Stability Analysis of Medial Axis Transform under Relative Hausdorff Distance*

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Abstract

Medial axis transform (MAT) *is a basic tool for shape analysis. But, in spite of its usefulness, it has some drawbacks, one of which is its instability under the boundary perturbation. To handle this problem in practical situations, various "pruning" methods have been proposed, which are usually heuristic in their nature and without sufficient error analyses. In this paper, we show that, although medial axis transform is unstable with respect to standard measures such as the Hausdorff distance, it is stable in a measure called relative Hausdorff distance for some "smoothed out" domains called injective domains. In fact, we obtain an upper bound of the relative Hausdorff distance of the MAT of an injective domain with respect to the MAT of an arbitrary domain which is in small Hausdorff distance from the original injective domain.*

One consequence of the above result is that, by approximating a given domain with injective domains, we can extract the most "essential part" of the MAT within the prescribed error bound in Hausdorff distance. This introduces a new pruning strategy with precise error estimation. We illustrate our results with an example.

1. INTRODUCTION

Medial axis transform ($\mathbf{MAT}$ for short) is one of the basic tools widely used in shape analysis. Being natural conceptually, it extracts thinned features of a shape, which is homotopically equivalent to the original shape [1],[2],[3]. It has a graph structure, which is simple to store and process in a computer. For these advantages, medial axis transform has a wide range of applications, such as biological shape recognition [4], character recognition and representation [5], fingerprint classification [6], and visual analysis of circuit boards [7], to name a few.

The *medial axis* ($\mathbf{MA}$) of a plane domain is the set of the centers of the maximal inscribed circles contained in the given domain. The set of all the pairs of the medial axis point and the radius of the corresponding inscribed circle, is called the *medial axis transform*, which can be used to reconstruct the original domain. More explicitly, the medial axis transform $\mathbf{MAT}(\Omega)$ and the medial axis $\mathbf{MA}(\Omega)$ of a plane domain Ω is defined by

$$\mathbf{MAT}(\Omega) = \{\ (p,r) \in \mathbb{R}^2 \times [0,\infty) \mid B_r(p) \text{ is a maximal ball contained in } \Omega\ \},$$

$$\mathbf{MA}(\Omega) = \left\{ p \in \mathbb{R}^2 \mid \exists r \geq 0,\ \text{s.t. } (p,r) \in \mathbf{MAT}(\Omega) \right\}.$$

One nuisance when dealing with medial axis (transform) is that it is not stable under the perturbation of the domain [8],[9].

In many practical situations, the domains in question are often given with small noises. This would produce undesirable results, since $\mathbf{MA}$ (and $\mathbf{MAT}$) are not stable. A traditional alleviation of such phenomenon is the "pruning" methods which "cut off" the less important part of $\mathbf{MA}$ (and $\mathbf{MAT}$) according to some measures [8],[10]. But in general, these methods are heuristic in determining what is the important part of $\mathbf{MA}$ (and $\mathbf{MAT}$), and they often provide no underlying error estimations.

In this paper, we show that $\mathbf{MA}$ and $\mathbf{MAT}$ are indeed stable, if we concentrate on *relative Hausdorff distance* instead of Hausdorff distance. We will prove that, when a plane domain Ω satisfies a certain smoothness condition called the *injectivity*, then the *relative Hausdorff distance* of $\mathbf{MA}(\Omega)$ (*resp.*, $\mathbf{MAT}(\Omega)$) with respect to $\mathbf{MA}(\Omega')$ (*resp.*, $\mathbf{MAT}(\Omega')$) goes to zero, if the Hausdorff distances between $\partial\Omega$, $\partial\Omega'$ and between $\partial\Omega$, $\partial\Omega'$ go to zero for arbitrary domain Ω'. This is achieved by first deriving an upper bound formula for $\mathbf{MA}$ and $\mathbf{MAT}$ errors, which depends on the given injective domain.

We show how this result can be utilized in a new pruning strategy; by approximating a given domain with injective

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domains, we can guarantee that the $\mathbf{MA}$ (and the $\mathbf{MAT}$) of the approximating domain are the most "essential parts" of the $\mathbf{MA}$ (and the $\mathbf{MAT}$) of the original domain within a prescribed error bound in Hausdorff distance.

In principle, this amounts to a pruning by boundary smoothing. But we stress here that, unlike the previous known methods such as Curvature Flow Smoothing [11],[12], our method is equipped with precise error analysis.

2. PRELIMINARIES

2.1 NORMAL DOMAINS AND THEIR MAT

For reasonable behavior of MAT, we restrict the kind of the domains we deal with to what we call the *normal domains*. Although it is often neglected in literature, this restriction is necessary for reasonable behaviours of $\mathbf{MA}$ and $\mathbf{MAT}$. We will call a subset Ω of $\mathbb{R}^2$ a *normal domain*, if it satisfies the following two conditions:

- Ω is compact, or equivalently, Ω is closed and bounded.
- The boundary $\partial\Omega$ of Ω is a (disjoint) union of finite number of simple closed curves, each of which in turn consists of finite number of real analytic curve pieces.

With normal domains, the following expected behaviour of $\mathbf{MA}$ and $\mathbf{MAT}$ turns out to be true.

Proposition 1 ([3]) *Let Ω be a normal domain in $\mathbb{R}^2$. Then both $\mathbf{MA}(\Omega)$ and $\mathbf{MAT}(\Omega)$ have finite graph structures.*

2.2 RELATIVE HAUSDORFF DISTANCE

Hausdorff distance is one of the most popular and intuitively appealing measures of the difference between two sets [13]. We also introduce the concept *relative Hausdorff distance*, with which we later show the stability of $\mathbf{MA}$ and $\mathbf{MAT}$.

Let A and B be two (closed) subsets in $\mathbb{R}^n$ for some $n \geq 1$. The *relative Hausdorff distance* of A with respect to B, $\mathcal{H}(A|B)$, is defined by

$$\mathcal{H}(A|B) = \max_{p \in A} d(p, B),$$

where $d(\cdot,\cdot)$ is the usual Euclidean distance in $\mathbb{R}^n$.

The *Hausdorff distance* between A and B, $\mathcal{H}(A,B)$, is defined by

$$\mathcal{H}(A,B) = \max\{\mathcal{H}(A|B), \mathcal{H}(B|A)\}.$$

3. STABILITY OF MAT

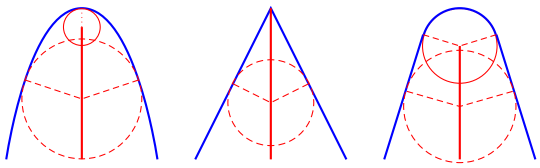
In this section, we show that $\mathbf{MA}$ and $\mathbf{MAT}$ are stable under relative Hausdorff distance, for some special kind of plane domains which we will call *injective*. In a sense, injective domains are some kinds of "smoothed out" or "rounded off" ones of ordinary normal domains.

Let Ω be a normal domain, and let $(p,r) \in \mathbf{MAT}(\Omega)$. We will introduce two domain-dependent constants ρ_Ω and θ_Ω as follows [14] :

$$\rho_\Omega = \inf_{(p,r)\in\mathbf{MAT}(\Omega)} r, \qquad \theta_\Omega = \inf\{\theta(p) \,:\, p \in G(\Omega)\}.$$

Definition 1 (Injective Domain)
A normal domain Ω in $\mathbb{R}^2$ is called injective, if $\rho_\Omega > 0$ and $\theta_\Omega > 0$.

Let Ω be a normal domain. It is easy to see that there are exactly three kinds of 1-prong points [3] in $\mathbf{MA}(\Omega)$. See Figure 1: Type (a) is the center of a maximal circle with only one contact point at which the circle osculates the boundary. Type (b) is a sharp corner. Type (c) is a 1-prong point with a contact arc.



(a) (b) (c)

Figure 1. Three types of 1-prong points

Suppose Ω is an injective domain. Then every 1-prong point of $\mathbf{MA}(\Omega)$ cannot be of Type (a) or Type (b) in Figure 1: If $\mathbf{MA}(\Omega)$ has a 1-prong point of Type (a) or Type (b), then we would have either $\theta_\Omega = 0$ or $\rho_\Omega = 0$. Conversely, it obvious that Ω is injective if every 1-prong point of $\mathbf{MA}(\Omega)$ is of Type (c) in Figure 1. Thus Ω is injective, if and only if every 1-prong point of $\mathbf{MA}(\Omega)$ is of Type (c) in Figure 1.

For every injective domain Ω, we define $\eta_\Omega : [0,\infty) \to [0,\infty)$ by

$$\eta_\Omega(x) = \frac{\rho_\Omega \cdot x}{\rho_\Omega \sin^2(\theta_\Omega/2) - x\cos^2(\theta_\Omega/2)}.$$

Note that $\eta_\Omega(x) > 0$ when $0 < x < \rho_\Omega \tan^2\theta_\Omega/2$, and $\lim_{x\to 0}\eta_\Omega(x) = 0$.

Now we present the main theorem of this section. See [14] for a proof.

Theorem 1 *Let Ω be an injective domain, and let Ω' be a normal domain with $\mathcal{H}(\partial\Omega,\partial\Omega') < \epsilon$ and $\mathcal{H}(\Omega,\Omega') < \epsilon$, where $\epsilon < \min\{\rho_\Omega\tan^2\theta_\Omega/2, \rho_\Omega/2\}$. Then we have*

$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) < \eta_\Omega(\epsilon),$$

and

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) < \sqrt{\eta_\Omega(\epsilon)^2 + \{\epsilon + \eta_\Omega(\epsilon)\}^2}.$$

The above theorem tells us that $\mathbf{MA}$ and $\mathbf{MAT}$ of injective domains are indeed stable under relative Hausdorff distance. Thus, when we deform an injective domain continuously (with respect to Hausdorff distance), the $\mathbf{MA}$ and the $\mathbf{MAT}$ of the original injective domain deviates continuously (with respect to relative Hausdorff distance) from the $\mathbf{MA}$s and the $\mathbf{MAT}$s of the deformed domains.

Theorem 1 also says that $\mathbf{MA}$ (*resp.*, $\mathbf{MAT}$) of an injective domain is approximately a *maximal common* part of the $\mathbf{MA}$ (*resp.*, the $\mathbf{MAT}$) of any domain sufficiently close to the original injective domain in Hausdorff distance.

4. APPLICATION TO PRUNING : EXTRACTING THE MOST ESSENTIAL PART

We have seen that $\mathbf{MA}$ and $\mathbf{MAT}$ of injective domains are stable under relative Hausdorff distance. But still, it is true that they are unstable under Hausdorff distance. However, their stability under relative Hausdorff distance can be utilized to extract the "most essential" part of a normal domain approximately.

Suppose we perturb an injective domain with domains which are also injective. In this case, $\mathbf{MA}$ and $\mathbf{MAT}$ become stable under "Hausdorff distance", not only under relative Hausdorff distance.

Theorem 2 *Let Ω_1 and Ω_2 be two injective domains, and let $\rho = \min\{\rho_{\Omega_1}, \rho_{\Omega_2}\}$, $\theta = \min\{\theta_{\Omega_1}, \theta_{\Omega_2}\}$. Suppose $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) < \epsilon$ and $\mathcal{H}(\Omega_1, \Omega_2) < \epsilon$, where $0 < \epsilon < \min\{\rho\tan^2\theta/2, \rho/2\}$. Then we have*

$$\mathcal{H}(\mathbf{MA}(\Omega_1), \mathbf{MA}(\Omega_2)) < \eta$$

and

$$\mathcal{H}(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) < \sqrt{\eta^2 + (\epsilon + \eta)^2},$$

where

$$\eta = \frac{\rho\epsilon}{\rho\sin^2\theta - \epsilon\cos^2\theta}.$$

Proof. This follows by applying Theorem 1 symmetrically to Ω_1 and Ω_2. □

Corollary 1 *Let Ω be a normal domain, and let Ω_1 and Ω_2 be two injective domains. Let $\rho = \min\{\rho_{\Omega_1}, \rho_{\Omega_2}\}$ and $\theta = \min\{\theta_{\Omega_1}, \theta_{\Omega_2}\}$. Suppose $\mathcal{H}(\partial\Omega_i, \partial\Omega) < \epsilon$ and $\mathcal{H}(\Omega_i, \Omega) < \epsilon$ for $i = 1, 2$, where $0 < 2\epsilon < \min\{\rho\tan^2\theta/2, \rho/2\}$. Then we have*

$$\mathcal{H}\left(\mathbf{MA}(\Omega_1), \mathbf{MA}(\Omega_2)\right) < \eta$$

and

$$\mathcal{H}(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) < \sqrt{\eta^2 + (2\epsilon + \eta)^2},$$

where

$$\eta = \frac{2\rho\epsilon}{\rho\sin^2\theta/2 - 2\epsilon\cos^2\theta/2}.$$

In short, with respect to Hausdorff distance, $\mathbf{MA}$ and $\mathbf{MAT}$ change continuously, when we change the approximating injective domain continuously. Thus the choice of the injective domain to approximate a normal domain does not affect the resulting $\mathbf{MA}$ and $\mathbf{MAT}$ much. An important point of Corollary 1 is that it sets an error bound of $\mathbf{MA}$ and $\mathbf{MAT}$ in Hausdorff distance, which may arise from the choice of the approximating injective domain.

¿From these considerations, we can propose a new pruning strategy which approximates the original domain with appropriate injective domains. This makes it possible to extract the maximal common, or the most essential part of $\mathbf{MA}$ and $\mathbf{MAT}$ within an error bound that depend on the constants such as ρ, θ, ϵ. Of course, as we vary these constants (that is, as we vary the approximating injective domains), we can obtain the essential part of $\mathbf{MA}$ and $\mathbf{MAT}$ with varying degrees of fine details.

For example, let Ω be the domain depicted in Figure 2. The $\mathbf{MA}$ shown has many hairy parts due to the zigzag nature of the boundary, which is a common characteristic of the bitmap figures. We approximated Ω with an injective domain Ω' with $\rho_{\Omega'} = 1200$ and $\theta_{\Omega'} = 54.19425477°$ shown in Figure 3. The Hausdorff distances $\mathcal{H}(\partial\Omega, \partial\Omega')$ and $\mathcal{H}(\Omega, \Omega')$ are less than $\epsilon = 37.5$. Now by Theorem 1, it follows that $\mathcal{H}(\mathbf{MA}(\Omega')|\mathbf{MA}(\Omega)) < 205.2384901$ and $\mathcal{H}(\mathbf{MAT}(\Omega')|\mathbf{MAT}(\Omega)) < 317.8754668$. Furthermore, if we approximate Ω with another injective domain Ω'' with $\rho_{\Omega''} \geq \rho_{\Omega'}$, $\theta_{\Omega''} \geq \theta_{\Omega'}$, $\mathcal{H}(\partial\Omega, \partial\Omega'') < \epsilon$ and $\mathcal{H}(\Omega, \Omega'') < \epsilon$, then Corollary 1 guarantees that $\mathcal{H}(\mathbf{MA}(\Omega'), \mathbf{MA}(\Omega'')) < 474.8397268$ and $\mathcal{H}(\mathbf{MAT}(\Omega'), \mathbf{MAT}(\Omega'')) < 726.4960366$. Thus we can say that $\mathbf{MA}(\Omega')$ (*resp.*, $\mathbf{MAT}(\Omega')$) is the most essential part of $\mathbf{MA}(\Omega)$ (*resp.*, $\mathbf{MAT}(\Omega)$) within the upper bound of Hausdorff distance 474.8397268 (*resp.*, 726.4960366) with the fine details specified by the constants ϵ, $\rho_{\Omega'}$ and $\theta_{\Omega'}$.

5. CONCLUSION

We showed that the instability of medial axis transform can be resolved, by measuring the differences with respect to the relative Hausdorff distance. It turned out that $\mathbf{MA}$ and $\mathbf{MAT}$ of injective domains are stable under relative Hausdorff distance. Furthermore, we derived a quantitative relation between the Hausdorff distance between the

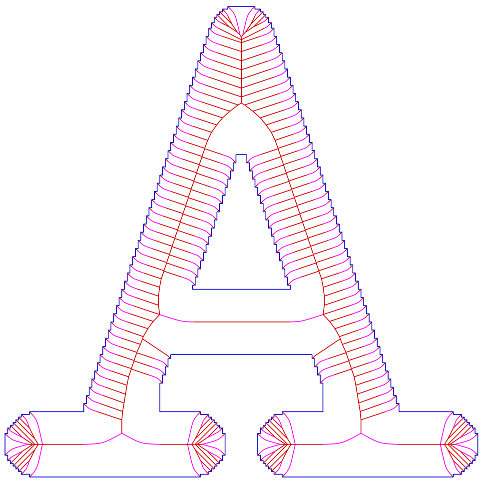


Figure 2. The Letter A (a): Ω with its $\mathbf{MA}$

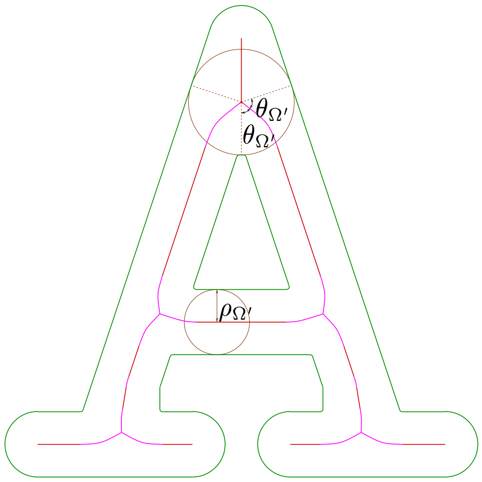


Figure 3. The Letter A (b): Ω' with its $\mathbf{MA}$

domains and the relative Hausdorff distance of one $\mathbf{MA}$ (*resp.*, $\mathbf{MAT}$) with respect to the other $\mathbf{MA}$ (*resp.*, $\mathbf{MAT}$).

By approximating a domain with an injective domain, this result makes it possible to extract the most essential part of $\mathbf{MA}$ and $\mathbf{MAT}$ of the domain within a prescribed error bound. In consequence, we can use this technique to devise a new pruning method with a precise upper bound for the errors, while there have been few significant error analyses in the previously known pruning methods.

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