Two-dimensional Offsets via Medial Axis Transform I: Mathematical Theory*

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Abstract

This paper is the first half of the two-part installment of our result dealing with the two-dimensional offset curves by utilizing the medial axis transform. In this paper, a mathematical theory of two-dimensional offset curves of a planar domain is presented: their relation with the medial axis transform is revealed and all the possible geometric configuration is classified. Every level of the offset curves has similar relation with the medial axis transform as the original boundary curve does. Hence, once the medial axis transform is available—although it is computationally expensive to compute the medial axis

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transform of a free-form domain—the offset curves can be easily constructed using the envelope formula. We give a complete list of all the possible local shape of the offset curves. These results will serve as a theoretical foundation of our algorithm which to be presented in the accompanying paper[1]. Without this priori knowledge of qualitative nature of offset curves, however, mere numerical approximation scheme is prone to miss its subtle topological feature such as where and how the offset curve bifurcates.

1 Introduction

Mathematically, an offset curve is the curve of fixed distance from a given curve. This simple enough definition begets very complicated problems. The complication can be crudely classified into two types: one is the curve representation problem of the so-called untrimmed offsets, and the other is a more global one, namely, the trimming process of the untrimmed offset curves.

It is easy to see that even if the curve is a polynomial curve, the offset curve need not be a rational curve (of the original parameter). Although one can easily write down the formula for the offset curve with the original curve parameter, the resulting representation is not in general rational. This may cause some serious problems in handling the offset curves in the computer aided geometric design. The first kind, i.e., the representation problem, thus received a lot of attention, as far as we know, since Klass [19] had approximated offset curves of cubic splines by another cubic splines. Since then, the spline approximation problem of the offset curve has been extensively studied. For example, to find the control polygon of the offset spline of non-uniform rational B-spline, Tiller and Hanson [27] used the offset lines of the original control polygon and Coquillart [8] computed the offset of the control vertex of the original curve using its closest point on the curve. Hoschek [18] used various geometric continuity conditions to determine approximating spline curves, and Pham [23] provided an interactive algorithm using uniform B-splines. Recently, Lee et al. [20] presented another method using approximation of the unit circle with piecewise quadratic polynomial curve segments. For more information on all of the above approaches, one is referred to the survey by Pham [24] and the comparison result by Elber et al. [9]. Also notable study on conic offsets were developed by Farouki [10].

In addition to these efforts to approximate offset curves with existing spline curves, there have been attempts to invent new types of spline curves whose offsets are easy to handle. Meek and Walton [21] studied the offsets of

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curves consisting of clothoidal splines. In this regard, an important pioneering work has been done by Farouki and Sakkalis [14]. They introduced the Pythagorean hodographs which are special kinds of splines admitting rational offset curves. This fact makes it easier to manipulate the offset curves. Later, Pottmann [25, 26] introduced a rational generalization. Namely, he characterized the rational curves with rational offsets.

The second kind, the trimming problem, is of somewhat different nature. This kind of problem occurs due to the presence of cut or focal loci. An obvious approach is to compute the intersection of the untrimmed offset curves and decide which portion should be removed to generate the genuine offset curve. Hoschek [17] devised an intersection algorithm and criterion to delete the undesirable portion of the untrimmed offset. Tiller and Hanson [27] and Meek and Walton [21] also considered this issue. Lee et al. [20] used polygonal approximation of the offset curve segments to determine self-intersection points.

A well-known way of getting around this difficulty is to use the medial axis. For domains whose boundary consists of circular arcs and line segments, *Voronoi diagram* which is almost equivalent to the medial axis, has been widely used for a long time in the NC machining industry since the work of Persson [22]. Chou and Cohen [7] elaborated this idea and one can find thorough treatments in Held [15] and Held et al. [16]. The trouble, however, is that most algorithms of finding the medial axis are restricted to the domain whose boundary is made up of circular arcs and line segments. And when the boundary of the domain is free-form, finding the medial axis has been an even more difficult problem, thereby begging the question.

Some studies of the medial axis transform when the boundary curves are free-form have been done more carefully by some of the present authors: In [3] and [4], a new mathematical theory and algorithm for approximately finding the medial axis transform are presented. It is based on the so-called Domain Decomposition Lemma, which enables one to decompose a complicated domain into simpler, hence easier to handle, subdomains. It is to be noted that there also has been some work on this feature by Chiang *et al.* [1].

This paper is the first installment of our two-part series of papers that deal with the mathematical as well as algorithmic issues related to the offset curves. In this paper, we concentrate on the purely *intrinsic geometric* issues without paying attention to algorithmic or the curve representation issues. It should be noted that somewhat related theoretical works from different viewpoint are also done by Farouki et al. [11, 12, 13]. Our approach to this problem relies heavily on the domain decomposition lemma as was the case in [3] and [4]. In fact, we are using a modified version of the domain decomposition lemma. Our overriding viewpoint is the attention we pay to the close relationship between the offset curves and the medial axis transform regarded as an embedded geometric graph in $\mathbb{R}^3$. This way, a lot of geometric information can be easily deciphered from the three-dimensional data of the medial axis transform. As a result, we are able, among others, to classify all possible local geometric shapes of the inner offset curves of the planar domains. This result will be the mathematical foundation on which our algorithm for computing the offset curves is based. This new algorithm is sketched in [5], and the full details will be published in [2]. (In this regard, another algorithm combining the medial axis transform and the *Minkowski Pythagorean hodograph* recently developed by the authors [6].)

We develop our theory as follows. In Section 2 we briefly present the basic definition of the medial axis transform and the assumptions on the domains. The mathematical relation between the offset curve and the medial axis transform of a domain is discussed in Section 3. In Section 4 we enrich our knowledge on the medial axis transform by studying the geometric property of its intersection with horizontal planes. With these abundant information on the medial axis transform and a slightly modified version of the domain decomposition lemma, tailored for offset curves in Section 5, we categorize all the possible geometric configuration of the offset curves in Section 6. Finally, some concluding remarks follow in Section 7.

2 Preliminaries

The offset curve considered throughout our discussion is an *inner* offset curve of a planar region. We investigate the offset curve by analyzing the medial axis transform of the region. Since the concept of the medial axis transform is so widely used in many research area in as many different forms, we first fix the relating terminologies of the medial axis transform to avoid the reader's confusion. We also introduce the specification of the region we will deal with. This section is concluded by showing our special treatment for the corner points of the region's boundary—the extended boundary.

Definition 1. Let S be a general subset of $\mathbb{R}^2$ and $B_r(\mathbf{p})$ denote the *closed* disk of radius r centered at $\mathbf{p}$. We define a set $\mathcal{D}(S)$ by

$$\mathcal{D}(S) = \{B_r(\mathbf{p}) \,|\, B_r(\mathbf{p}) \subset \overline{S} \text{ and } \mathrm{int}(B_r(\mathbf{p})) \subset \mathrm{int}(S)\},$$

where $\overline{X}$ and $\mathrm{int}(X)$ denote the set-theoretic closure and interior of the set X, respectively.

Remark 1. Throughout this paper, all disks are assumed to be closed disks unless stated otherwise. The radius of a disk is allowed to be zero, in which case the disk is just one point.

Definition 2. The *core* $\mathbf{CORE}(S)$ of a set S is the set of all maximal elements in $\mathcal{D}(S)$, that is,

$$\mathbf{CORE}(S) = \{B_r(\mathbf{p}) \in \mathcal{D}(S) \mid \\ B_r(\mathbf{p}) = B_s(\mathbf{q}) \text{ if } B_s(\mathbf{q}) \in \mathcal{D}(S) \text{ and } B_s(\mathbf{q}) \supset B_r(\mathbf{p})\}.$$

If $B_r(\mathbf{p}) \in \mathbf{CORE}(S)$, we call $B_r(\mathbf{p})$ a *maximal*, or *contact disk* and $\partial B_r(\mathbf{p})$ a *maximal*, or *contact circle*.

Definition 3 (Medial axis transform). The *medial axis* of a set $S \subset \mathbb{R}^2$ is the set of the centers of disks in $\mathbf{CORE}(S)$. That is,

$$\mathbf{MA}(S) = \{\mathbf{p} \in \mathbb{R}^2 \mid B_r(\mathbf{p}) \in \mathbf{CORE}(S)\}.$$

The *medial axis transform* of a domain S is the set of the ordered pairs of centers and radii of disks in $\mathbf{CORE}(S)$. That is,

$$\mathbf{MAT}(S) = \{(\mathbf{p}, r) \in \mathbb{R}^2 \times \overline{\mathbb{R}_+} \mid B_r(\mathbf{p}) \in \mathbf{CORE}(S)\},$$

where $\overline{\mathbb{R}_+}$ is the set of all non-negative real numbers.

Remark 2. Note that the radius value r is allowed to be zero. Such a case occurs at a sharp corner point of ∂S.

The apparently redundant condition in Definition 1 is in fact intentional. Consider for example the set

$$S = \{(x, y) \mid x^2 + y^2 < 1 \text{ and } (x, y) \neq (c, 0) \text{ for any } c \text{ with } 0 < c < 1\}.$$

Without the closure inclusion, every point outside $\overline{S}$ is a medial axis point with a zero radius, which is far from the expected property of the medial axis transform. But just the closure inclusion without the interior inclusion drastically distort the nature of $\mathcal{D}(S)$ as $\overline{S}$ is just the unit disk. So the above double-checking enables the medial axis transform to faithfully describe the geometry of S.

Definition 4. A curve $\mathbf{r} : (a, b) \to \mathbb{R}^n$ ($n = 1, 2, \ldots$), is a C^k *curve*, if there is a reparameterization $\overline{\mathbf{r}(t)} = (x_1(t), \ldots, x_n(t))$ of $\mathbf{r}$ by the arc length, such that $x_1(t), \ldots, x_n(t)$ are C^k functions of t, where $k = 1, 2, \ldots, \infty$, or $k = \omega$ when it is real analytic.

Definition 5. A curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ ($n = 1, 2, \ldots$) is a C^k curve, if there exists a C^k curve $\tilde{\mathbf{r}} : (a - \epsilon, b + \epsilon) \to \mathbb{R}^n$ for some $\epsilon > 0$ such that $\tilde{\mathbf{r}}(t) = \mathbf{r}(t)$ for all $t \in [a, b]$, where $k = 1, 2, \ldots, \infty, \omega$. A C^ω curve will also be called a *real analytic curve*.

Remark 3. A curve $\mathbf{r} : (a, b) \to \mathbb{R}^n$ $(n = 1, 2, \ldots)$, is a C^k curve, if and only if, for any point $\mathbf{p}$ on $\mathbf{r}$, there exists a small number $\epsilon > 0$ such that the curve $\mathbf{r}$ can be represented as the graph of a C^k function with respect to the tangent line at $\mathbf{p}$ inside the ϵ-disk $B_\epsilon(\mathbf{p})$ in $\mathbb{R}^n$, where $k = 1, 2, \ldots, \infty, \omega$.

We now list the assumptions which a planar region Ω in our study should satisfy.

Assumption 1. Ω is the closure of a connected and bounded open subset in $\mathbb{R}^2$ bounded by finite number of mutually disjoint simple closed curves. (Here a simple closed curve means an embedding of the unit circle in $\mathbb{R}^2$.)

The closedness and the connectedness of the domain simplifies much of the technical arguments. And the boundedness is not only appropriate for the practical settings, but also essential sometimes in our analysis of the medial axis transform, since it, together with the closedness, implies the compactness. The simple closed curve of $\partial\Omega$ which bounds the unbounded connected component of $\mathbb{R}^2 \setminus \Omega$, is called the *outer* boundary (curve), and the rest of the simple closed curves of $\partial\Omega$ are called the *inner* boundary (curves). The number of the inner boundary curves of $\partial\Omega$ is called the *genus* of Ω.

Assumption 2. Each simple closed curve in $\partial\Omega$ consists of finite number of pieces of real analytic curves.

Let us be clearer about this assumption: Assumption 2 means that each simple closed curve of $\partial\Omega$ is represented as a closed curve $\mathbf{r} : [a, b] \to \mathbb{R}^2$ such that there exist finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ such that $\mathbf{r}_{[t_{i-1}, t_i]}$ is a real analytic curve for $i = 1, \ldots, n$. We call $\mathbf{r}_{[t_{i-1}, t_i]}$ a *real analytic piece* (of $\mathbf{r}$). It should be noted that $\mathbf{r}$ is C^1 at a point where two real analytic pieces join as long as the unit tangent vector fields along each piece coincide at that point. However, it is not realistic to expect it to be C^2 in general because the curvature of the two pieces may not coincide at the joint point. But, assuming each piece should be a polynomial curve is too restrictive, because, for example, conic sections, such as ellipses, parabolas, or hyperbolas, which are frequently used as NURBS curves are in general not polynomial curves. (Note that their x and y coordinates may have a polynomial relation, but the x and y coordinates of these conic sections cannot be polynomial functions of some parameter simultaneously.) Thus the right kinds of general class of curves are real analytic pieces joined in the C^1 manner except at corner points.

Standing assumption. From now on, by the term *domain*, we mean a non-circular domain satisfying the above two assumptions unless otherwise stated.

The reason we usually exclude the circular domain, i.e., the disk, is that the disk poses an exception to many of our technical results. But since it is a trivial domain with the medial axis consisting of one point, everything is known. Thus there is no real loss of generality in excluding the disk. It is also worthwhile to remember that a domain is always a *closed* set.

Definition 6 (Corner). A boundary point is a *corner* (point) if the unit tangent vector field is discontinuous at that point. It is called a *sharp* (resp., *dull*) corner if the interior angle is strictly less (resp., greater) than π.

In our analysis, we need to analyze the correspondence between the points of $\partial\Omega$ and the points of $\mathbf{MAT}(\Omega)$. For each point $\mathbf{q}$ of $\partial\Omega$, which is not a corner point, we can define the corresponding medial axis transform point as the one which is contacting at $\mathbf{q}$. For a sharp corner point, the correspondent will be the sharp corner point itself. (Remember we allowed zero-radius disk.) For a dull corner point $\mathbf{p}$, however, there are infinitely many disks contacting at $\mathbf{p}$. The following is a precise description of this case.

Definition 7 (Inward unit cone). Let $\mathbf{r}(t)$ be a piecewise real analytic curve which is a part of the boundary of Ω. We assume that $\mathbf{r}(t)$ is oriented in such a way that the interior of Ω is always on the left of $\mathbf{r}(t)$. Let $\mathbf{p} = \mathbf{r}(0)$ and suppose $\mathbf{p}$ is not a sharp corner. We define the *inward unit cone* $\mathrm{IC}(\mathbf{p})$ at $\mathbf{p}$ by

$$\mathrm{IC}(\mathbf{p}) = \{\mathbf{v} : |\mathbf{v}| = 1 \text{ and } \mathbf{v} \text{ is an inward pointing vector at } \mathbf{p} \\ \text{such that } \mathbf{v} \cdot \mathbf{r}'(0+) \leq 0 \text{ and } \mathbf{v} \cdot \mathbf{r}'(0-) \geq 0\}.$$

See Figure 1. Thus if $\mathbf{r}$ is differentiable at $\mathbf{p}$, $\mathrm{IC}(\mathbf{p})$ consists of the single inward unit normal vector.

Now there corresponds a contact disk for each inward pointing vector of a dull corner, and this prevents us to define a map from $\partial\Omega$ to $\mathbf{MAT}(\Omega)$. Thus we need some concept generalizing the real boundary $\partial\Omega$, which we call the extended boundary.

Definition 8 (Extended boundary). Let $\mathbf{p}$ be a dull corner of $\partial\Omega$ and ϕ be the angle of $\mathrm{IC}(\mathbf{p})$. Let us "parameterize" the dull corner $\mathbf{p}$ by agreeing to define $\mathbf{p}(\theta) = \mathbf{p}$ for $\theta \in [0, \phi]$ and the "tangent" vector $\mathbf{p}'(\theta)$ to be the tangent vector of $\mathrm{IC}(\mathbf{p})$ at the angle θ. (See Figure 1.) When each dull corner of $\partial\Omega$ is "parameterized" in this manner, we call it the *extended boundary* of Ω.

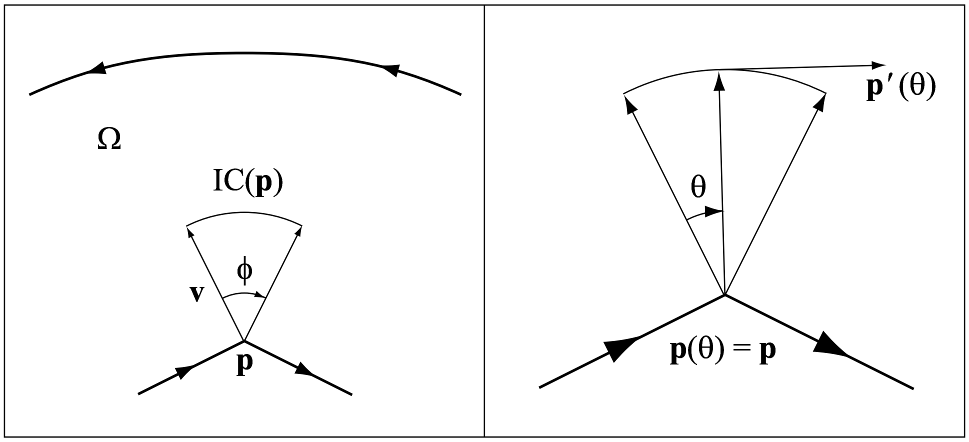

Figure 1: An inward unit cone and the extended boundary

The concept of extended boundary plays a crucial role in [3] when developing various regularity properties of the medial axis transform by using the medial axis transform map along the extended boundary. The merit of having the extend boundary is that the vectors of the inward unit cone can be treated in the same manner as the smooth points in the subsequent discussions. One is referred to [3] for more detailed discussion.

Remark 4. We will just say the "boundary" for both the *real* boundary and the *extended* boundary since it will not cause any misunderstanding.

3 Medial axis transform and offset

We present here basic definitions of the offset of a domain and its fundamental relationship with the medial axis transform.

Definition 9. Let Ω be a domain and $d > 0$. Then we define the *d-offset curve* $\mathcal{O}_d(\Omega)$ as the inner offset curve to the boundary $\partial\Omega$ with a distance d, i.e., the set

$$\mathcal{O}_d(\Omega) = \{\mathbf{p} \in \Omega \,|\, \mathrm{dist}(\mathbf{p}, \partial\Omega) = d\},$$

where $\mathrm{dist}(A, B)$ is the Euclidean distance between two sets A and B.

Definition 10. Let Ω be a domain and $d > 0$. Then we define the *d-offset region* Ω_d as the set inside the d-offset curve, i.e.,

$$\Omega_d = \{\mathbf{p} \in \Omega \,|\, \mathrm{dist}(\mathbf{p}, \partial\Omega) \geq d\}.$$

A set in $\mathbb{R}^2$ is called an offset region, if it is a d-offset region of a domain for some $d > 0$.

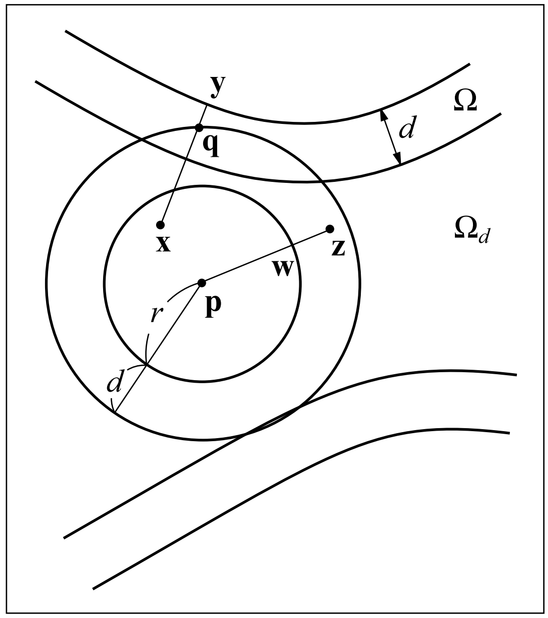

Figure 2: $B_{r+d}(\mathbf{p}) \in \mathcal{D}(\Omega) \Leftrightarrow B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$

Definition 11. Let A be a subset of $\mathbb{R}^2 \times \overline{\mathbb{R}_+}$. If we consider that each point $(\mathbf{p}, r)$ of A represents a disk $B_r(\mathbf{p})$ in $\mathbb{R}^2$, we denote by $\mathcal{R}(A)$ the region recovered by the disks of A, i.e., the set

$$\mathcal{R}(A) = \bigcup_{(\mathbf{p},r)\in A} B_r(\mathbf{p}) = \{\mathbf{q} \in \mathbb{R}^2 \mid \exists (\mathbf{p}, r) \in A, \text{ s.t. } |\overline{\mathbf{pq}}| \leq r\}.$$

Although $\mathcal{R}(A)$ may not be closed in general, $\mathcal{R}(\mathbf{MAT}(\Omega))$ is closed if Ω is a domain. It is obvious by the definition of the medial axis transform that

$$\Omega = \mathcal{R}(\mathbf{MAT}(\Omega)),$$

i.e., the domain is the union of all of its maximally inscribed disks.

Lemma 1. *Let Ω be a domain in $\mathbb{R}^2$, $r \geq 0$, and $d > 0$. Then $B_{r+d}(\mathbf{p}) \in \mathcal{D}(\Omega)$, if and only if $B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$.*

Proof. ($\Rightarrow$) Suppose $B_{r+d}(\mathbf{p}) \in \mathcal{D}(\Omega)$ and choose any $\mathbf{x} \in B_r(\mathbf{p})$. Then clearly $\text{dist}(\mathbf{x}, \partial B_{r+d}(\mathbf{p})) \geq d$. Let $\mathbf{y} \in \partial\Omega$ be a point such that $|\overline{\mathbf{xy}}| = \text{dist}(\mathbf{x}, \partial\Omega)$. (See Figure 2.) Thus there exists a (unique) point $\mathbf{q} \in \partial B_{r+d}(\mathbf{p})$ which lies on the line segment $\overline{\mathbf{xy}}$. Now

$$\text{dist}(\mathbf{x}, \partial\Omega) = |\overline{\mathbf{xy}}| \geq |\overline{\mathbf{xq}}| \geq \text{dist}(\mathbf{x}, \partial B_{r+d}(\mathbf{p})) \geq d.$$

Thus $\mathbf{x} \in \Omega_d$ and hence $B_r(\mathbf{p}) \subset \Omega_d$, which means that $B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$ since Ω_d is closed.
($\Leftarrow$) Suppose $B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$, i.e., $B_r(\mathbf{p}) \subset \Omega_d$, and $\mathbf{z} \in B_{r+d}(\mathbf{p})$. Then $|\overline{\mathbf{zp}}| \leq r + d$. Suppose $\mathbf{z} \notin \Omega$. Then $|\overline{\mathbf{zp}}| > r$, since $B_r(\mathbf{p}) \subset \Omega_d$. So there exists a point $\mathbf{w} \in \partial B_r(\mathbf{p})$ which lies on the line segment $\overline{\mathbf{zp}}$, and $|\overline{\mathbf{zw}}| \leq d$. But since $\mathbf{w}$ is in Ω_d, the ball $B_d(\mathbf{w})$ is contained in Ω, which means that $\mathbf{z}$ is in Ω. Thus $B_{r+d}(\mathbf{p}) \subset \Omega$. $\square$

Definition 12. The d-cutoff of $A \subset \mathbb{R}^2 \times \overline{\mathbb{R}_+}$ for $d \geq 0$, denoted by A_d, is defined by

$$A_d = \{(\mathbf{p}, r) \in \mathbb{R}^2 \times \overline{\mathbb{R}_+} \mid (\mathbf{p}, r + d) \in A\}.$$

That is, to get A_d, we first pull A down in the negative z-axis by d, then remove what is below the xy-plane. Then, the remainder is A_d.

Even if Ω is a domain, Ω_d may not be a domain satisfying Standing assumption of Section 2. However, since the medial axis transform is defined for general subsets of $\mathbb{R}^2$, it still makes sense to talk about $\mathbf{MAT}(\Omega_d)$. Now we state the relation between the medial axis transform and the offset region.

Lemma 2. *Let Ω be a domain in $\mathbb{R}^2$ and $d > 0$. Then*

$$\mathbf{MAT}(\Omega)_d = \mathbf{MAT}(\Omega_d).$$

That is, the medial axis transform of d-offset region is the d-cutoff of the medial axis transform of the original domain.

Proof. ($\supseteq$) Suppose $(\mathbf{p}, r)$ is in $\mathbf{MAT}(\Omega_d)$. That is, $B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$. By Lemma 1, we have $B_{r+d}(\mathbf{p}) \in \mathcal{D}(\Omega)$. It suffices to show that $B_{r+d}(\mathbf{p}) \in \mathbf{CORE}(\Omega)$, i.e., $(\mathbf{p}, r + d) \in \mathbf{MAT}(\Omega)$.

Suppose $B_{r+d}(\mathbf{p})$ is not a maximal element of $\mathcal{D}(\Omega)$. Then there exists a ball $B_{r'+d}(\mathbf{p}')$ of $\mathcal{D}(\Omega)$ such that $B_{r+d}(\mathbf{p}) \subsetneq B_{r'+d}(\mathbf{p}')$ with $r < r'$. Note that we then have $B_r(\mathbf{p}) \subsetneq B_{r'}(\mathbf{p}')$. On the other hand, By Lemma 1, we also have $B_{r'}(\mathbf{p}') \in \mathcal{D}(\Omega_d)$. But this contradicts the fact that $B_r(\mathbf{p})$ is a maximal element of $\mathcal{D}(\Omega_d)$.
($\subseteq$) The other half is just the same. Suppose $(\mathbf{p}, r)$ is in $\mathbf{MAT}(\Omega)_d$. That is, $(\mathbf{p}, r + d)$ is in $\mathbf{MAT}(\Omega)$. Since $B_{r+d}(\mathbf{p}) \in \mathcal{D}(\Omega)$, we have $B_r(\mathbf{p}) \in \mathcal{D}(\Omega_d)$ by Lemma 1. Now it suffices to show that $B_r(\mathbf{p}) \in \mathbf{CORE}(\Omega_d)$.

Suppose there is a ball $B_{r'}(\mathbf{p}')$ of $\mathcal{D}(\Omega_d)$ such that $B_r(\mathbf{p}) \subsetneq B_{r'}(\mathbf{p}')$. Then we have $B_{r+d}(\mathbf{p}) \subsetneq B_{r'+d}(\mathbf{p}')$ and by Lemma 1 we have $B_{r'+d}(\mathbf{p}') \in \mathcal{D}(\Omega)$. But this contradicts the fact that $B_{r+d}(\mathbf{p})$ is a maximal element of $\mathcal{D}(\Omega)$. $\square$

The following is a characterization of the offset curve via the medial axis transform.

Theorem 1. *Let Ω be a domain and $d > 0$. Then we have*

$$\mathcal{O}_d(\Omega) = \partial\mathcal{R}(\mathbf{MAT}(\Omega)_d).$$

That is, the d-offset curve is the boundary (or envelope) of the region saturated by the disks corresponding to the d-cutoff of the medial axis transform of the original domain.

Proof. Note that $\mathcal{O}_d(\Omega) = \partial\Omega_d$. Also $\Omega_d = \mathcal{R}(\mathbf{MAT}(\Omega_d))$ as noted above, and thus $\mathcal{O}_d(\Omega) = \partial\mathcal{R}(\mathbf{MAT}(\Omega_d))$. Thus the result follows from Lemma 2. $\square$

4 Local geometry of medial axis transform

The various geometric aspects of the medial axis transform is revealed in [3]. Especially, we could characterize the medial axis transform as a *geometric graph*: A geometric graph is a usual topological graph with a finite number of vertices and edges, where a vertex is a point in $\mathbb{R}^3$ and an edge is a real analytic curve with finite length and whose limits of tangents at the end points exist.

As a continuing investigation of the geometry of the medial axis transform, here we explore some local geometric property of the cross sections of the medial axis transform cut by horizontal planes. To classify all the possible configurations of the offset curve around its self-intersection points, it is crucial to understand the geometric nature of the cross section, since the self-intersection points of the d-offset curve fall on the cross section by the horizontal plane with height d. Before we start, let us fix some terminologies that will be used in the subsequent discussion.

Let Ω be a domain and let $(\mathbf{p}, r) \in \mathbf{MAT}(\Omega)$. Let $T(\mathbf{p})$ be the union of the line segments joining $\mathbf{p}$ and the contact points of $B_r(\mathbf{p})$. Suppose $B_r(\mathbf{p}) \setminus T(\mathbf{p})$ has n connected components. We denote them by $U_1, \ldots, U_n$. See Figure 3. In [3], it is proved that the medial axis emanating from $\mathbf{p}$ consists, near $\mathbf{p}$, of exactly n curves $\mathbf{s}_1, \ldots, \mathbf{s}_n$. Thus we called $(\mathbf{p}, r)$ an *n-prong* point. We may assume each $\mathbf{s}_i$ is contained in U_i for $i = 1, \ldots, n$. We define $\theta_i > 0$ to be the angle of U_i at $\mathbf{p}$ for $i = 1, \ldots, n$. Let $\mathbf{r}_i$ be the segment of the medial axis transform emanating from $(\mathbf{p}, r)$ corresponding to $\mathbf{s}_i$. (See Figure 4.) Let α_i be the angle at $(\mathbf{p}, r)$ between $\mathbf{r}_i$ and the xy-plane. The sign of α_i is chosen to be positive if $\mathbf{r}_i$ lies above the plane $\mathbb{R}^2 \times \{r\}$ near $(\mathbf{p}, r)$. Then, as shown in [3], we have

$$\tan\alpha_i = -\cos\frac{\theta_i}{2}, \tag{1}$$

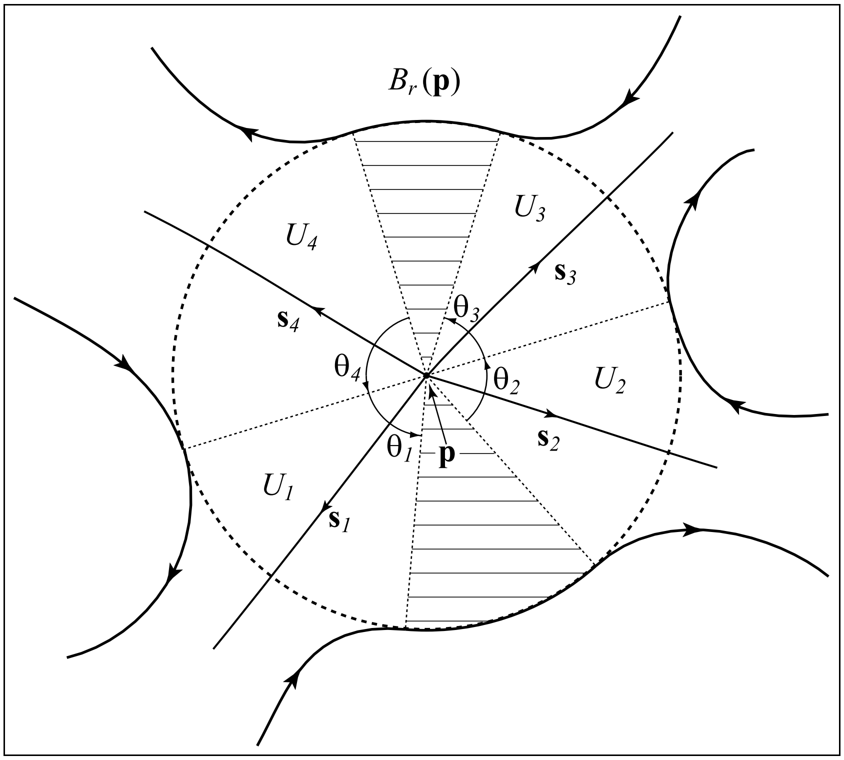

Figure 3: A contact circle

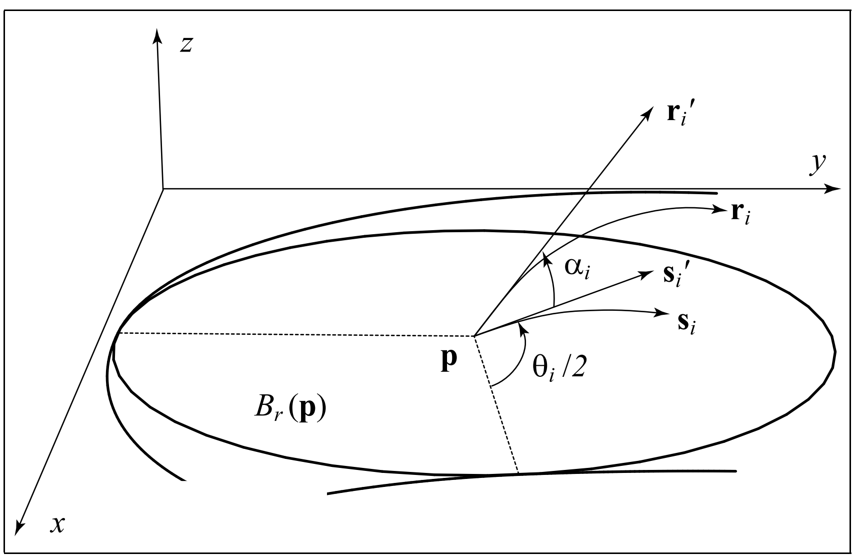

Figure 4: A 3D view of the contact circle

for $i = 1, \dots, n$.

Lemma 3. *Let Ω be a domain and $(\mathbf{p}, r) \in \mathbf{MAT}(\Omega)$ be an n-prong point such that $n \geq 2$. For $1 \leq i \leq n$, let $\mathbf{r}_i : [0, \epsilon] \to \mathbf{MAT}(\Omega)$ and θ_i, α_i be given as in Figure 3 and 4. Suppose one of the α_i's, say α_1, is greater than 0, then $\alpha_i < 0$ for $2 \leq i \leq n$.*

Proof. By Equation (1) and the fact $\sum_{i=1}^{n} \theta_i \leq 2\pi$, the proof follows immediately. □

Theorem 2. *Let Ω be a domain and $(\mathbf{p}, r) \in \mathbf{MAT}(\Omega)$ be an n-prong point with $n \geq 2$. For $1 \leq i \leq n$, let $\mathbf{r}_i : [0, \epsilon] \to \mathbf{MAT}(\Omega)$ and θ_i, α_i be given as in Figure 3 and 4 such that $\mathbf{r}_i(0) = (\mathbf{p}, r)$. Suppose that at least two of the α_i's, say α_1 and α_2, are non-negative. Then $(\mathbf{p}, r)$ is a 2-prong point i.e., $n = 2$ and we have $\alpha_1 = \alpha_2 = 0$. Furthermore, $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$ point in the opposite directions and each of the two contact components of $(\mathbf{p}, r)$ is an isolated contact point. Thus $\mathbf{MAT}(\Omega)$ is a C^1 curve in $\mathbb{R}^2 \times \overline{\mathbb{R}_+}$ near $(\mathbf{p}, r)$.*

Proof. Observe that if one of the α_i's is greater than 0, say $\alpha_1 > 0$, then $\theta_1 > \pi$, which implies the rest of the θ_i's are all less than π. This in turn implies that $\alpha_i < 0$ for $i = 2, \dots, n$. Thus we should have $\alpha_1 = \alpha_2 = 0$. But then $\theta_1 = \theta_2 = \pi$ by Equation (1), so n must be 2 from the fact that $\sum_{i=1}^{n} \theta_i \leq 2\pi$, and $\theta_i > 0$ for $1 \leq i \leq n$. So the two contact components of $(\mathbf{p}, r)$ must be isolated contact points. For the C^1 connectivity near $(\mathbf{p}, r)$, one may refer to Section 8.2 in [3]). □

Definition 13 (Horizontal section). Let Ω be a domain and $c \geq 0$. A c-horizontal section of $\mathbf{MAT}(\Omega)$ is a connected component of the set

$$\{(\mathbf{p}, r) \in \mathbf{MAT}(\Omega) \,|\, r = c\}.$$

Since $\mathbf{MAT}(\Omega)$ is a finite geometric graph embedded in $\mathbb{R}^3$ as mentioned earlier, it is easy to see that there are finite number of c-horizontal sections of $\mathbf{MAT}(\Omega)$ for given $c \geq 0$, and each horizontal section is either a point or a graph (i.e., a finite geometric graph which is a subset of $\mathbf{MAT}(\Omega)$). In fact, we can tell more from the above results.

Lemma 4. *Let Ω be a domain and let $c > 0$. Let H be a c-horizontal section of $\mathbf{MAT}(\Omega)$. Then H is either a point or a C^1 curve segment (possibly a closed curve). Moreover, if H is a curve segment (or a closed curve), every point in H, with possible exception at the end points, is a 2-prong point of $\mathbf{MAT}(\Omega)$.*

Proof. Suppose H is not a point. Suppose $(\mathbf{p}, r) \in H$ is an n-prong point of $\mathbf{MAT}(\Omega)$, but not an end point of H. For $1 \le i \le n$, let $\mathbf{r}_i$ and θ_i, α_i be given as in Figure 3 and 4. Note that $n \ge 2$ and at least two of α_i's, say α_1 and α_2, are non-negative, since $(\mathbf{p}, r)$ is not an end point of H. Thus by Theorem 2, $(\mathbf{p}, r)$ is a 2-prong point of $\mathbf{MAT}(\Omega)$, and $\mathbf{MAT}(\Omega)$ is locally a C^1 curve near $(\mathbf{p}, r)$. $\square$

Remark 5. An end point of a horizontal section may not be a 2-prong point of $\mathbf{MAT}(\Omega)$. For example, let Ω be the rectangular domain defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid -2c \le x \le 2c, -c \le y \le c\},$$

for some $c > 0$. Then the set $H = \{(x, y, r) \in \mathbb{R}^2 \times \overline{\mathbb{R}_+} \mid -c \le x \le c, y = 0, r = c\}$ is a c-horizontal section of $\mathbf{MAT}(\Omega)$, but the end points $(-c, 0, c)$ and $(c, 0, c)$ of H are 3-prongs of $\mathbf{MAT}(\Omega)$.

Remark 6. A c-horizontal section may be a closed curve, when, for example, Ω is an annulus and c is the half of the annulus' width.

Definition 14 (Peak, valley, and slope). Let Ω be a domain and H is a c-horizontal section of $\mathbf{MAT}(\Omega)$ for $c \ge 0$. H is called a *peak*, if it is locally maximal in r, i.e., there exists a neighborhood N of H in $\mathbb{R}^2 \times \overline{\mathbb{R}_+}$ such that $r \le c$ for any $(\mathbf{p}, r)$ in $N \cap \mathbf{MAT}(\Omega)$. Similarly, H is called a *valley*, if it is locally minimal in r. Finally if H is neither a peak nor a valley, it is called a *slope*.

Remark 7. Note that the results in [3] easily implies that the number of peaks and valleys of $\mathbf{MAT}(\Omega)$ is finite.

Remark 8. Note that a c-horizontal section can be a peak and a valley at the same time. This is the case when, for example, Ω is a stadium defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid -a \le y \le a, -\sqrt{a^2 - y^2} - a \le x \le \sqrt{a^2 - y^2} + a\},$$

and $c = a$, the half of the stadium's width.

Theorem 3 (Shape of valley). *Let Ω be a domain. Suppose V is a valley of $\mathbf{MAT}(\Omega)$ and not a 0-horizontal section of $\mathbf{MAT}(\Omega)$. Then V has the following properties:*

(1) *V is either a point or a C^1 (possibly closed) curve segment. if V is a (closed) curve segment, each point on V, which is not an end point of V, is a 2-prong point of $\mathbf{MAT}(\Omega)$.*

(2) *None of the points in V (including the possible end points) is a bifurcation point of $\mathbf{MAT}(\Omega)$.*

(3) *Suppose V is a curve segment. If an end point $(\mathbf{p}, r)$ of V is a 2-prong point of $\mathbf{MAT}(\Omega)$, then $\mathbf{MAT}(\Omega)$ is a C^1 curve near $(\mathbf{p}, r)$.*

(4) *If V is a connected component of $\mathbf{MAT}(\Omega)$, then it is either a closed curve or a curve segment both of whose end points are 1-prong points of $\mathbf{MAT}(\Omega)$.*

Proof. The first property is just Lemma 4. Let $(\mathbf{p}, r)$ be a point in V. We will show that $(\mathbf{p}, r)$ cannot be a bifurcation point of $\mathbf{MAT}(\Omega)$. We can assume that V is a curve segment and $(\mathbf{p}, r)$ is an end point of V. Suppose $(\mathbf{p}, r)$ is an n-prong point of $\mathbf{MAT}(\Omega)$ with $n \geq 1$. For $1 \leq i \leq n$, let $\mathbf{r}_i$, θ_i, and α_i be given as in Figure 3 and 4. Note that $\alpha_i \geq 0$ for each i, since V is a valley. Thus by Equation (1) and the fact that $\sum_{i=1}^{n} \theta_i \leq 2\pi$, we have $n \leq 2$. The third property immediately follows from Theorem 2.

Now assume V is a connected component of $\mathbf{MAT}(\Omega)$. Note that if V is a closed curve then it is a connected component of $\mathbf{MAT}(\Omega)$. Suppose V is not a closed curve. Then by the second property, each of the two end points of V is either a 1-prong point or a 2-prong point of $\mathbf{MAT}(\Omega)$. But if one of the end points of V is a 2-prong point of $\mathbf{MAT}(\Omega)$, then V is not a component of $\mathbf{MAT}(\Omega)$. So both of the two end points of V are 1-prong points of $\mathbf{MAT}(\Omega)$. $\square$

Lemma 5. *Let Ω be a domain and let $\mathbf{r} : [0, 1] \to \mathbf{MAT}(\Omega)$, $\mathbf{r}(t) = (\mathbf{p}(t), r(t))$ be a continuous path with no self-intersections. (Here we allow the possibility that $\mathbf{r}(0) = \mathbf{r}(1)$.) Suppose there exist a and b $(0 < a \leq b < 1)$ such that the function $r(t)$ takes a local minimum on $[a, b]$. That is, r is constant on $[a, b]$ and there exists a neighborhood N of $[a, b]$ in $[0, 1]$ such that $r(t) < r(\tau)$ for every $t \in [a, b]$ and $\tau \in N \setminus [a, b]$. Then $\mathbf{r}([a, b])$ is a valley of $\mathbf{MAT}(\Omega)$.*

Proof. We may assume $r([a, b]) > 0$.The assumption says that $\mathbf{r}([a, b])$ is a local minimum in the "path" $\mathbf{r}([0, 1])$. It remains to show that $\mathbf{r}([a, b])$ is also a local minimum in the "whole" $\mathbf{MAT}(\Omega)$. To show that, it remains to prove that there is no bifurcation point on $\mathbf{r}([a, b])$. Let $(\mathbf{p}, r)$ be any n-prong point of $\mathbf{MAT}(\Omega)$ on $\mathbf{r}([a, b])$. For $1 \leq i \leq n$, let $\mathbf{r}_i : [0, \epsilon] \to \mathbf{MAT}(\Omega)$ and θ_i, α_i be given as in Figure 3 and 4. By the assumption, there are at least two α_i's, say α_1 and α_2, greater than 0. By Theorem 2, $(\mathbf{p}, r)$ is a 2-prong point. $\square$

Theorem 4 (Existence of valley between two peaks). *Let Ω be a domain and P_1 and P_2 be two (not necessarily distinct) peaks of $\mathbf{MAT}(\Omega)$. Suppose that $\mathbf{r} : [0, 1] \to \mathbf{MAT}(\Omega)$, $\mathbf{r}(t) = (\mathbf{p}(t), r(t))$ is a continuous, not* *self-intersecting path (possibly closed if $P_1 = P_2$) in $\mathbf{MAT}(\Omega)$ with the following properties:*

(1) *The path $\mathbf{r}$ connects P_1 and P_2, i.e., we have $\mathbf{r}(0) \in P_1$ and $\mathbf{r}(1) \in P_2$.*

(2) *There exists a $t_0 \in [0, 1]$ such that $\mathbf{r}(t_0) \notin P_1 \cup P_2$.*

Then, there is a valley of $\mathbf{MAT}(\Omega)$ on the path $\mathbf{r}$.

Proof. In view of Lemma 5, it is sufficient to show that there exist a and b ($0 < a \leq b < 1$) such that $r(t)$ takes a local minimum on $[a, b]$ in the sense of the lemma. Suppose not. Then $r(t)$ must be either non-decreasing or non-increasing. Note that $r(t)$ is not constant since there exists a point on $\mathbf{r}$ which is not in $P_1 \cup P_2$. But if $r(t)$ is non-decreasing (respectively, non-increasing), P_1 (respectively, P_2) cannot be a peak. $\square$

5 Domain decomposition lemma and offset

One of the weak points of the medial axis transform is that it is very sensitive to the perturbation of the domain's boundary. If you wrinkle a smooth segment of the boundary, you will see many branches of the medial axis transform coming out toward the wrinkled boundary. However, the dependence of the medial axis transform on the domain's boundary is in substance local. That is, the wrinkled boundary affects only the contact disks that does contact the boundary. In addition to this localized property, on the other hand, the global information about the medial axis transform can be obtained by combining local information about the medial axis transform. The following is an abstract of the above statements.

Theorem 5 (Domain decomposition lemma). *For any fixed medial axis point $\mathbf{p} \in \mathbf{MA}(\Omega)$, let $B_r(\mathbf{p})$ be the corresponding contact disk. Suppose $A_1, \ldots, A_m$ are the connected components of $\Omega \setminus B_r(\mathbf{p})$. Denote $\Omega_i = A_i \cup B_r(\mathbf{p})$ for $i = 1, \ldots, m$. Then we have*

$$\begin{aligned} \mathbf{MA}(\Omega) &= \bigcup_{i=1}^{m} \mathbf{MA}(\Omega_i), \\ \mathbf{MAT}(\Omega) &= \bigcup_{i=1}^{m} \mathbf{MAT}(\Omega_i). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathbf{MA}(\Omega_i) \quad \cap \quad \mathbf{MA}(\Omega_j) \quad &= \quad \{\mathbf{p}\}, \\ \mathbf{MAT}(\Omega_i) \quad \cap \quad \mathbf{MAT}(\Omega_j) \quad &= \quad \{(\mathbf{p}, r)\} \end{aligned}$$

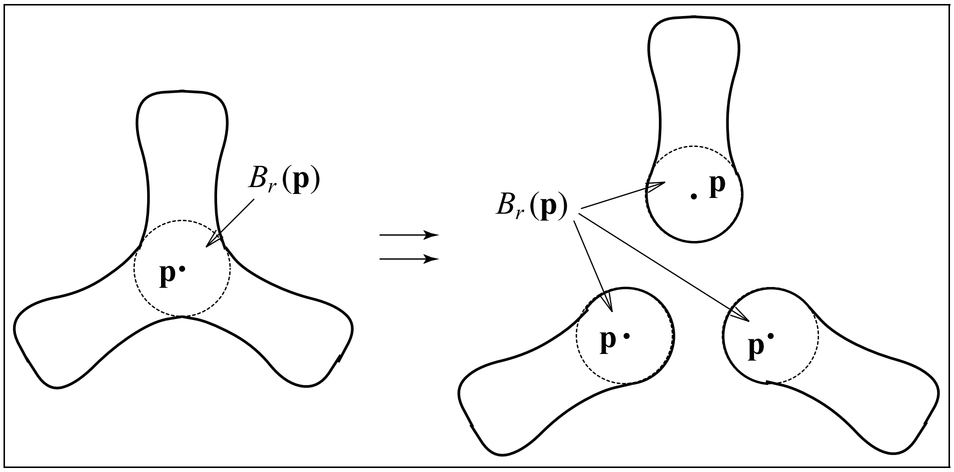

Figure 5: Domain decomposition at $(\mathbf{p}, r)$

for every distinct i and j. See Figure 5.

Proof. See [3]. $\square$

The domain decomposition method was the basic tool for the construction of the medial axis transform in [4]. Now we have at hand the relation between the medial axis transform and the offset curve, it is advantageous if we know how the domain decomposition method can be applied to the offset curve construction.

Let $(\mathbf{p}, r)$ be an n-prong point of $\mathbf{MAT}(\Omega)$ and let $C_1, \ldots, C_n$ be the corresponding contact components. Let $A_1, \ldots, A_m$ and $\Omega_1, \ldots, \Omega_m$ be the sets as given in Theorem 5. Though in general we have $m \leq n$, let us suppose we have $m = n$ for the clearer argument. (The general $m < n$ case is almost the same.) Let $S_i = \partial B_r(\mathbf{p}) \cap \overline{A_i}$, a segment of $\partial B_r(\mathbf{p})$. (See Figure 6.) Let U_i be the union of the closed sectors of $B_r(\mathbf{p})$ corresponding to S_i. That is, $U_i = \{\mathbf{z} \in \overline{\mathbf{pq}} \,|\, \mathbf{q} \in S_i\}$. By setting $A'_i = A_i \cup U_i$, we can divide $\mathcal{O}_d(\Omega_i)$ into two sets: one in A'_i and the other in $\Omega_i \setminus A'_i$. (See Figure 7.) It is easy to see that the segment of $\mathcal{O}_d(\Omega_i)$ in $\Omega_i \setminus A'_i$ is the d-offset curve to complementary arc $\tilde{S}_i$ $(= \partial B_r(\mathbf{p}) \setminus S_i)$ of S_i, which is just the same arc as $\tilde{S}_i$ with the same center but smaller radius $r - d$ (if $d > r$, no such arc exists). Note that $\tilde{S}_i$ is an open arc, i.e., it does not contain its end points.

Lemma 6. *For any $\mathbf{z} \in A'_i$, we have* $\mathrm{dist}(\mathbf{z}, \partial\Omega_i) = \mathrm{dist}(\mathbf{z}, \partial\Omega)$. *Hence we have $\mathcal{O}_d(\Omega_i) \cap A'_i = \mathcal{O}_d(\Omega) \cap A'_i$.*

Proof. Denote $\mathrm{dist}(\mathbf{z}, \partial\Omega_i)$ by d. Let $\mathbf{w}$ be a foot point of $\mathbf{z}$ on $\partial\Omega_i$. We first show that $\mathbf{w}$ is on $\partial\Omega$ and thus we have $\mathrm{dist}(\mathbf{z}, \partial\Omega_i) \geq \mathrm{dist}(\mathbf{z}, \partial\Omega)$.

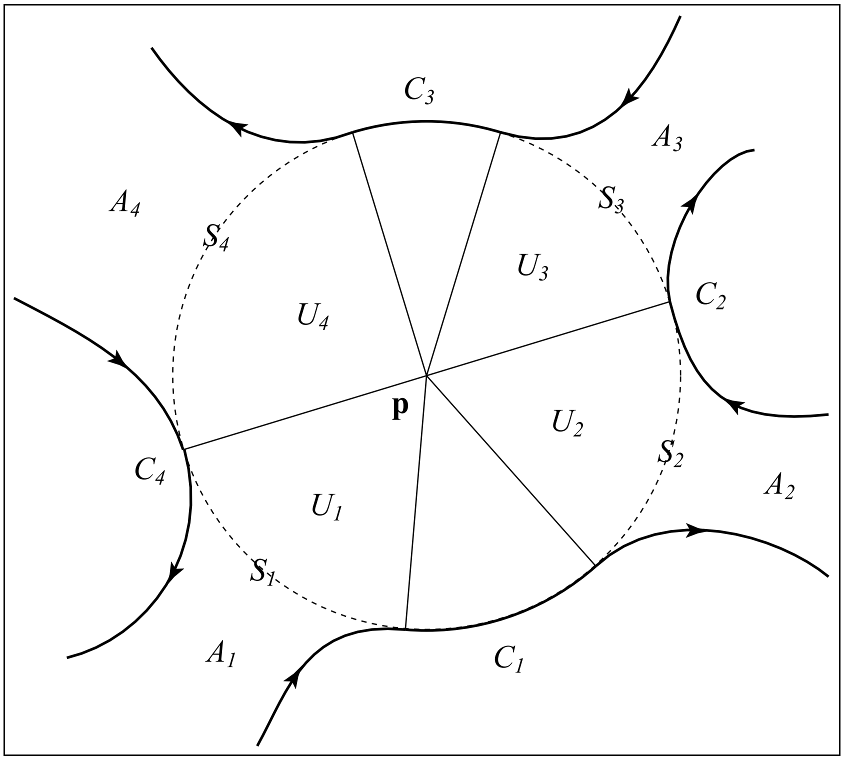

Figure 6: The partition of the contact circle

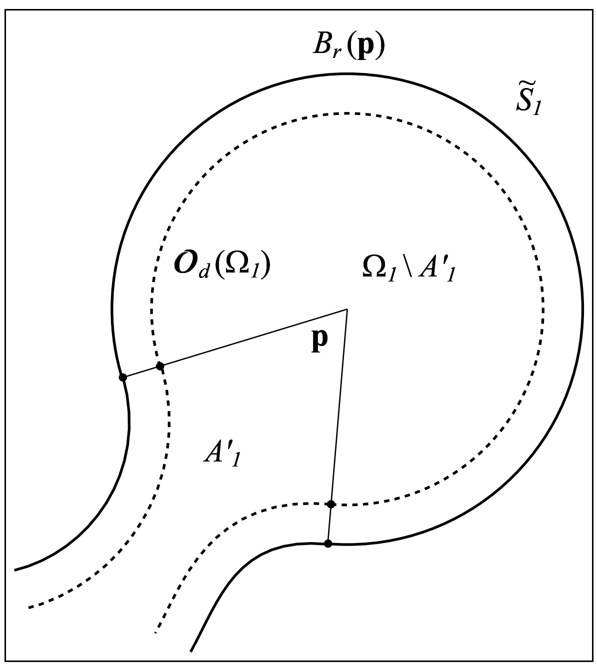

Figure 7: d-offset curve of the subdomain Ω_i

Suppose $\mathbf{w}$ is not on $\partial\Omega$ but on $\tilde{S}_i$. Since $\mathbf{z}$ is inside A'_i (including the boundary) but $\mathbf{w}$ is outside A'_i, $\overline{\mathbf{zw}}$ intersects the boundary of A'_i, which is $(\partial A_i \cap \partial\Omega) \cup \overline{\mathbf{pq}_1} \cup \overline{\mathbf{pq}_2}$, where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the end points of S_i. By the assumption, $\overline{\mathbf{zw}}$ must intersect $\overline{\mathbf{pq}_1}$ or $\overline{\mathbf{pq}_2}$. (Otherwise, the intersection point on $\partial A_i \cap \partial\Omega$ realizes $\mathrm{dist}(\mathbf{z}, \partial\Omega_i)$ with a smaller value than d.) Then it is easy to see that there are points on $\tilde{S}_i$ realizing $\mathrm{dist}(\mathbf{z}, \partial\Omega_i)$ with a smaller value than d.

Now we show that $\mathrm{dist}(\mathbf{z}, \partial\Omega_i) \leq \mathrm{dist}(\mathbf{z}, \partial\Omega)$. Let $\mathbf{v}$ be a foot point of $\mathbf{z}$ on $\partial\Omega$ realizing $\mathrm{dist}(\mathbf{z}, \partial\Omega)$. If $\mathbf{v}$ is on $\partial\Omega_i$, we have $\mathrm{dist}(\mathbf{z}, \partial\Omega_i) \leq |\overline{\mathbf{zv}}| = \mathrm{dist}(\mathbf{z}, \partial\Omega)$. If $\mathbf{v}$ is not on $\partial\Omega_i$, then $\overline{\mathbf{zv}}$ must intersect $\partial\Omega_i$. Let $\mathbf{y}$ be the intersection point. Then we have $\mathrm{dist}(\mathbf{z}, \partial\Omega_i) \leq |\overline{\mathbf{zy}}| \leq |\overline{\mathbf{zv}}| = \mathrm{dist}(\mathbf{z}, \partial\Omega)$. □

Theorem 6 (Domain decomposition lemma for offset). *At any contact disk $B_r(\mathbf{p})$, the d-offset curve $\mathcal{O}_d(\Omega)$ can be decomposed into the following components;*

(1) $\mathcal{O}_d(\Omega_i) \cap A'_i$ *for* $i = 1, \ldots, m$,

(2) *d-offset curves to the arcs C_j, for $j = 1, \ldots, n$.*

Furthermore, each pairs of the above components can at best share their end points. See Figure 8.

Proof. Let T_j be the sector of $\partial B_r(\mathbf{p})$ corresponding to the contact component C_j, i.e., $T_j = \{\mathbf{z} \in \overline{\mathbf{pq}} \,|\, \mathbf{q} \in C_j\}$. (See Figure 6.) It is easy to see that $\mathcal{O}_d(\Omega) \cap T_j$ is the d-offset curve to the arc C_j, which is just the same arc as C_j with the same center but $r - d$ radius. (If $d > r$, no such arc exists.)

Since the union of all A'_i and T_j is the whole domain Ω, we have

$$\mathcal{O}_d(\Omega) = \bigcup_{i=1}^{m} (\mathcal{O}_d(\Omega) \cap A'_i) \cup \bigcup_{j=1}^{n} (\mathcal{O}_d(\Omega) \cap T_j)\,.$$

By Lemma 6, we have $\mathcal{O}_d(\Omega) \cap A'_i = \mathcal{O}_d(\Omega_i) \cap A'_i$. Finally, note that any pair of distinct A'_i and A'_k or A'_i and T_j can at best share their boundary line segments. □

6 Classification of points on offset curves

Now we are ready to investigate the local geometry of the offset curves of a domain. For a real analytic curve "segment", the geometric and topological features of its offset curve are already familiar in the CAGD context. When

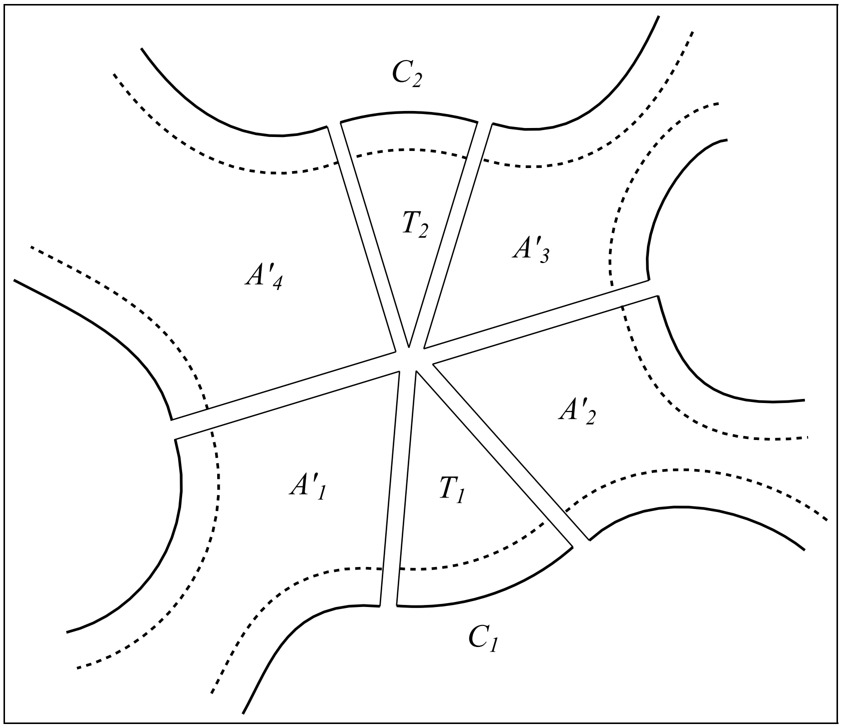

Figure 8: The partition of the offset curve

the offset distance d is relatively small, its offset curve is very similar to the original one. As d increases, however, the offset curve exhibits irregular points such as cusps, infinite curvature points, or self-intersections. For more information on the irregularity of offset curves, one is referred to [11, 12].

The domain of our concern is bounded by many real analytic curve segments. So the offset curve of the domain is a subset of the union of the offset curve segments to the boundary curve segments. In addition to the irregularity of the offset curve segments, caused by the corresponding boundary segments by itself, now the offset curve segments can intersect themselves (depending on the global geometry of the domain) to introduce another kind of irregularity.

Observing that the irregular points always occurs on the medial axis of the domain, we will separately treat points of the offset curve that are *on* the medial axis and *off* the medial axis after a brief review of the offset of a curve segment.

6.1 Untrimmed offset curves

Definition 15. By the *untrimmed* d-offset curve to a boundary curve segment $\mathbf{r}(t)$ for $d > 0$, we mean the curve

$$\mathbf{r}_d(t) = \mathbf{r}(t) + d\,\mathbf{n}(t),$$

where $\mathbf{n}(t)$ is the unit normal vector of $\mathbf{r}(t)$ pointing inside the domain. Due to the orientation convention of the boundary, the normal vector $\mathbf{n}(t)$ can be obtained by rotating the vector $\mathbf{r}'(t)/|\mathbf{r}'(t)|$ counterclockwise by $\pi/2$.

The unit tangent vector $\boldsymbol{\tau}_d(t)$ and curvature $\kappa_d(t)$ of the untrimmed offset curve $\mathbf{r}_d(t)$ can be readily computed [12] by

$$\boldsymbol{\tau}_d(t) = \frac{1 - d\kappa(t)}{|1 - d\kappa(t)|}\,\boldsymbol{\tau}(t) \tag{2}$$

$$\kappa_d(t) = \frac{\kappa(t)}{|1 - d\kappa(t)|} \tag{3}$$

where $\boldsymbol{\tau}(t)$ is the unit tangent vector and $\kappa(t)$ is the curvature of $\mathbf{r}(t)$. Note that $\mathbf{r}_d(t)$ exhibits singularities in its tangent vector (abrupt change of its direction) and curvature (infinite curvature) where $\kappa(t) = 1/d$, i.e., at the center of curvature of $\mathbf{r}(t)$. For more analytic properties of the untrimmed offset curve, one is referred to [12].

6.2 Points of offset curve off medial axis

We first consider points of the offset curve which is not on the medial axis.

Theorem 7. *Let $\mathbf{p}$ be a point on $\mathcal{O}_d(\Omega)$. If $\mathbf{p}$ is not on $\mathbf{MA}(\Omega)$, the following are true:*

(1) *There is a unique foot point $\mathbf{q}$ of $\mathbf{p}$ on $\partial\Omega$.*

(2) *Let $\mathbf{r}(t)$ be a parameterization of $\partial\Omega$ near $\mathbf{q}$ with $\mathbf{r}(0) = \mathbf{q}$. There exists an $\epsilon > 0$ such that the untrimmed offset curve $\mathbf{r}_d(t)$ for $t \in (-\epsilon, \epsilon)$ is $\mathcal{O}_d(\Omega)$ near $\mathbf{p}$.*

(3) *$\mathbf{p}$ is not a singular point of $\mathbf{r}_d(t)$.*

Proof. If there are more than one foot point of $\mathbf{p}$ on $\partial\Omega$, we should have $\mathbf{p} \in \mathbf{MA}(\Omega)$. Now we prove the second claim. Let $B_r(\mathbf{x})$ be the maximal disk of Ω tangent to $\partial\Omega$ at $\mathbf{q}$. Clearly, we have $r > d$. Since the mapping from the boundary to the medial axis transform (the *medial axis transform map* in [3])

prong	valley	peak	slope
1	yes	yes	no
2	yes	yes	yes
$n(\geq 3)$	no	yes	yes

Table 1: Single-point horizontal section

is continuous, there exists an $\epsilon > 0$ such that the corresponding maximal disk contacting at $\mathbf{r}(t)$ has a larger radius value than d for $t \in (-\epsilon, \epsilon)$. We show that the untrimmed offset curve $\mathbf{r}_d(t)$ for $t \in (-\epsilon, \epsilon)$ is the d-offset curve of Ω near $\mathbf{p}$. It suffices to show that for any point $\mathbf{r}_d(t)$, we have $\mathrm{dist}(\mathbf{r}_d(t), \partial\Omega) = d$. Draw a disk $B_d(\mathbf{r}_d(t))$ and $B_{r'}(\mathbf{p}')$, where $B_{r'}(\mathbf{p}')$ is the maximal disk at $\mathbf{r}(t)$. By our choice of ϵ, we have $r' > d$. Since the two disks $B_d(\mathbf{r}_d(t))$ and $B_{r'}(\mathbf{p}')$ are tangent to each other at $\mathbf{r}(t)$, we have $B_d(\mathbf{r}_d(t)) \subset B_{r'}(\mathbf{p}')$. Hence there is no other point of $\partial\Omega$ which is closer to $\mathbf{r}_d(t)$ than $\mathbf{r}(t)$.

To prove the last claim, it suffices to show $\kappa(0) < 1/d$, where $\kappa(t)$ is the curvature of $\mathbf{r}(t)$. Since we have $r \leq 1/\kappa(0)$ (the radius of the contact disk cannot be larger than the radius of curvature), we have $d < r \leq 1/\kappa(0)$. □

In summary, around the point which is off the medial axis, the offset curve of the domain is essentially the untrimmed offset curve for some boundary curve segments of the domain and the tangent vector and the curvature are well defined by Equation (2) and (3).

6.3 Points of offset curve on medial axis

Now we study points of the offset curve which is on the medial axis. Let $\mathbf{p}$ be a point on $\mathcal{O}_d(\Omega)$ and $\mathbf{MA}(\Omega)$. Note that $\mathbf{p}$ is basically a self-intersection point of the offset curve. Let H be the d-horizontal section of $\mathbf{MAT}(\Omega)$ having $(\mathbf{p}, d)$. By Lemma 4, H is either a single point or a C^1 curve segment. We first consider the case in which H is a single point.

Single point horizontal section

Let $\mathbf{p}$ be a point on $\mathcal{O}_d(\Omega)$. Suppose the single point $(\mathbf{p}, d)$ is a d-horizontal section of $\mathbf{MAT}(\Omega)$. We list the possible status of $(\mathbf{p}, d)$ as a medial axis transform point in Table 1. Here, there is no 1-prong slope by definition and 3-prong valley is also impossible by Theorem 3. Other possible cases are depicted in Figure 9.

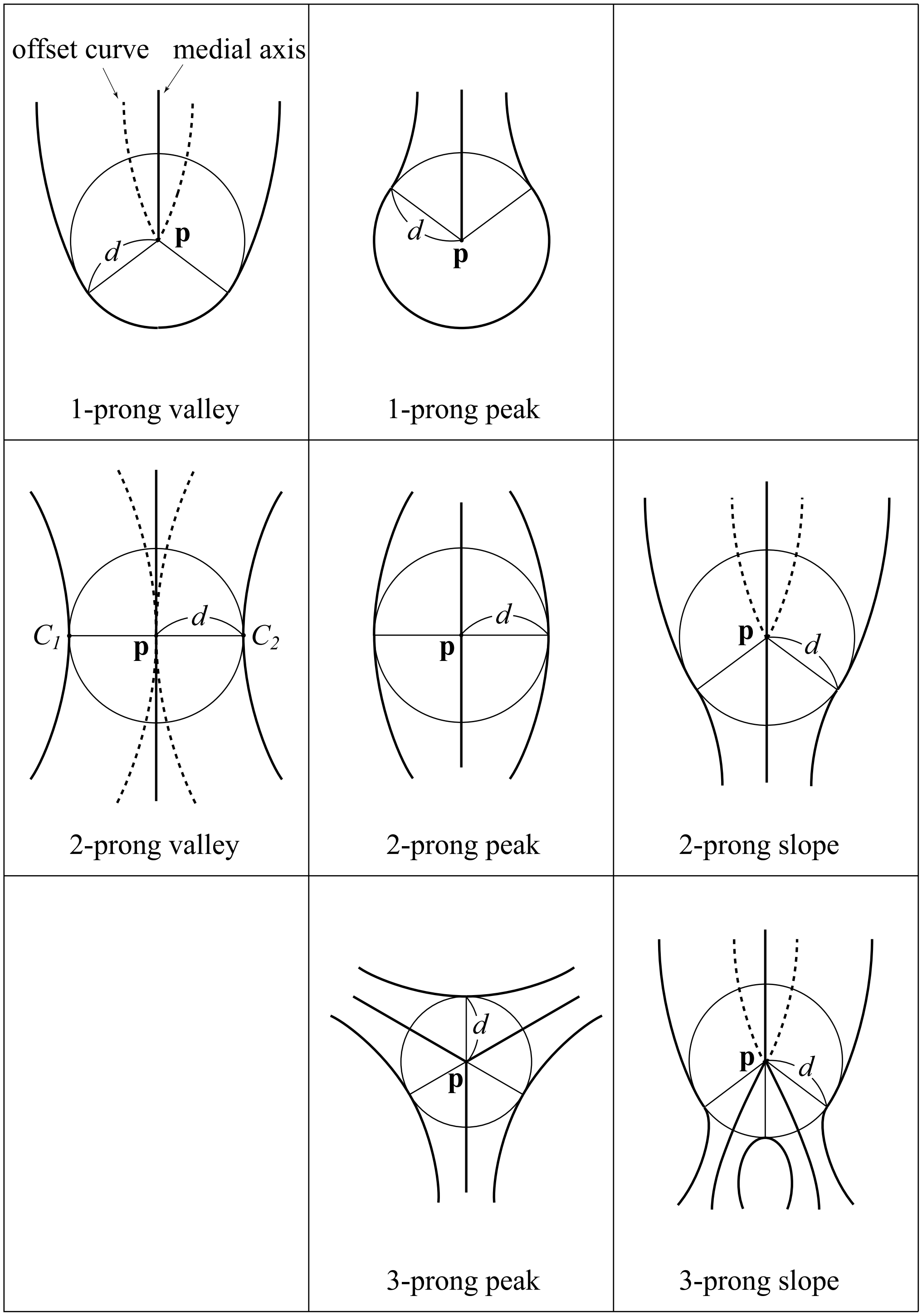

Figure 9: Typical figures near single-point horizontal sections

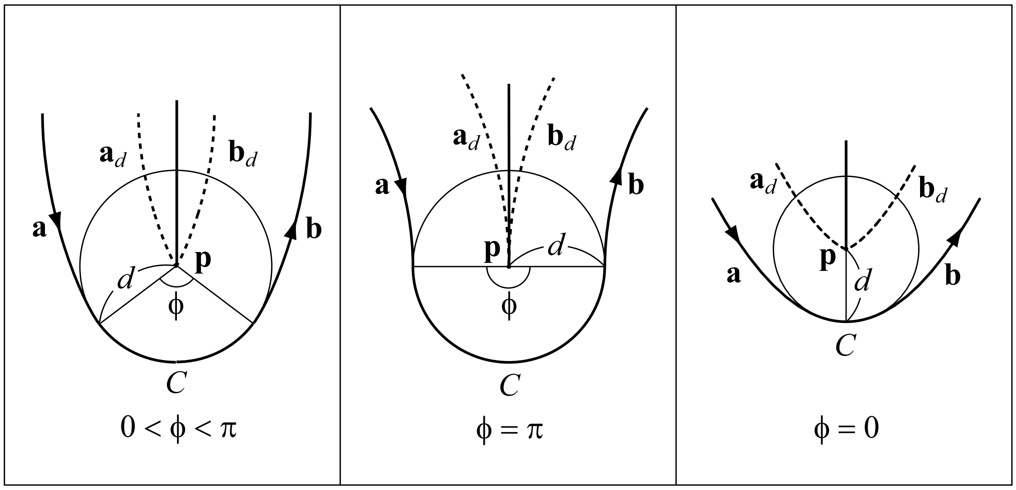

Figure 10: Typical figures near 1-prong valleys

Case 1 (1-prong valley). The contact disk $B_d(\mathbf{p})$ corresponding to $(\mathbf{p}, d)$ is an inscribed osculating disk. The (only one) contact component C of $B_d(\mathbf{p})$ is either a point or an arc, but cannot be the whole circle. Let $C(t)$ be a parameterization of C for $t \in [0, \phi]$, where ϕ is the angle over which the arc C extends. (See Figure 10.) Since $(\mathbf{p}, d)$ is a valley, we have $0 \leq \phi \leq \pi$. The case $\phi = 0$ occurs when C is a point. Let $\mathbf{a}$ and $\mathbf{b}$ be the boundary curve segments of Ω connected by C. (See Figure 10.) Assume that $\mathbf{a}$ and $\mathbf{b}$ are parameterized in the intervals $(-\epsilon, 0]$ and $[0, \epsilon)$ respectively such that $\mathbf{a}(0) = C(0)$ and $\mathbf{b}(0) = C(\phi)$.

Theorem 8. *The d-offset curve of Ω near $\mathbf{p}$ is the union of the untrimmed offset curve $\mathbf{a}_d$ and $\mathbf{b}_d$.*

Proof. Since $(\mathbf{p}, d)$ is a valley point and the mapping from the boundary to the medial axis transform is continuous, we can assume that the contact disk at $\mathbf{a}(t)$ or $\mathbf{b}(t)$ has greater radius than d for $t \neq 0$. Now the proof can go the same as in that of Theorem 7. $\square$

Remark 9. Note that $\mathbf{a}_d$ or $\mathbf{b}_d$ has infinite curvature at $\mathbf{p}$ if the curvature of $\mathbf{a}(t)$ or $\mathbf{b}(t)$ is $1/d$ at $t = 0$. Hence if $\phi = 0$, at least one of $\mathbf{a}_d$ and $\mathbf{b}_d$ must have infinite curvature at $\mathbf{p}$ while having continuous tangent direction. And if $\phi = \pi$, one can easily show that none of $\mathbf{a}_d$ and $\mathbf{b}_d$ can have infinite curvature at $\mathbf{p}$.

In summary, the d-offset curve near $\mathbf{p}$ can be decomposed into two curve segments at $\mathbf{p}$. The interior angle of the offset curve at $\mathbf{p}$ is $\pi - \phi$. If $\phi = 0$, in particular, the offset curve has continuous tangent direction but infinite curvature at $\mathbf{p}$.

Case 2 (1-prong peak). Since $(\mathbf{p}, d)$ is a peak point, we can assume that any contact disk near $\mathbf{p}$ (except $B_d(\mathbf{p})$ itself) has smaller radius value than d. This means that any point near $\mathbf{p}$ (except $\mathbf{p}$ itself) cannot have d distance to $\partial\Omega$. Hence d-offset curve near $\mathbf{p}$ is just $\mathbf{p}$ itself.

Case 3 (2-prong valley). If we decompose Ω at $B_d(\mathbf{p})$ into Ω_1 and Ω_2, then $(\mathbf{p}, d)$ is a single 1-prong valley point for each Ω_i. Now we can apply Theorem 6. Since the contact components C_1 and C_2 are isolated points (by Theorem 2), $\mathcal{O}_d(\Omega)$ is the union of $\mathcal{O}_d(\Omega_1) \cap A_1'$ and $\mathcal{O}_d(\Omega_2) \cap A_2'$. Note that we have examined each $\mathcal{O}(\Omega_i)$ in Case 1 with $\phi = \pi$. Hence, the d-offset curve near $\mathbf{p}$ is composed of four untrimmed offset curve segments emanating at $\mathbf{p}$ with the same tangent line. None of them have infinite curvature at $\mathbf{p}$. See Figure 9.

Case 4 (2-prong peak). Applying Theorem 6 and Case 2, we can see that the d-offset curve near $\mathbf{p}$ is just $\mathbf{p}$ itself.

Case 5 (2-prong slope). Decompose Ω at $B_d(\mathbf{p})$ into Ω_1 and Ω_2 such that $(\mathbf{p}, d)$ is a valley for $\mathbf{MAT}(\Omega_1)$ and a peak for $\mathbf{MAT}(\Omega_2)$. By Case 1 and 2, we can see that $\mathcal{O}_d(\Omega)$ near $\mathbf{p}$ is just the same as $\mathcal{O}_d(\Omega_1)$ near $\mathbf{p}$, which we have described in Case 1 with $\phi \neq 0$.

Case 6 (*n*-prong peak $(n \geq 3)$). Applying Theorem 6 and Case 2, we can see that the d-offset curve near $\mathbf{p}$ is just $\mathbf{p}$ itself.

Case 7 (*n*-prong slope $(n \geq 3)$). Decompose Ω at $B_d(\mathbf{p})$ into $\Omega_1, \ldots, \Omega_n$ where $(\mathbf{p}, d)$ is a valley for only one of the medial axis transforms, say $\mathbf{MAT}(\Omega_1)$ and a peak for $\mathbf{MAT}(\Omega_2), \ldots, \mathbf{MAT}(\Omega_n)$. By Case 1 and 2, we can see that $\mathcal{O}_d(\Omega)$ near $\mathbf{p}$ is just the same as $\mathcal{O}_d(\Omega_1)$ near $\mathbf{p}$, which we have described in Case 1 with $\phi \neq 0$.

Curve segment horizontal section

Now we consider the case in which H is a C^1 curve segment. First, suppose $(\mathbf{p}, d)$ is not an end point of H. Since H is a d-horizontal section, for every point $(\mathbf{q}, d)$ of H we have $\mathrm{dist}(\mathbf{q}, \partial\Omega) = d$, i.e., the whole H is a d-offset curve of Ω. Since every point of H except the end points is a 2-prong point, there are two corresponding boundary segments near $(\mathbf{p}, d)$. (See Figure 11.) And H and the corresponding two boundary curve segments are d-offset curves to each other. In summary, the d-offset curves of Ω near $\mathbf{p}$ is a C^1 untrimmed offset curve to the corresponding boundary curve segments.

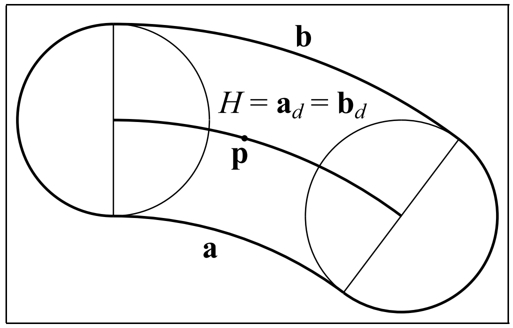

Figure 11: Near a curve segment horizontal section

prong	"valley"	"peak"	"slope"
1	-	-	-
2	yes	yes	-
$n(\geq 3)$	no	yes	-

Table 2: At the end point of horizontal section

Case 8 (Curve segment horizontal section). If $(\mathbf{p}, d)$ is in the C^1 curve segment d-horizontal section H but it is not an end point of H, then the d-offset curves of Ω near $\mathbf{p}$ is a C^1 untrimmed offset curve to the corresponding boundary curve segments.

Now it remains to investigate the case $\mathbf{p}$ is an end point of H. We can classify this case as in Table 1. Here we temporarily modify the definitions of the peak, valley, and slope. Note that they were defined for the *whole* horizontal section. We will call a *single point* $(\mathbf{p}, r)$ a "peak" of $\mathbf{MAT}(\Omega)$, if there is a neighborhood N of $(\mathbf{p}, r)$ such that r is a locally maximal radius value in $N \cap \mathbf{MAT}(\Omega)$. A "valley" and "slope" are similarly modified for the single point case.

We list the possible status of $(\mathbf{p}, d)$, which is an end point of the C^1 curve segment horizontal section H, in Table 2. Note that the concept "slope" is irrelevant here since the radius value of $\mathbf{MAT}(\Omega)$ is constant in the direction of H near $(\mathbf{p}, d)$. And for the 1-prong case, there is no need to artificially divide it into a "valley" and "peak". Typical examples of Table 2 are depicted in Figure 12.

Case 9 (1-prong end point). This case is just Case 8 except that there is only one d-offset curve segment emanating from $\mathbf{p}$. That is, the d-offset curve of Ω near $\mathbf{p}$ is a C^1 untrimmed offset curve to the corresponding boundary curve segments emanating from $\mathbf{p}$.

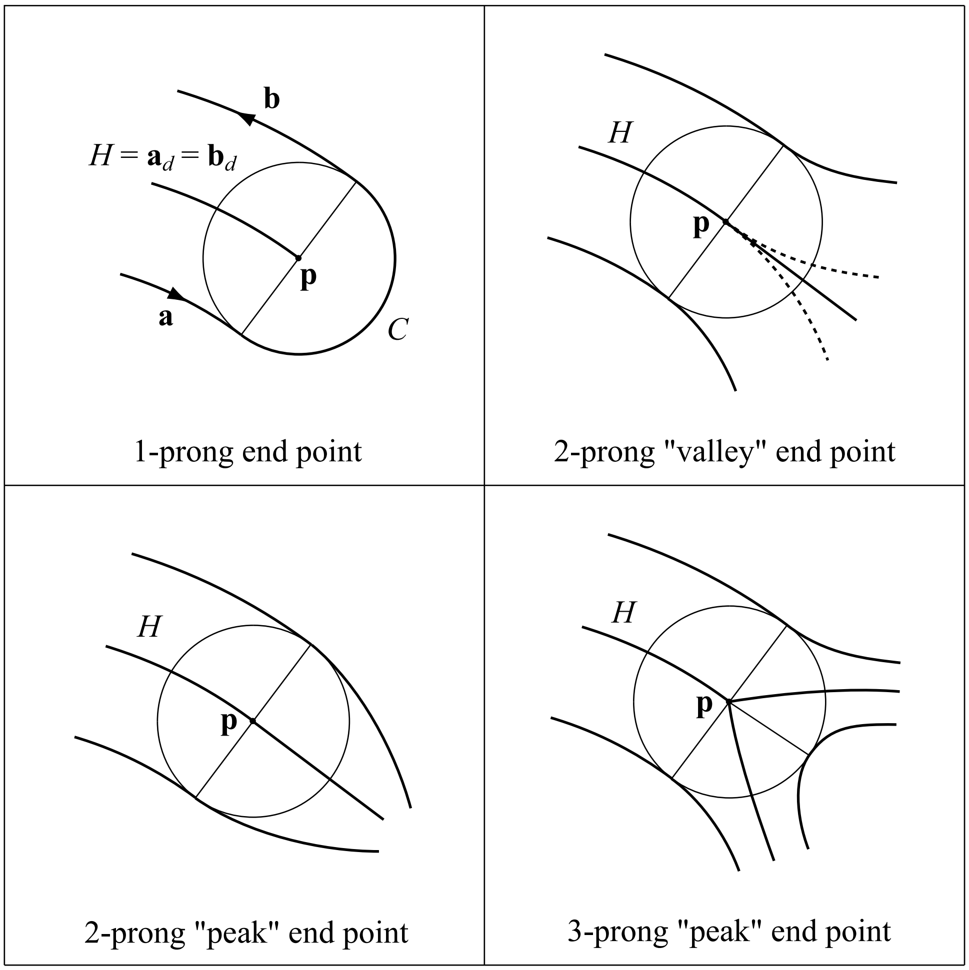

Figure 12: Typical figures near end points of C^1 curve horizontal sections

Case 10 (2-prong "valley" end point). By dividing the domain at $B_d(\mathbf{p})$, we have two subdomains Ω_1 and Ω_2 such that $\mathbf{p}$ is a 1-prong valley point of $\mathbf{MAT}(\Omega_1)$ with $\phi = \pi$ (Case 1) and a 1-prong end point of $\mathbf{MAT}(\Omega_2)$ as in Case 9. Hence, the d-offset curve of the domain near $\mathbf{p}$ consists of three untrimmed offset curve segments emanating at $\mathbf{p}$ with the same tangent line and none of them have infinite curvature at $\mathbf{p}$. See Figure 12.

Case 11 (2-prong "peak" end point). Decompose Ω at $B_d(\mathbf{p})$ into Ω_1 and Ω_2 such that $\mathbf{MAT}(\Omega_1) = H$. Since $(\mathbf{p}, d)$ is a peak for Ω_2, there is no contribution to the d-offset curve from the subdomain Ω_2 except the point $\mathbf{p}$ itself. Hence the d-offset curve near $\mathbf{p}$ is the same as in Case 9.

Case 12 (n-prong "peak" end point $(n \geq 3)$). Decompose Ω at $B_d(\mathbf{p})$ into $\Omega_1, \ldots, \Omega_n$ such that $\mathbf{MAT}(\Omega_1) = H$. Now $(\mathbf{p}, d)$ is a peak for all other $\Omega_2, \ldots, \Omega_n$. As we have seen in Case 1, there is no contribution to the d-offset curve from the subdomains $\Omega_2, \ldots, \Omega_n$ except the point $\mathbf{p}$ itself. Hence the d-offset curve near $\mathbf{p}$ is the same as in Case 9.

Now that we have exhausted all the possible cases of the offset curve points of a domain, we can summarize the classification in the following theorem. For the convenience of the enumeration, we introduce the following term.

Definition 16. A point $\mathbf{p}$ of $\mathcal{O}_d(\Omega)$ is an *n-fork* point if there are n pieces of curve segments of $\mathcal{O}_d(\Omega)$ emanating from $\mathbf{p}$.

Remark 10. We use the term "fork" to distinguish it from the term "prong" used in the description of the medial axis transform in [3, 4].

Theorem 9 (Classification of points on offset curves). *Let* $\mathbf{p}$ *be a point of the d-offset curve of a domain with* $d > 0$*. Then* $\mathbf{p}$ *is one of the following types; (See Figure 13.)*

(0) *a 0-fork point, i.e., an isolated point;*

(1) *a 1-fork point, where a* C^1 *curve segment is emanating with finite curvature;*

(2) *a 2-fork point,*

  (a) *where two* C^1 *curve segments are emanating with finite curvature and opposite tangent direction;*

  (b) *where two* C^1 *curve segments are emanating with opposite tangent direction and at least one of them having infinite curvature;*

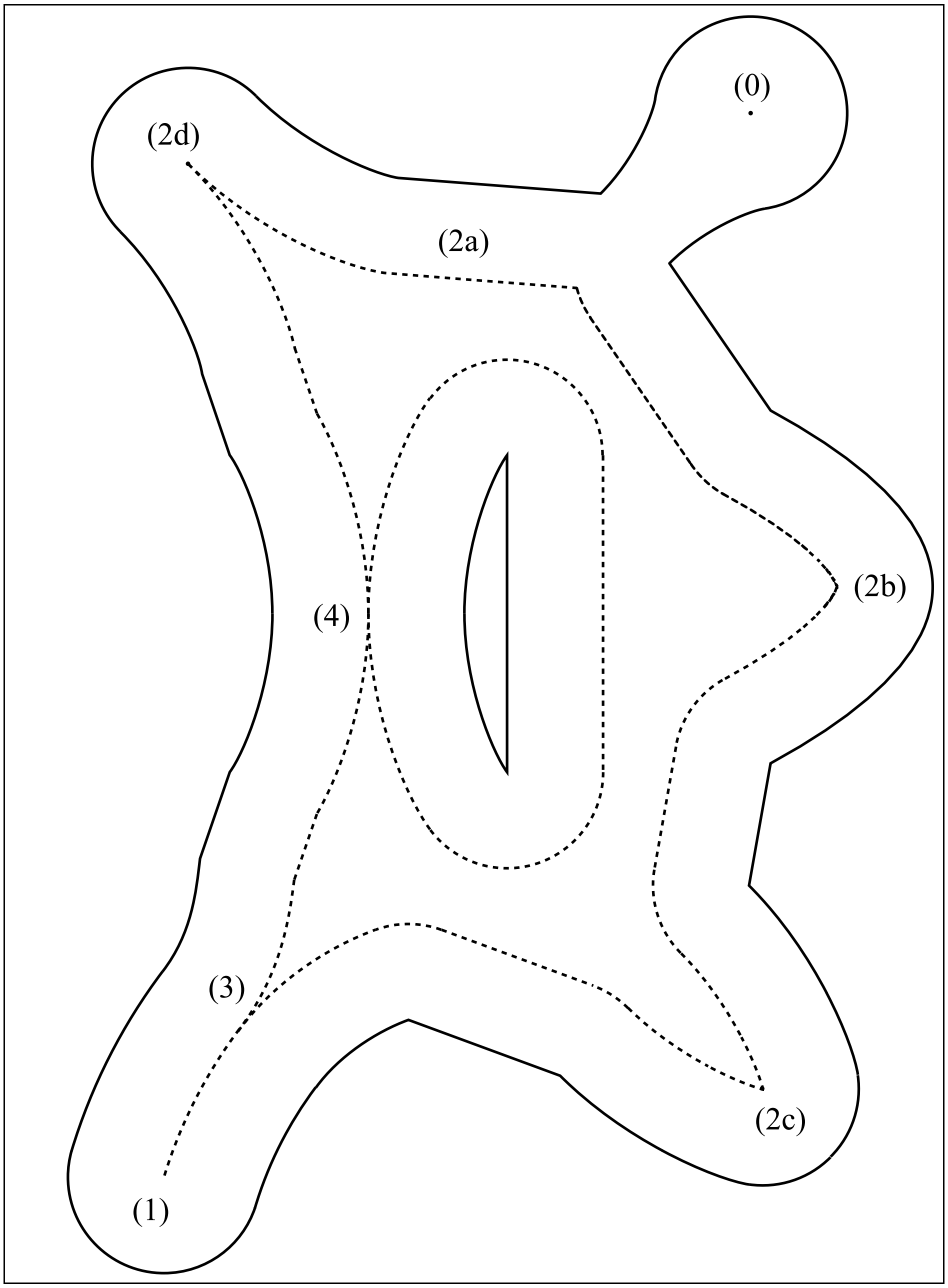

Figure 13: A domain and the offset curve

(c) *where two* C^1 *curve segments are emanating with interior angle* $0 < \phi < \pi$ *and each of them may or may not has infinite curvature;*

(d) *where two* C^1 *curve segments are emanating with the same tangent direction and finite curvature;*

(3) *a 3-fork point where three* C^1 *curve segments are emanating with finite curvature such that two of them have the same tangent direction but the third has the opposite tangent direction;*

(4) *a 4-fork point where four* C^1 *curve segments are emanating with finite curvature such that two of them have a common tangent direction and the other two have also a common tangent direction but opposite to the first one.*

Remark 11. Near a 2-fork point with discontinuous tangent direction, the d-offset region is on one side of d-offset curve such that the interior angle at that point is less than π, i.e., there is no "dull" corner point on the boundary of d-offset region.

Remark 12. One can show that the number of fork points except (2a) is finite.

7 Concluding remarks

For one curve segment, there have been a lot of literature on the geometric and topological features of its offset curve. However, for many pieces of curve segments bounding a planar region, the interference of the offset curve segments with each other makes the analysis much more cumbersome. Using the medial axis transform, which holds the complete data of the region, we were able to rigorously give a full description of the offset curve of the region, albeit people in the CAGD community are well aware by experience. In a subsequent paper, we will present an algorithm to compute the offset curve using the medial axis transform.

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