Hyperbolic Hausdorff Distance for Medial Axis Transform

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Although the Hausdorff distance is a popular device to measure the differences between sets, it is not natural for some specific classes of sets, especially for the medial axis transform which is defined as the set of all pairs of the centers and the radii of the maximal balls contained in another set. In spite of its many advantages and possible applications, the medial axis transform has one great weakness, namely its instability under the Hausdorff distance when the boundary of the original set is perturbed. Though many attempts have been made for the resolution of this phenomenon, most of them are heuristic in nature and lack precise error analysis.

In this paper, we show that this instability can be remedied by introducing a new metric called the *hyperbolic Hausdorff distance*, which is most natural for measuring the differences between medial axis transforms. Using the hyperbolic Hausdorff distance, we obtain error bounds, which make the operation of medial axis transform almost an *isometry*. By various examples, we also show that the bounds obtained are sharp. In doing so, we show that bounding both the Hausdorff distance between domains and the Hausdorff distance between their boundaries is necessary and sufficient for bounding the hyperbolic Hausdorff distance between their medial axis transforms. These results drastically improve the previous results, and open a new way to practically control the Hausdorff distance error of the domains under their medial axis transform error, and *vice versa*.

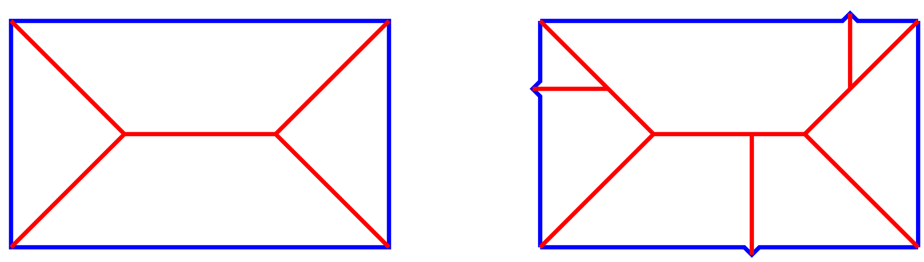
Key Words: Hausdorff distance, hyperbolic Hausdorff distance, medial axis transform, instability, Minkowski space-time, error bound.

1. INTRODUCTION

Among the many descriptors of shape, the medial axis transform is one of the most fundamental and widely-used ones. It has natural definitions, and is homotopically equivalent to the the original shape, while decreasing the dimension by one [4, 26]. It is also the set of the singularities of the distance function from the boundary, and the meeting points of the waves starting from the boundary [2]. It could also be considered as a limit of Voronoi diagram [29] as the number of the generating points becomes infinite. Due to these nice properties, medial axis transform has been a focus of many applications in such diverse fields as computational geometry [20], computer vision [12], shape modeling [22, 23, 15], mechanical engineering [14], optics [10, 11], biological shape recognition [19, 21], character recognition and representation [28, 27], fingerprint classification [16], visual analysis of circuit boards [30].

In this paper, we define the medial axis transform as the set of all pairs of the centers and the radii of the maximal inscribed balls in a domain. One merit of including the radii is that we can completely reconstruct the original domain with its medial axis transform. In contrast to the other literature, we generalize the domains of the definition to the general compact sets in $\mathbb{R}^n$ for $n = 1, 2, \cdots$, for we will show that our results hold in this general context.

One of the problems with the medial axis transform is its instability to noises. Medial axis transform may change very unstably, even when the boundary of the domain has only a slight perturbation. This phenomenon is illustrated in Figure 1: When measured by the usual Hausdorff distance, the domains (a) and (b) are very close to each other, but their medial axis transforms differ much. In fact, an infinitesimally small difference between the domains can result in a drastic difference between their medial axis transforms. In other words, the map **MAT** : {domains} $\rightarrow$ {medial axis transforms}, which corresponds to taking the medial axis transforms from the domains is not continuous under the usual Hausdorff metric.



(a) (b)

FIG. 1. Instability of Medial Axis Transform

Obviously, this instability could produce many problems, especially when one wants to get the medial axis transform of an input which might have noise. In fact, this is the case in most practical situations. So there have been many attempts to get around this unplausible phenomenon. Mainly, there has been the so-called "pruning" approach [25, 8], which prunes the less important part of the medial axis transform, leaving only the essential part. Some have also tried to smooth the boundary of the domain so that the resulting medial axis transform become more simple, hopefully capturing only important features [17, 24, 13]. But one common drawback of these methods is that they seldom provide precise error analysis, which makes them heuristic in most cases.

Recently, there has been an attempt [6] to obtain error bounds for the difference between the medial axis transforms when the domains are perturbed. They showed that the *one-sided* Hausdorff distance $\mathcal{H}(\mathbf{MAT}(\Omega_1)|\mathbf{MAT}(\Omega_2))$ of the medial axis transform $\mathbf{MAT}(\Omega_1)$ of a plane domain Ω_1 satisfying some regularity condition on the shape with respect to the medial axis transform $\mathbf{MAT}(\Omega_2)$ of any reasonable plane domain Ω_2, is bounded as follows:

$$\mathcal{H}(\mathbf{MAT}(\Omega_1)|\mathbf{MAT}(\Omega_2)) < \sqrt{\eta^2 + (\epsilon + \eta)^2},$$

for every $\max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\} < \epsilon < \min\{\rho \tan^2 \theta/2, \rho/2\}$. Here $\eta = \rho\epsilon/(\rho \sin^2 \theta/2 - \epsilon \cos^2 \theta/2)$, where ρ and θ are the positive constants depending only on Ω_1, and $\mathcal{H}(\cdot, \cdot)$ is the usual *two-sided* Hausdorff distance.

In this paper, instead of using the usual Hausdorff distance, we introduce a new metric called the *hyperbolic Hausdorff distance* to measure the difference between the medial axis transforms. We show that, if we endow this new metric on the space of the medial axis transforms, then the process of taking medial axis transform becomes almost an isometry. More specifically, let Ω_1, Ω_2 be compact sets in $\mathbb{R}^n$ such that their respective medial axis transforms $\mathbf{MAT}(\Omega_1)$, $\mathbf{MAT}(\Omega_2)$ be compact. Then we show that

$$\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) \leq 3 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}, \tag{1}$$

and

$$\max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\} \leq \mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)), \tag{2}$$

where $\mathcal{H}_h(\cdot, \cdot)$ denotes the hyperbolic Hausdorff distance. Thus, as a result of switching to the hyperbolic Hausdorff distance, we get much stronger results, which implies that the hyperbolic Hausdorff distance is a most natural metric for the medial axis transforms. As a result, we can now effectively control the perturbation of medial axis transform by that of the domain and its boundary, and conversely, the perturbation of the domain the boundary by that of the medial axis transform. We also show that the above bounds are sharp by various examples. Note that these results are more symmetric than those in [6], since we have no differences of the assumptions on Ω_1 and Ω_2. Also, the assumption itself is very general, in that only the compactness is required, and the considerations are in $\mathbb{R}^n$ for $n = 1, 2, \cdots$, rather than only in $\mathbb{R}^2$.

One byproduct of the above results is that we have a characterization of the medial axis transforms being close to each others under the hyperbolic Hausdorff distance, in terms of the usual Hausdorff distances between the original domains: There are two traditional methods to measure the difference between the domains with the Hausdorff distance. One is to measure the Hausdorff distance between the domains themselves, and the other is to measure the Hausdorff distance between their boundaries. We show that the hyperbolic Hausdorff distance between the medial axis transforms are enough to bound both the Hausdorff distance between the original domains and the Hausdorff distance between their boundaries. Conversely, we show that both types of the Hausdorff distances are needed for bounding the hyperbolic Hausdorff distance between their medial axis transforms. Thus the two types of measuring the Hausdorff distance between the domains are both necessary and sufficient for bounding the hyperbolic Hausdorff distance between their medial axis transforms.

We also mention that the definition of the hyperbolic Hausdorff distance is simple and natural, taking its motivation from Lorentz metric for the hyperbolic spaces [18]. The fact that the hyperbolic Hausdorff distance requires essentially the same computational effort compared to the usual Hausdorff distance, is expected to be a great advantage in applying the hyperbolic Hausdorff distance to many practical applications.

The rest of this paper is organized as follows: In Section 2, we introduce the Hausdorff distance in $\mathbb{R}^n$. Especially, we show, by examples, that both types of the Hausdorff distances, *i.e.*, that between the domains and that between their boundaries, are important. Then we introduce the hyperbolic Hausdorff distance and show some of its basic properties in Section 3. In Section 4, the medial axis transform for the general compact set in $\mathbb{R}^n$ is introduced. In Section 5, we obtain the bound (2), and show this is sharp by examples, and in Section 6, we obtain the bound (1), and also show its sharpness by examples. Together, they show that the process of taking the medial axis transform is almost an isometry under the Hausdorff distance for the domains and the hyperbolic Hausdorff distance for the medial axis transforms. Finally, we summarize our results and discuss some implications and applications of them in Section 7.

2. HAUSDORFF DISTANCE OF DOMAINS *VS.* BOUNDARIES

Let Ω_1 and Ω_2 be two nonempty compact sets in $\mathbb{R}^n$, where $n = 1, 2, \cdots$. The *Hausdorff distance* $\mathcal{H}(\Omega_1, \Omega_2)$ between Ω_1 and Ω_2 is defined by

$$\mathcal{H}(\Omega_1, \Omega_2) = \max\{\mathcal{H}(\Omega_1|\Omega_2), \mathcal{H}(\Omega_2|\Omega_1)\},$$

where the *one-sided Hausdorff distance* $\mathcal{H}(\Omega_1|\Omega_2)$ of Ω_1 with respect to Ω_2 is defined by

$$\mathcal{H}(\Omega_1|\Omega_2) = \max_{p_1 \in \Omega_1} d(p_1, \Omega_2).$$

Here, we denote by $d(\cdot, \cdot)$ the usual Euclidean distance in $\mathbb{R}^n$.

The following is a basic property of the Hausdorff distance.

PROPOSITION 2.1. ([1])
For each $n = 1, 2, \cdots$, the Hausdorff distance is a complete metric on the space of all nonempty compact sets in $\mathbb{R}^n$.

Usually, the Hausdorff distance is considered as a good device to measure the differences between two sets. Meanwhile, in many situations, especially where the sets are represented by their boundaries, it is customary to measure the differences between the sets by the Hausdorff distance between the boundaries of the sets. In this section, we will discuss the difference between these two methods, and show by examples that both of them can be sometimes misleading when used alone.

EXAMPLE 2.1. Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^2$ as depicted in Figure 2. Note that $\mathcal{H}(\Omega_1, \Omega_2)$ can be made arbitrarily small, although $\mathcal{H}(\partial\Omega_1, \partial\Omega_2)$ can be made converge to some positive number. This example sharply shows that the Hausdorff distance between the domains may overlook the differences which the Hausdorff distance between the boundaries considers to be important.

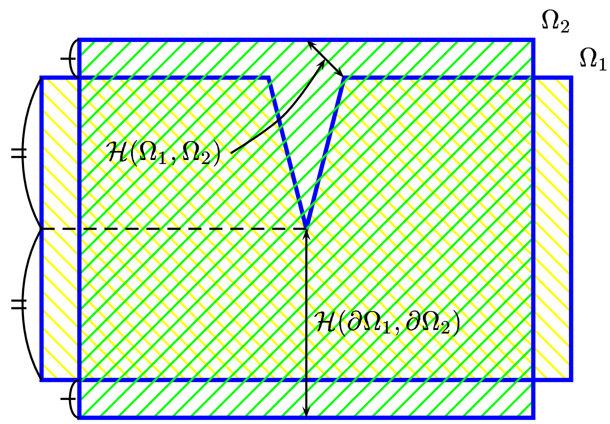


FIG. 2. Importance of the Hausdorff distance between the boundaries; Here we have $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) \gg \mathcal{H}(\Omega_1, \Omega_2)$.

Here is the opposite extreme.

EXAMPLE 2.2. Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^2$ as depicted in Figure 3. Here, Ω_2 is a thin ring around Ω_1. Note that $\mathcal{H}(\partial\Omega_1, \partial\Omega_2)$ can be made arbitrarily small, although $\mathcal{H}(\Omega_1, \Omega_2)$ can be made converge to some positive number. This example sharply shows that the Hausdorff distance between the boundaries may overlook the differences which the Hausdorff distance between the domains does not.

The examples in this section clearly show that neither the Hausdorff distance between the sets nor the Hausdorff distance between the boundaries alone is not enough to measure the difference of shapes. Later, we will see that both are needed to bound the *hyperbolic Hausdorff distance* between the medial axis transforms, and conversely, they are bounded by the hyperbolic Hausdorff distance between their medial axis transforms.

3. HYPERBOLIC HAUSDORFF DISTANCE

In this section, we introduce the hyperbolic Hausdorff distance, and show some of its basic properties.

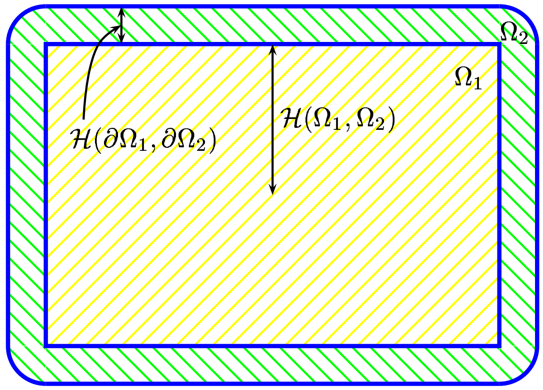


FIG. 3. Importance of the Hausdorff distance between the domains; Here we have $\mathcal{H}(\Omega_1, \Omega_2) \gg \mathcal{H}(\partial\Omega_1, \partial\Omega_2)$.

We will denote $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$. Let $(p, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, where $n = 1, 2, \cdots$. By $B_r(p)$, we denote the *closed* ball in $\mathbb{R}^n$ centered at p with the radius r, *i.e.*, $B_r(p) = \{x \in \mathbb{R}^n \mid d(x, p) \leq r\}$. Note that, when the radius is zero, a ball consists of only one point (the center).

DEFINITION 3.1. **(Hyperbolic Distance)**
Let $P_1 = (p_1, r_1), P_2 = (p_2, r_2)$ be in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, where $n = 1, 2, \cdots$. Then the *hyperbolic distance* $d_h(P_1|P_2)$ *from* P_1 *to* P_2 is defined by

$$d_h(P_1|P_2) = \max\{0, d(p_2, p_1) - (r_2 - r_1)\}.$$

Figure 4 illustrates the hyperbolic distance for various cases.

Here, we mention a motivation for the name *hyperbolic*: The usual Euclidean distance is associated to the standard Euclidean metric $dx_1^2 + \cdots + dx_n^2$ on $\mathbb{R}^n$. Now it can be clearly seen that the hyperbolic distance has an analogous association to the hyperbolic or Lorentz metric [18] $dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$ on $\mathbb{R}^{n+1}$.

The following is a basic property of the hyperbolic distance.

LEMMA 3.1. *Let* $P_1 = (p_1, r_1)$, $P_2 = (p_2, r_2)$ *be in* $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, *where* $n = 1, 2, \cdots$. *Then we have* $d_h(P_1|P_2) = \mathcal{H}(B_{r_1}(p_1)|B_{r_2}(p_2))$. *Suppose* $r_1 \geq \epsilon$ *for some* $\epsilon \geq 0$. *Then the following two conditions are equivalent:*
(1) $d_h(P_1|P_2) \leq \epsilon$,
(2) $B_{r_1 - \epsilon}(p_1) \subset B_{r_2}(p_2)$.

Proof. Easy. See Figure 4. ■

DEFINITION 3.2. **(Hyperbolic Hausdorff Distance)**
Let M_1, M_2 be nonempty compact sets in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, where $n = 1, 2, \cdots$. Then the *one-sided hyperbolic Hausdorff distance* $\mathcal{H}_h(M_1|M_2)$ *from* M_1 *to* M_2 is defined

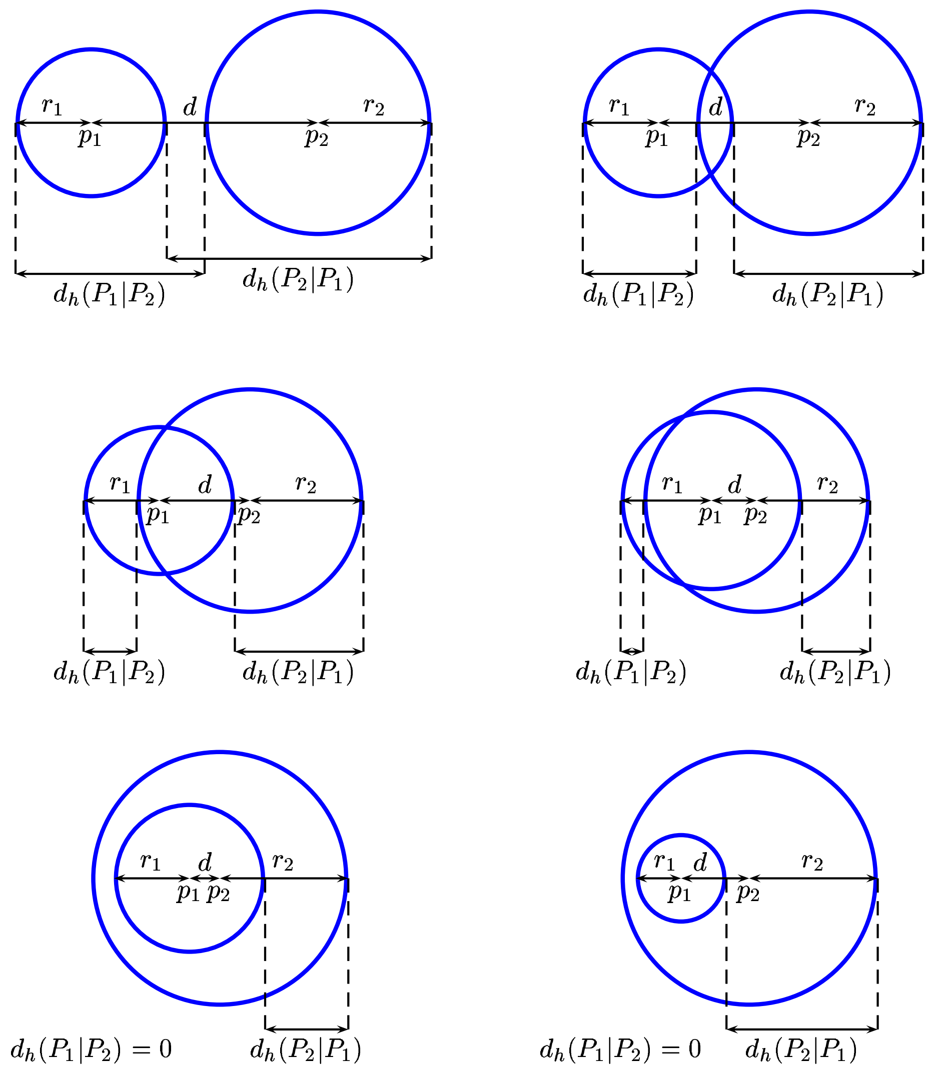


FIG. 4. Hyperbolic Hausdorff distance for various cases; Here, we assume $r_1 \leq r_2$ with no loss of generality.

by

$$\mathcal{H}_h(M_1|M_2) = \max_{P_1 \in M_1} \left\{ \min_{P_2 \in M_2} d_h(P_1|P_2) \right\}.$$

The *hyperbolic Hausdorff distance* between M_1 and M_2 is defined by

$$\mathcal{H}_h(M_1, M_2) = \max \{\mathcal{H}_h(M_1|M_2), \mathcal{H}_h(M_2|M_1)\}.$$

LEMMA 3.2. **(Comparison with Hausdorff Distance)**
Let $n = 1, 2, \cdots$.
(1) For every $P_1, P_2 \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, we have

$$d(P_1, P_2) \leq \max \{d_h(P_1|P_2), d_h(P_2|P_1)\} \leq \sqrt{2} \cdot d(P_1, P_2).$$

(2) For every compact sets $M_1, M_2 \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, we have

$$\mathcal{H}_h(M_1, M_2) \leq \sqrt{2} \cdot \mathcal{H}(M_1, M_2).$$

Proof. (1) Let $P_i = ((x_{i,1}, \cdots, x_{i,n}), r_i)$ for $i = 1, 2$. Note that

$$\begin{aligned} \max \{d_h(P_1|P_2), d_h(P_2|P_1)\} &= d(p_1, p_2) + |r_1 - r_2| \\ &= \sqrt{(x_{1,1} - x_{2,1})^2 + \cdots + (x_{1,n} - x_{2,n})^2} \\ &\quad + \sqrt{(r_1 - r_2)^2}. \end{aligned}$$

Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \leq \sqrt{2} \cdot \sqrt{a+b}$ for every $a, b \geq 0$, we have

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_{1,1} - x_{2,1})^2 + \cdots + (x_{1,n} - x_{2,n})^2 + (r_1 - r_2)^2} \\ &\leq \sqrt{(x_{1,1} - x_{2,1})^2 + \cdots + (x_{1,n} - x_{2,n})^2} + \sqrt{(r_1 - r_2)^2} \\ &= \max \{d_h(P_1|P_2), d_h(P_2|P_1)\} \\ &\leq \sqrt{2} \cdot \sqrt{(x_{1,1} - x_{2,1})^2 + \cdots + (x_{1,n} - x_{2,n})^2 + (r_1 - r_2)^2} \\ &= \sqrt{2} \cdot d(P_1, P_2). \end{aligned}$$

(2) From (1), we have

$$\begin{aligned} \mathcal{H}_h(M_1, M_2) &= \max \left\{ \max_{P_1 \in M_1} \left\{ \min_{P_2 \in M_2} d_h(P_1|P_2) \right\}, \max_{P_2 \in M_2} \left\{ \min_{P_1 \in M_1} d_h(P_2|P_1) \right\} \right\} \\ &\leq \max \left\{ \max_{P_1 \in M_1} \left\{ \min_{P_2 \in M_2} \sqrt{2} \cdot d(P_1, P_2) \right\}, \right. \\ &\qquad \left. \max_{P_2 \in M_2} \left\{ \min_{P_1 \in M_1} \sqrt{2} \cdot d(P_2, P_1) \right\} \right\} \\ &= \sqrt{2} \cdot \mathcal{H}(M_1, M_2). \end{aligned}$$

■

Remark. The bound in (2) is sharp, which can be seen from the following example: Let $M_1 = \{((x,y),r) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \mid -1 \leq x \leq 1, y = 0, r = 1\}$, and let $M_2 = \{((x,y),r) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \mid -1 \leq x \leq 1, y = 1, r = 2\}$. It is easy to see that $\mathcal{H}_h(M_1, M_2) = 2$ and $\mathcal{H}(M_1, M_2) = \sqrt{2}$.

Remark. There is no positive constant k such that $\mathcal{H}(M_1, M_2) \leq k \cdot \mathcal{H}_h(M_1, M_2)$ for every compact sets M_1, M_2 in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. For example, let $M_1 = \{((0,0),1)\} \subset \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$, and let $M_2 = \{((x,y),r) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \mid \frac{x}{a} + r = 1, y = 0, r \leq 1\}$ for $a > 1$. Then it is easy to see that $\mathcal{H}(M_1, M_2) = \sqrt{1+a^2}$, whereas $\mathcal{H}_h(M_1, M_2) = a - 1$. So, as $a \to 1$, we have $\mathcal{H}(M_1, M_2) \to \sqrt{2}$, whereas $\mathcal{H}_h(M_1, M_2) \to 0$.

4. MEDIAL AXIS TRANSFORM

Usually, medial axis transforms are defined for well-behaved domains, whose boundaries consist of curves with sufficient piecewise differentiability. One reason for this is that the medial axis transform has been used mainly in the application-oriented areas, where more pathologically-shaped domains are outside of the interests.

Another reason is that the medial axis transform of a set without sufficient regularity of their shape may lose the finite graph structure, which is an implicit assumption in most applications. In fact, Choi *et al.* [4] showed that the medial axis transform of a compact set Ω in $\mathbb{R}^2$ can exhibit quite anomalous behaviours like infinitely many prongs or infinitely many branches, if Ω does not satisfy the following rather strict condition: $\partial\Omega$ is a disjoint union of finitely many simple closed curves, each of which consists of finitely many real-analytic curves. They also showed that, if a set Ω satisfies the above assumption, its medial axis transform is shaped as expected:

PROPOSITION 4.1. ([4])
Suppose a compact set $\Omega \subset \mathbb{R}^2$ satisfies the above assumption. Then its medial axis transform has a finite graph structure.

See also [3] for a similar result.

In this paper, we define the medial axis transform for the general compact sets in $\mathbb{R}^n$, since the results we show are independent of the regularity of the shapes.

Let $n = 1, 2, \cdots$. We will denote by $\mathcal{C}_n$ the set of all nonempty compact sets in $\mathbb{R}^n$. By $\mathcal{C}_{n,1}$, we denote the set of all nonempty compact sets in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. For every Ω in $\mathcal{C}_n$, we define the *medial axis transform* $\mathbf{MAT}(\Omega)$ of Ω by

$$\begin{aligned}\mathbf{MAT}(\Omega) \;=\; \{(p,r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid B_r(p) \subset \Omega, \\ B_r(p) \subset B_{r'}(p') \subset \Omega \Rightarrow (p,r) = (p',r')\}.\end{aligned}$$

Unfortunately, **MAT**(Ω) may not be compact, even if Ω is compact. This can be seen from the following example.

EXAMPLE 4.1. For $n = 1, 2, \cdots$, let Ω_n be the domain in $\mathbb{R}^2$ as depicted in Figure 5. Here we assume $\sum_{n=1}^{\infty} r_n < \infty$.

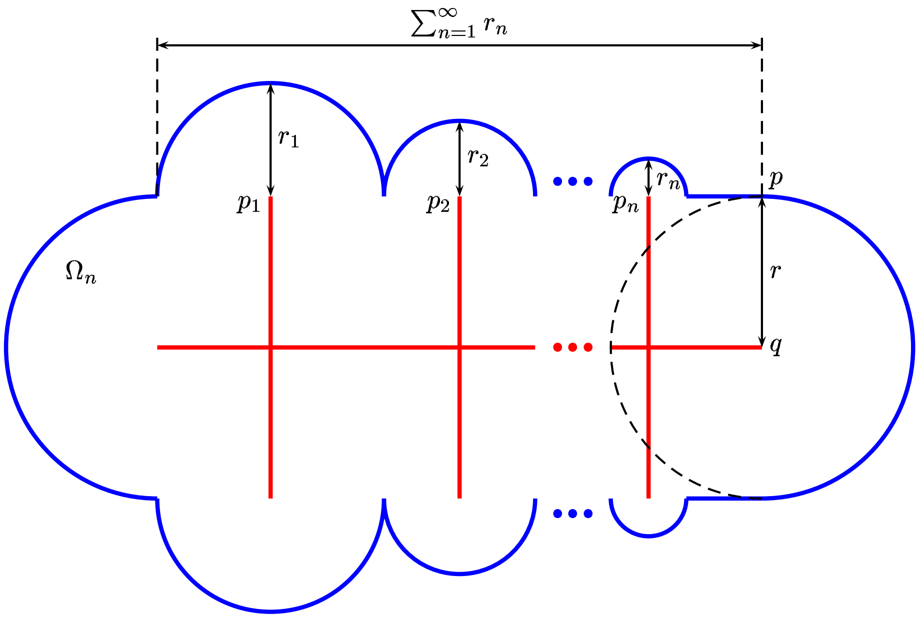


FIG. 5. Ω is compact, since $\Omega_n \to \Omega$. But **MAT**(Ω) is not compact.

Clearly, Ω_n is a compact set for every n. Note that $\mathcal{H}(\Omega_i, \Omega_j) \to 0$ as $i, j \to \infty$. So, by Proposition 2.1, there exists a compact set Ω such that $\mathcal{H}(\Omega_n, \Omega) \to 0$ as $n \to \infty$. Now, it is easy to see that every (p_n, r_n) is in **MAT**(Ω), and $(p_n, r_n) \to (p, 0)$ in the usual Euclidean metric in $\mathbb{R}^3$. But the ball $B_r(q)$, which is contained in Ω, strictly contains $B_0(p)$. So $(p, 0) \notin$ **MAT**(Ω). Thus **MAT**(Ω) is not closed, and hence not compact.

For every M in $\mathcal{C}_{n,1}$, we define

$$\begin{aligned} \mathbf{TAM}(M) &= \{x \in \mathbb{R}^n \mid \exists (p,r) \in M \text{ s.t. } x \in B_r(p)\} \\ &= \bigcup_{(p,r)\in M} B_r(p). \end{aligned}$$

In the case of **TAM**, we can show that **TAM**(M) is compact for every $M \in \mathcal{C}_{n,1}$, and so, **TAM** is a map from $\mathcal{C}_{n,1}$ to $\mathcal{C}_n$.

LEMMA 4.1. *Let $M \in \mathcal{C}_{n,1}$, where $n = 1, 2, \cdots$. Then* **TAM**(M) *is in $\mathcal{C}_n$.*

Proof. It is clear from the definition that **TAM**(M) is bounded, since M is bounded. So we only have to show that **TAM**(M) is closed. Suppose $q_n \to q$, where $q_n \in \mathbf{TAM}(M)$ for $n = 1, 2, \cdots$. Obviously, there exists $(p_n, r_n) \in M$ such that $q_n \in B_{r_n}(p_n)$ for $n = 1, 2, \cdots$. Since M is bounded, we can choose a subsequence (p_{n_k}, r_{n_k}) of (p_n, r_n) such that $(p_{n_k}, r_{n_k}) \to (p, r)$ for some $(p, r) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ in the usual Euclidean metric in $\mathbb{R}^{n+1}$. Now it is easy to see that $q \in B_r(p)$. Since M is closed, (p, r) is in M. So it follows that $q \in \mathbf{TAM}(M)$. Thus we conclude that $\mathbf{TAM}(M)$ is closed, and hence, is compact. ■

Note that $(\mathbf{TAM} \circ \mathbf{MAT})(\Omega) = \Omega$ and $(\mathbf{MAT} \circ \mathbf{TAM})(M) = M$ for every $\Omega \in \mathcal{C}_n$ and for every $M \in \mathbf{MAT}(\mathcal{C}_n)$. So $\mathbf{TAM}$ can be considered as an inverse of the map $\mathbf{MAT}$. In fact, $\mathbf{TAM}$ corresponds to the reconstruction of the original domain from its medial axis transform. Note also that $\mathbf{MAT}$ and $\mathbf{TAM}|_{\mathbf{MAT}(\mathcal{C}_n)}$ are one-to-one for $n = 1, 2, \cdots$.

5. BOUNDING DOMAIN/BOUNDARY PERTURBATION WITH MAT

In this section, we show that, if two compact sets in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ are close under the hyperbolic Hausdorff distance, then their images in $\mathbb{R}^n$ under the map $\mathbf{TAM}$ are close under the Hausdorff distance. Furthermore, we show that, when the sets are medial axis transforms, then the boundaries of their images are also close under the Hausdorff distance. Thus, if two medial axis transforms are close under the hyperbolic Hausdorff distance, then both the Hausdorff distance between the original domains and the Hausdorff distance between their boundaries are small.

We first start with the one-sided case.

LEMMA 5.1. *Let $n = 1, 2, \cdots$. For any $M_1, M_2 \in \mathcal{C}_{n,1}$, we have*

$$\mathcal{H}\left(\mathbf{TAM}(M_1) | \mathbf{TAM}(M_2)\right) \leq \mathcal{H}_h(M_1 | M_2).$$

Proof. Suppose $\mathcal{H}_h(M_1|M_2) \leq \epsilon$ for some $\epsilon \geq 0$. Let $\Omega_i = \mathbf{TAM}(M_i)$ for $i = 1, 2$. Let p be a point in Ω_1. From the definition of the map $\mathbf{TAM}$, it is clear that we can take $P_1 = (p_1, r_1)$ in M_1 such that $p \in B_{r_1}(p_1)$. Since $\mathcal{H}_h(M_1|M_2) \leq \epsilon$, we can take $P_2 = (p_2, r_2)$ in M_2 such that $d_h(P_1|P_2) \leq \epsilon$. From Lemma 3.1, it is easy to see that $d(p, B_{r_2}(p_2)) \leq \epsilon$, which implies that $d(p, \Omega_2) \leq \epsilon$. Since p is taken arbitrarily, we conclude that $\mathcal{H}(\Omega_1|\Omega_2) \leq \epsilon$ Now the proof follows, since ϵ is arbitrary. ■

From Lemma 5.1, we immediately have the two-sided result:

THEOREM 5.1. *Let $n = 1, 2, \cdots$. For any $M_1, M_2 \in \mathcal{C}_{n,1}$, we have*

$$\mathcal{H}\left(\mathbf{TAM}(M_1), \mathbf{TAM}(M_2)\right) \leq \mathcal{H}_h(M_1, M_2).$$

When M_1, M_2 are medial axis transforms, we can also bound the Hausdorff distance between the boundaries.

THEOREM 5.2. *Let $n = 1, 2, \cdots$. For any $M_1, M_2 \in \mathbf{MAT}(\mathcal{C}_n) \cap \mathcal{C}_{n,1}$, we have*

$$\mathcal{H}\left(\partial\left(\mathbf{TAM}(M_1)\right), \partial\left(\mathbf{TAM}(M_2)\right)\right) \leq \mathcal{H}_h(M_1, M_2).$$

Proof. Let $\Omega_i = \mathbf{TAM}(M_i)$ for $i = 1, 2$. Suppose $\mathcal{H}_h(M_1, M_2) \leq \epsilon$ for some $\epsilon \geq 0$. Suppose also $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) > \epsilon$. With no loss of generality, we can assume $\mathcal{H}(\partial\Omega_1|\partial\Omega_2) > \epsilon$. Then there exists $q_1 \in \partial\Omega_1$ such that $d(q_1, \partial\Omega_2) > \epsilon$. Take $q_2 \in \partial\Omega_2$ such that $d(q_1, q_2) = d(q_1, \partial\Omega_2) > \epsilon$. See Figure 6.

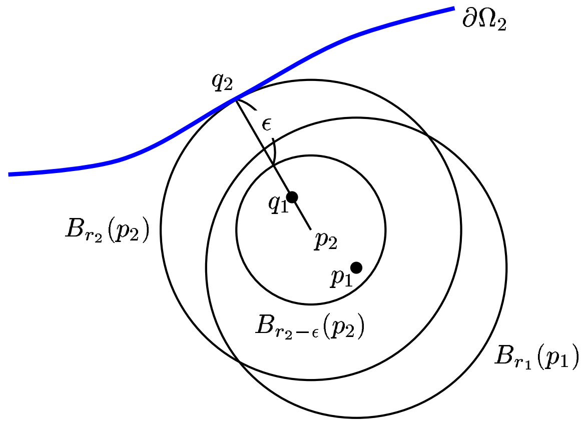


FIG. 6. Illustration for the proof of Theorem 5.2.

Note that $q_1 \in int\Omega_2$. Otherwise, we would have $\mathcal{H}(\Omega_1|\Omega_2) \geq d(q_1, \Omega_2) = d(q_1, \partial\Omega_2) > \epsilon$, contradicting the fact that $\mathcal{H}(\Omega_1, \Omega_2) \leq \epsilon$ (Theorem 5.1). Since M_2 is a medial axis transform, there exists unique $P_2 = (p_2, r_2)$ in M_2 such that $q_2 \in \partial B_{r_2}(p_2)$ and $\overline{q_1q_2} \subset \overline{p_2q_2}$. Note that $r_2 - \epsilon > 0$ and $q_1 \in intB_{r_2-\epsilon}(p_2)$. Since $\mathcal{H}_h\,(M_2|M_1) \leq \epsilon$, there exists $P_1 = (p_1, r_1)$ in M_1 such that $d_h\,(P_2|P_1) \leq \epsilon$. Now we have $q_1 \in intB_{r_1}(p_1) \subset int\Omega_1$, since $B_{r_2-\epsilon}(p_2) \subset B_{r_1}(p_1)$ by Lemma 3.1. This is a contradiction to the fact that $q_1 \in \partial\Omega_1$. So we must have $\mathcal{H}(\partial\Omega_1|\partial\Omega_2) \leq \epsilon$, which is again a contradiction. Thus we conclude that $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) \leq \epsilon$. Now the proof follows, since ϵ is arbitrary. ■

Remark. Theorem 5.2 does not hold for the general sets in $\mathcal{C}_{n,1}$. For example, let

$$\begin{aligned} M_1 &= \{((x,y),r) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \mid x^2 + y^2 = 1, r = 1\}, \\ M_2 &= \{((x,y),r) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \mid x^2 + y^2 = (1+\delta)^2, r = 1 - \delta\}, \end{aligned}$$

for $0 < \delta < 1$. Let $\Omega_i = \mathbf{TAM}(M_i)$ for $i = 1, 2$. Then,

$$\begin{aligned} \Omega_1 &= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2^2\}, \\ \Omega_2 &= \{(x,y) \in \mathbb{R}^2 \mid (2\delta)^2 \leq x^2 + y^2 \leq 2^2\}. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{H}_h(M_1, M_2) &= 2\delta, \\ \mathcal{H}(\partial\Omega_1, \partial\Omega_2) &= 2 - 2\delta. \end{aligned}$$

So, as $\delta \to 0$, we have $\mathcal{H}_h(M_1, M_2) \to 0$, but $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) \to 2$.

Remark. While Theorem 5.1 has its one-sided version, *i.e.*, Lemma 5.1, Theorem 5.2 has no one-sided counterpart. For example, let $\Omega_1 = B_1((0,0))$, $\Omega_2 = B_2((0,0))$. Let $M_i = \mathbf{MAT}(\Omega_i)$ for $i = 1, 2$. Then $M_1 = \{((0,0),1)\}$ and $M_2 = \{((0,0),2)\}$. Note that $\mathcal{H}_h(M_1|M_2) = 0$, while $\mathcal{H}(\partial\Omega_1|\partial\Omega_2) = 1$.

In fact, the inequalities in Theorem 5.1 and 5.2 are sharp in various ways; See [7] for the examples showing this.

6. BOUNDING MAT PERTURBATION WITH DOMAIN/ BOUNDARY

The opposite directions of the inequalities in Theorem 5.1 and 5.2 do not hold in general. This can be seen from Examples 6.1 and 6.2 below. Nevertheless, a slightly looser inequality turns out to be true.

THEOREM 6.1. *Let $n = 1, 2, \cdots$. For any $\Omega_1, \Omega_2, \in \mathcal{C}_n$ such that $\mathbf{MAT}(\Omega_1)$, $\mathbf{MAT}(\Omega_2)$ are compact, we have*

$$\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) \leq 3 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}.$$

Proof. Let $M_i = \mathbf{MAT}(\Omega_i)$ for $i = 1, 2$. First, note that the above inequality is trivially true when $\max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\} = 0$. Suppose $\max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\} \leq \epsilon$ for some $\epsilon > 0$. Let $P_1 = (p_1, r_1)$ be in M_1. Suppose first $r_1 > 2\epsilon$. Since $\mathcal{H}(\partial\Omega_1, \partial\Omega_2) \leq \epsilon$, we have $intB_{r_1-\epsilon}(p_1) \cap \partial\Omega_2 = \emptyset$. So either $intB_{r_1-\epsilon}(p_1) \subset int\Omega_2$ or $intB_{r_1-\epsilon}(p_1) \cap \Omega_2 = \emptyset$. But, if the latter is true, then we would have $d(p_1, \Omega_2) > \epsilon$, which contradicts the assumption $\mathcal{H}(\Omega_1, \Omega_2) \leq \epsilon$. So we must have $intB_{r_1-\epsilon}(p_1) \subset int\Omega_2$. Since M_2 is a medial axis transform, it is clear that there exists $P_2 = (p_2, r_2) \in M_2$ such that $B_{r_1-\epsilon}(p_1) \subset B_{r_2}(p_2)$. By Lemma 3.1, this means that $d_h(P_1|P_2) \leq \epsilon$, and hence, $d_h(P_1|P_2) \leq \epsilon \leq 3\epsilon$.

Suppose now $r_1 \leq 2\epsilon$. Since $\mathcal{H}(\Omega_1, \Omega_2) \leq \epsilon$, there exists $q_2 \in \Omega_2$ such that $d(p_1, q_2) \leq \epsilon$. Clearly, we can take $P_2 = (p_2, r_2) \in M_2$ such that $q_2 \in B_{r_2}(p_2)$. Now $d_h(P_1|P_2) = d(p_1, p_2) - (r_2 - r_1) \leq d(p_1, q_2) + d(q_2, p_2) - d(q_2, p_2) + 2\epsilon \leq 3\epsilon$. Thus we have $\mathcal{H}_h(M_1|M_2) < 3\epsilon$, since P_1 is arbitrary. Since ϵ is arbitrary, we conclude that $\mathcal{H}_h(M_1|M_2) \leq 3 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}$. By the symmetric argument, we can also show $\mathcal{H}_h(M_2|M_1) \leq 3 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}$. Thus the proof is complete. ∎

The proof of Theorem 6.1 also shows the following plausible result:

THEOREM 6.2. *Let $n = 1, 2, \cdots$. For any $\Omega_1, \Omega_2 \in \mathcal{C}_n$ such that $\mathbf{MAT}(\Omega_1)$, $\mathbf{MAT}(\Omega_2)$ are compact and $\rho_{\Omega_1}, \rho_{\Omega_2} > 2 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}$, we have*

$$\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) = \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}.$$

Here, we define $\rho_\Omega = \min\{r : (p, r) \in \mathbf{MAT}(\Omega)\}$ for every $\Omega \in \mathcal{C}_n$.

Proof. The proof of Theorem 6.1 shows the $\leq$ part, and the $\geq$ part follows from Theorem 5.1 and 5.2. ■

Theorem 6.2 tells that the transform **MAT** is in fact an *isometry* locally at every Ω with $\rho_\Omega > 0$. But note that the sets around such Ω's on which **MAT** is an isometry are in general not open under the Hausdorff distance.

The following examples show that the bound in Theorem 6.1 is sharp. Moreover, Example 6.1 shows that both $\mathcal{H}(\Omega_1, \Omega_2)$ and $\mathcal{H}(\partial\Omega_1, \partial\Omega_2)$ are crucial for bounding $\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2))$.

EXAMPLE 6.1. Define the two domains Ω_1, Ω_2 in $\mathbb{R}^2$ by

$$\begin{aligned}\Omega_1 &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2^2\},\\ \Omega_2 &= \{(x, y) \in \mathbb{R}^2 \mid (1+\delta)^2 \leq x^2 + y^2 \leq 2^2\}.\end{aligned}$$

See Figure 7. Here we assume $0 \leq |\delta| \ll 1$. Let $M_i = \mathbf{MAT}(\Omega_i)$ for $i = 1, 2$. It is easy to see that

$$\begin{aligned}\mathcal{H}(\Omega_1, \Omega_2) &= 1 + \delta\ ,\\ \mathcal{H}(\partial\Omega_1, \partial\Omega_2) &= 1 - \delta\ ,\\ \mathcal{H}_h(M_1, M_2) &= 3 + \delta\ .\end{aligned}$$

So, when $\delta > 0$, we have

$$3 \cdot \mathcal{H}(\partial\Omega_1, \partial\Omega_2) < \mathcal{H}_h(M_1, M_2) < 3 \cdot \mathcal{H}(\Omega_1, \Omega_2),$$

and, when $\delta < 0$, we have

$$3 \cdot \mathcal{H}(\Omega_1, \Omega_2) < \mathcal{H}_h(M_1, M_2) < 3 \cdot \mathcal{H}(\partial\Omega_1, \partial\Omega_2).$$

When $\delta = 0$, we have

$$\mathcal{H}_h(M_1, M_2) = 3 \cdot \mathcal{H}(\Omega_1, \Omega_2) = 3 \cdot \mathcal{H}(\partial\Omega_1, \partial\Omega_2).$$

EXAMPLE 6.2. Here we have a more realistic example. Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^2$ as depicted in Figure 8. Let $M_i = \mathbf{MAT}(\Omega_i)$ for $i = 1, 2$. Then it is clear that

$$\begin{aligned}\mathcal{H}(\Omega_1, \Omega_2) &= \delta\ ,\\ \mathcal{H}(\partial\Omega_1, \partial\Omega_2) &= \delta\ ,\\ \mathcal{H}_h(M_1, M_2) &= 3\delta\ ,\end{aligned}$$

for every $\delta > 0$. So we have

$$\mathcal{H}_h(M_1, M_2) = 3 \cdot \mathcal{H}(\Omega_1, \Omega_2) = 3 \cdot \mathcal{H}(\partial\Omega_1, \partial\Omega_2).$$

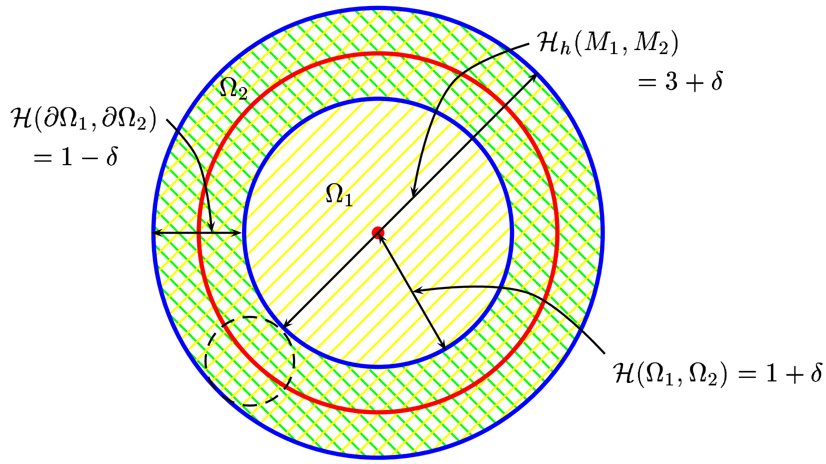


FIG. 7. Both $\mathcal{H}(\Omega_1, \Omega_2)$ and $\mathcal{H}(\partial\Omega_1, \partial\Omega_2)$ are needed for bounding $\mathcal{H}_h(M_1, M_2)$.

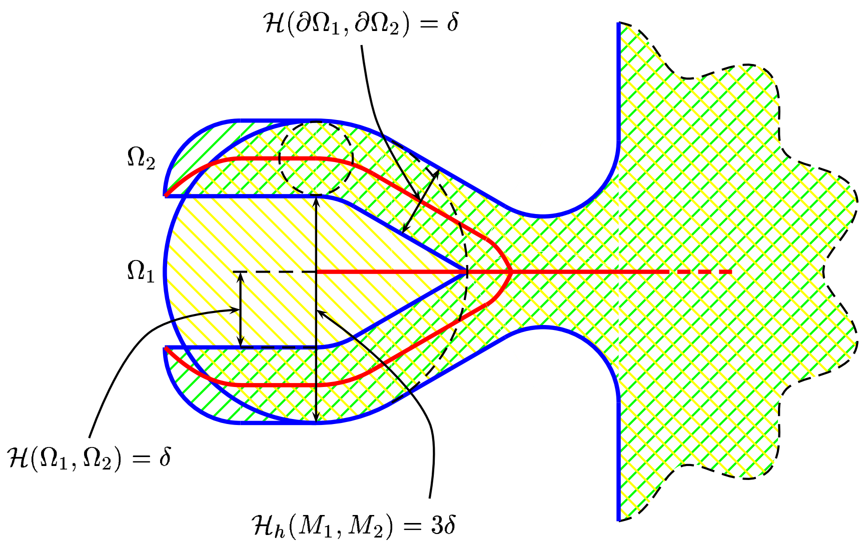


FIG. 8. A realistic example showing the sharpness of the inequality in Theorem 6.1.

7. SUMMARY

Now we summarize our results. For this purpose, we find it useful to interpret the results in terms of the properties of the maps **MAT** and **TAM**.

For each $n = 1, 2, \cdots$, define a new metric $\mathcal{H}_\partial$ on $\mathcal{C}_n$ by:

$$\mathcal{H}_\partial(\Omega_1, \Omega_2) = \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\},$$

for every $\Omega_1, \Omega_2 \in \mathcal{C}_n$. It is straightforward to check that $\mathcal{H}_\partial$ is indeed a metric on $\mathcal{C}_n$ for $n = 1, 2, \cdots$. But unfortunately, the metric space $(\mathcal{C}_n, \mathcal{H}_\partial)$ is not complete [7]. We observed in Section 4 that a medial axis transform may not be compact, even if its original domain is compact. This leads us to define the space $\mathcal{D}_n = \mathbf{TAM}(\mathcal{C}_{n,1} \cap \mathbf{MAT}(\mathcal{C}_n))$ for $n = 1, 2, \cdots$. Note that $\mathcal{D}_n \subset \mathcal{C}_n$ by Lemma 4.1. In fact, $\mathcal{D}_n$ is the largest reasonable subspace in $\mathcal{C}_n$ concerning compactness of the medial axis transform.

We view the maps **MAT** and **TAM** as bijections between the two metric spaces $(\mathcal{D}_n, \mathcal{H}_\partial)$ and $(\mathcal{M}_n, \mathcal{H}_h)$, where $\mathcal{M}_n$ denotes $\mathcal{C}_n \cap \mathbf{MAT}(\mathcal{C}_n)$ for $n = 1, 2, \cdots$. Then the two maps are the exact inverses to each other. Note that both of the above metric spaces are not complete.

Now Theorem 5.1, 5.2 and 6.1 together have the following implication:

THEOREM 7.1. *For every $n = 1, 2, \cdots$, the maps* $\mathbf{MAT} : (\mathcal{D}_n, \mathcal{H}_\partial) \to (\mathcal{M}_n, \mathcal{H}_h)$ *and* $\mathbf{TAM} : (\mathcal{M}_n, \mathcal{H}_h) \to (\mathcal{D}_n, \mathcal{H}_\partial)$, *which are the inverses to each other, are uniformly continuous. In fact, we have*

$$\mathcal{H}_h \left(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)\right) \leq 3 \cdot \mathcal{H}_\partial(\Omega_1, \Omega_2),$$

for every $\Omega_1, \Omega_2 \in \mathcal{D}_n$, and

$$\mathcal{H}_\partial \left(\mathbf{TAM}(M_1), \mathbf{TAM}(M_2)\right) \leq \mathcal{H}_h(M_1, M_2),$$

for every $M_1, M_2 \in \mathcal{M}_n$. In particular, **MAT** *(and thus* **TAM***) is a homeomorphism.*

The above result tells us that, when we introduce the hyperbolic Hausdorff distance, the process of taking the medial axis transform and its inverse reconstruction process can be made to be continuous. This is an important feature, since the continuity is an important requirement for any processes which we want to be under control. Moreover, the uniformity result means that we don't need *a priori* knowledge of the individual domains or the medial axis transforms to do the control, which is certainly another advance compared to the previous result in [6].

Suppose we approximate a given domain with other domains. Then the result of the approximation for their medial axis transforms will be almost the same under the hyperbolic Hausdorff distance. Conversely, if we approximate a given medial axis transform with other medial medial axis transforms under the hyperbolic Hausdorff distance, then the result of the approximation for the reconstructed domains will be exactly the same.

In fact, our bounds, which we saw are sharp in Sections 5 and 6, are good enough to make the maps **MAT** and **TAM** even *isometric* in some important cases. Remember that we defined $\rho_\Omega = \min\{r : (p, r) \in \mathbf{MAT}(\Omega)\}$ for $\Omega \in \mathcal{C}_n$, $n = 1, 2, \cdots$. Note that $\rho_\Omega > 0$ means that $\partial\Omega$ has a relatively smooth shape, *e.g.*, with no sharp corners, *etc.* For every $\Omega \in \mathcal{D}_n$ such that $\rho_\Omega > 0$, we define $\Omega \in \mathcal{O}_\Omega \subset \mathcal{D}_n$ by

$$\mathcal{O}_\Omega = \left\{ \Omega' \in \mathcal{D}_n \,|\, \mathcal{H}_\partial(\Omega', \Omega) < \frac{1}{4}\rho_\Omega,\ \rho_{\Omega'} \geq \rho_\Omega \right\}.$$

Intuitively speaking, $\mathcal{O}_\Omega$ is a set containing the domains which are close to Ω and have reasonable smoothness. Now Theorem 6.2 has the following consequence:

THEOREM 7.2. *Let $n = 1, 2, \cdots$. For every $\Omega \in \mathcal{D}_n$ such that $\rho_\Omega > 0$, $\mathbf{MAT} : (\mathcal{D}_n, \mathcal{H}_\partial) \to (\mathcal{M}_n, \mathcal{H}_h)$ is an isometry on $\mathcal{O}_\Omega$.*

Proof. Suppose $\Omega', \Omega'' \in \mathcal{O}_\Omega$. Then we have $\mathcal{H}_\partial(\Omega', \Omega'') \le \mathcal{H}_\partial(\Omega', \Omega) + \mathcal{H}_\partial(\Omega'', \Omega) < \frac{1}{2}\rho_\Omega \le \frac{1}{2}\cdot\min\{\rho_{\Omega'}, \rho_{\Omega''}\}$. Thus, $\mathcal{H}_h(\mathbf{MAT}(\Omega'), \mathbf{MAT}(\Omega'')) = \mathcal{H}_\partial(\Omega', \Omega'')$ by Theorem 6.2, which completes the proof. ■

Thus, the perturbation of the domains and that of the medial axis transforms are exactly same in quantity, provided that the domains have reasonable smoothness. Here, by the smoothness, we mean, of course, the condition $\rho_\Omega > 0$, and this includes the important and wide class of domains in [4] without sharp corners, which we mentioned in Section 4.

In fact, it can be shown that the hyperbolic Hausdorff distance is a complete metric on the canonical quotient space of $\mathcal{C}_{n,1}$ [7]. By this completeness, we can guarantee that an approximation always leads to a limit, though sometimes this limit may not be a reasonable one. Nevertheless, this may be valuable in many cases.

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After the submission of the manuscipt, the authors found that H. I. Choi, C. Y. Han and J.-H. Yoon [5] had recently introduced a metric which is similar to, but different from the hyperbolic Hausdorff distance.

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