

Independence of the Approximation

Let Ω', Ω_1, Ω_2 be normal domains and suppose Ω_1, Ω_2 are not weakly injective and their boundaries consist of quadratic splines. Suppose $\max\{\mathcal{H}(\Omega_i, \Omega'), \mathcal{H}(\partial\Omega_i, \partial\Omega')\} \leq \epsilon$ for $i = 1, 2$. Then we have

$$\begin{aligned}\mathcal{H}(\mathbf{MA}(\Omega_1), \mathbf{MA}(\Omega_2)) &\leq 18\sqrt{R} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),\\ \mathcal{H}(\mathbf{MA}(\Omega_1), \mathbf{MA}(\Omega_2)) &\leq 18\sqrt{R} \cdot \sqrt{\epsilon} + o(\sqrt{\epsilon}),\end{aligned}$$

where $R = \max\{R_{\Omega_1}, R_{\Omega_2}\}$.

Conclusions

- Analyzed quantitatively the effect of the boundary perturbation on medial axis transform.

- Proved the stability of medial axis transform under one-sided Hausdorff distance.

- Suggested a new pruning strategy with precise error analysis.