One-sided Stability of Medial Axis Transform

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Abstract. Medial axis transform (MAT) is very sensitive to the noise, in the sense that, even if a shape is perturbed only slightly, the Hausdorff distance between the MATs of the original shape and the perturbed one may be large. But it turns out that MAT is stable, if we view this phenomenon with the one-sided Hausdorff distance, rather than with the two-sided Hausdorff distance. In this paper, we show that, if the original domain is weakly injective, which means that the MAT of the domain has no end point which is the center of an inscribed circle osculating the boundary at only one point, the one-sided Hausdorff distance of the original domain's MAT with respect to that of the perturbed one is bounded linearly with the Hausdorff distance of the perturbation. We also show by example that the linearity of this bound cannot be achieved for the domains which are not weakly injective. In particular, these results apply to the domains with the sharp corners, which were excluded in the past. One consequence of these results is that we can clarify theoretically the notion of extracting "the essential part of the MAT", which is the heart of the existing pruning methods.

1 Introduction

The *medial axis* (**MA**) of a plane domain is defined as the set of the centers of the maximal inscribed circles contained in the given domain. The *medial axis transform* (**MAT**) is defined as the set of all the pairs of the medial axis point and the radius of the corresponding inscribed circle. Because of the additional radius information, **MAT** can be used to reconstruct the original domain. More explicitly, the medial axis transform $\mathbf{MAT}(\Omega)$ and the medial axis $\mathbf{MA}(\Omega)$ of a plane domain Ω is defined by $\mathbf{MAT}(\Omega) = \{ (p,r) \in \mathbb{R}^2 \times [0,\infty) \,|\, B_r(p)$: maximal ball in Ω }, $\mathbf{MA}(\Omega) = \{p \in \mathbb{R}^2 \,|\, \exists r \geq 0,$ s.t. $(p,r) \in \mathbf{MAT}(\Omega)\}$.

Medial axis (transform) is one of the most widely-used tools in shape analysis. It has a natural definition, and has a graph structure which preserves the original shape homotopically [2], [3]. But the medial axis transform has one weak point; It is not stable under the perturbation of the domain [12], [5], [1]. See Figure 6. Even when the domain in (b) is slightly perturbed to the domain in (a) (that is, the Hausdorff distance between the domains in (a) and (b) is small), the **MAT** (**MA**) changes drastically, which results in a large value of the Hausdorff distance between the **MAT**s (**MA**s) of the domains in (a) and (b).

This seemingly unplausible phenomenon can produce a lot of problems, especially in the recognition fields, since the data representing the domains have inevitable noises. So there has been many attempts to reduce the complexity of the **MAT** by "pruning" out what is considered less important, or considered to be caused by the noise [11], [13], [9].

One important observation that can be made from Figure 6 is that the **MAT** (**MA**) in (b) is contained approximately in the **MAT** (**MA**) in (a). In other words, although the two-sided Hausdorff distance of the **MAT**s in (a) and (b) is large, the *one-sided* Hausdorff distance of the **MAT** in (b) with respect to that in (a) is still small.

In this paper, we analyze this phenomenon, and show that **MA** and **MAT** are indeed stable, if we measure the change by the one-sided Hausdorff distance instead of the two-sided Hausdorff distance. We will show that, when a plane domain Ω satisfies a certain smoothness condition which we call the *weak-injectivity*, then the one-sided Hausdorff distance of $\mathbf{MA}(\Omega)$ (*resp.*, $\mathbf{MAT}(\Omega)$) with respect to $\mathbf{MA}(\Omega')$ (*resp.*, $\mathbf{MAT}(\Omega')$) has an upper bound which is *linear* with the Hausdorff distances between Ω, Ω' and between $\partial\Omega$, $\partial\Omega'$ for arbitrary domain Ω'. In particular, the weak-injectivity is shown to be essential for having the linear bound. This result extends the previous one for the *injective* domains [5]; We now can allow the sharp corners in the domains for which the linear one-sided stability is valid.

It turns out that the coefficient of the linear bound grows as the angle θ_Ω (See Section 2) characteristic to a weakly injective domain Ω decreases. An important consequence of this is that we can approximately measure the degree of the "detailed-ness" of a domain Ω by the value θ_Ω. Along with this, we will discuss about the relation between our result and the pruning of **MAT**.

2 Preliminaries

2.1 Normal Domains

Contrary to the common belief, $\mathbf{MAT}(\Omega)$ and $\mathbf{MA}(\Omega)$ may not be graphs with finite structure, unless the original domain Ω satisfies the following rather strong condition [3]: Ω is compact, or equivalently, Ω is closed and bounded, and the boundary $\partial\Omega$ of Ω is a (disjoint) union of finite number of simple closed curves, each of which in turn consists of finite number of real-analytic curve pieces. So we will consider only the domains satisfying this condition, which we call *normal.*

Let Ω be a normal domain. Then, except for some finite number of the special points, the maximal ball $B_r(p)$ for every $P = (p, r) \in \mathbf{MAT}(\Omega)$ has exactly two contact points with the boundary $\partial\Omega$. It is well known that $\mathbf{MA}(\Omega)$ is a C^1 curve around such p in $\mathbb{R}^2$. See Figure 1. We will denote the set of all such *generic* points in $\mathbf{MA}(\Omega)$ by $G(\Omega)$, and, for every $p \in G(\Omega)$, define $0 < \theta(p) \leq \frac{\pi}{2}$ to be the angle between $\overline{pq_1}$ (or equivalently $\overline{pq_2}$) and $\mathbf{MA}(\Omega)$ at p, where q_1, q_2 are the two contact points.

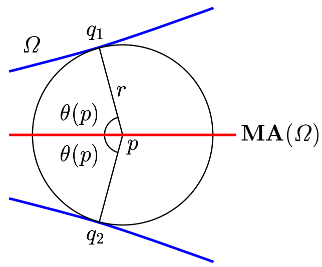

Fig. 1. Local geometry of **MA** around a generic point

Now, for every normal domain Ω, we define $\theta_\Omega = \inf\{\theta(p) : p \in G(\Omega)\}$. Note that $0 \leq \theta_\Omega \leq \frac{\pi}{2}$. We also define $\rho_\Omega = \min\{r : (p, r) \in \mathbf{MAT}(\Omega)\}$, that is, ρ_Ω is the smallest radius of the maximal balls contained in Ω.

We call an end point of **MA** a 1-*prong point*. There are exactly three kinds of the 1-prong points in **MA**, which are depicted in Figure 2; Type (a) is the center of a maximal circle with only one contact point at which the circle osculates the boundary. Type (b) is a sharp corner. Type (c) is a 1-prong point with a contact arc. It is easy to see that $\theta_\Omega = 0$, if and only if $\mathbf{MA}(\Omega)$ has a 1-prong point of the type (a), and $\rho_\Omega = 0$, if and only if $\mathbf{MA}(\Omega)$ has a 1-prong point of the type (b).

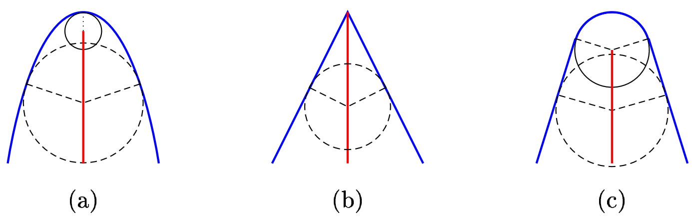

Fig. 2. Three types of 1-prong points

We call a normal domain Ω *injective*, if $\theta_\Omega > 0$ and $\rho_\Omega > 0$, and *weakly injective*, if $\theta_\Omega > 0$. Thus, Ω is injective, if and only if every end point of $\mathbf{MA}(\Omega)$ is of the type (c), and it is weakly injective, if and only if $\mathbf{MA}(\Omega)$ does not have the end points of the type (a). Note that a weakly injective domain may have a sharp corner (*i.e.*, the type (b)), while an injective domain may not.

For more details on the properties of the medial axis transform, see [2], [3], [10].

2.2 Hausdorff Distance : Euclidean *vs.* Hyperbolic

Although sometimes it might be misleading [6], the Hausdorff distance is a natural device to measure the difference between two shapes. Let A and B be two (compact) sets in $\mathbb{R}^2$. The *one-sided Hausdorff distance* of A with respect to B, $\mathcal{H}(A|B)$, is defined by $\mathcal{H}(A|B) = \max_{p \in A} d(p, B)$, where $d(\cdot, \cdot)$ is the usual Euclidean distance. The *(two-sided) Hausdorff distance* between A and B, $\mathcal{H}(A, B)$, is defined by $\mathcal{H}(A, B) = \max\{\mathcal{H}(A|B), \mathcal{H}(B|A)\}$. Note that, whereas the two-sided Hausdorff distance measures the difference between two sets, the one-sided Hausdorff distance measures how approximately one set is contained in another set.

Though the Hausdorff distance is intuitively appealing, it cannot capture well the seemingly unstable behaviour of **MAT** under the perturbation. Recently, there has been the introduction of a new measure called the *hyperbolic Hausdorff distance*, so that **MAT** (and **MA**) becomes stable, if the difference between two **MAT**s is measured by this measure [6] (See Proposition 1 below).

Let $P_1 = (p_1, r_1), P_2 = (p_2, r_2)$ be in $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$, where we denote $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. Then the *hyperbolic distance* $d_h(P_1|P_2)$ *from* P_1 *to* P_2 is defined by $d_h(P_1|P_2) = \max\{0, d(p_1, p_2) - (r_2 - r_1)\}$. Let M_1, M_2 be compact sets in $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. Then the *one-sided hyperbolic Hausdorff distance* $\mathcal{H}_h(M_1|M_2)$ of M_1 with respect to M_2 is defined by $\mathcal{H}_h(M_1|M_2) = \max_{P_1 \in M_1}\{\min_{P_2 \in M_2} d_h(P_1|P_2)\}$, and the *(two-sided) hyperbolic Hausdorff distance* between M_1 and M_2 is defined by $\mathcal{H}_h(M_1, M_2) = \max\{\mathcal{H}_h(M_1|M_2), \mathcal{H}_h(M_2|M_1)\}$.

Now we have the following result which plays an important role in showing the main result (Theorem 1) of this paper.

Proposition 1. **([6])** *For any normal domains Ω_1 and Ω_2, we have*

$$\max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\} \leq \mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)),$$
$$\mathcal{H}_h(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)) \leq 3 \cdot \max\{\mathcal{H}(\Omega_1, \Omega_2), \mathcal{H}(\partial\Omega_1, \partial\Omega_2)\}.$$

3 Perturbation of Weakly Injective Domain

We first review the previous result for the injective domains.

Proposition 2. **(Infinitesimal Perturbation of Injective Domain, [5], [7])** *Let Ω be an injective domain. Then we have*

$$\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) \leq \frac{2}{1 - \cos\theta_\Omega} \cdot \epsilon + o(\epsilon),$$
$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) \leq \frac{\sqrt{4 + (3 - \cos\theta_\Omega)^2}}{1 - \cos\theta_\Omega} \cdot \epsilon + o(\epsilon),$$

for every $\epsilon \geq 0$ and normal domain Ω' with $\max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\} \leq \epsilon$.

We show that, infinitesimally, the one-sided Hausdorff distance of **MAT** (and **MA**) of a weakly injective domain is bounded linearly by the magnitude of the perturbation. Define a function $g : (0, \pi/2] \to \mathbb{R}$ by $g(\theta) = 3\left(1 + \frac{2\sqrt{1+\cos^2\theta}}{1-\cos\theta}\right)$.

Theorem 1. (Infinitesimal Perturbation of Weakly Injective Domain) *Let Ω be a weakly injective domain. Then we have*

$$\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')), \mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) \leq g(\theta_\Omega) \cdot \epsilon + o(\epsilon),$$

for every $\epsilon \geq 0$ and normal domain Ω' with $\max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\} \leq \epsilon$.

Proof. See [7].

Example 1. Let Ω be a weakly injective domain with a sharp corner P_1 depicted as in Figure 3. Let Ω' be the domain obtained by smoothing Ω near P_1 so that $\mathbf{MAT}(\Omega) = \overline{P_2P_3}$. Let $P_i = (p_i, r_i)$ for $i = 1, 2, 3$. Note that $\mathcal{H}(\Omega, \Omega') = \mathcal{H}(\partial\Omega, \partial\Omega') = \epsilon$, and $\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) = d(p_1, p_2) = \frac{1}{1-\cos\theta_\Omega} \cdot \epsilon$, $\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) = d(P_1, P_2) = \frac{\sqrt{1+\cos^2\theta_\Omega}}{1-\cos\theta_\Omega} \cdot \epsilon$.

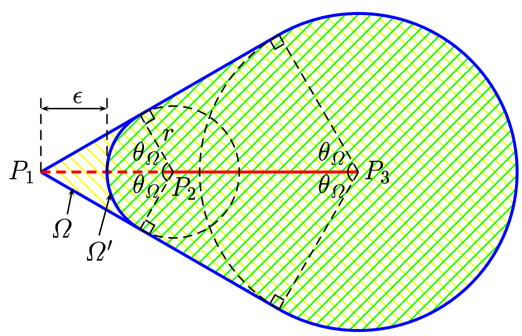

Fig. 3. One-sided stability for weakly injective domain

This example shows that the factor $\frac{1}{1-\cos\theta_\Omega}$ in $g(\theta_\Omega)$, which blows up as $\theta_\Omega \to 0$, is indeed unavoidable. One important consequence is that the class of the weakly injective domains is the largest possible class for which we have a linear bound for the one-sided Hausdorff distance of **MAT** (and **MA**) with respect to the perturbation.

4 Illustrating Examples

Now we will consider a few examples, and calculate explicitly the constants θ_Ω and $g(\theta_\Omega)$ for each of them.

Example 2. Consider an equilateral triangle and a star-shaped domain depicted respectively as in Figure 4 (a) and (b). Note that $\theta_\Omega = \frac{\pi}{3}$ for (a), and $\theta_\Omega = \frac{\pi}{5}$ for (b). So $g(\theta_\Omega) = 3(1 + 2\sqrt{5}) = 16.416408\ldots$ for (a), and $g(\theta_\Omega) = 43.410203\ldots$ for (b).

Example 3. Consider the rectangular domains with the constant widths depicted as in Figure 5. Note that $\theta_\Omega = \frac{\pi}{4}$, and hence $g(\theta_\Omega) = 3\left(1 + 2\sqrt{3}(1 + \sqrt{2})\right) = 28.089243\ldots$ for all cases.

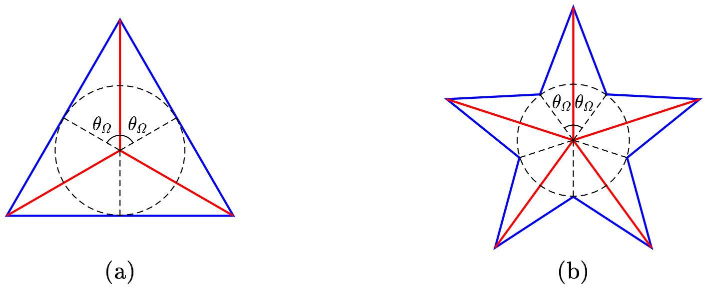

Fig. 4. (a) Equilateral triangle; $\theta_\Omega = \frac{\pi}{3}$ and $g(\theta_\Omega) = 16.416408\ldots$, (b) five-sided star; $\theta_\Omega = \frac{\pi}{5}$ and $g(\theta_\Omega) = 43.410203\ldots$.

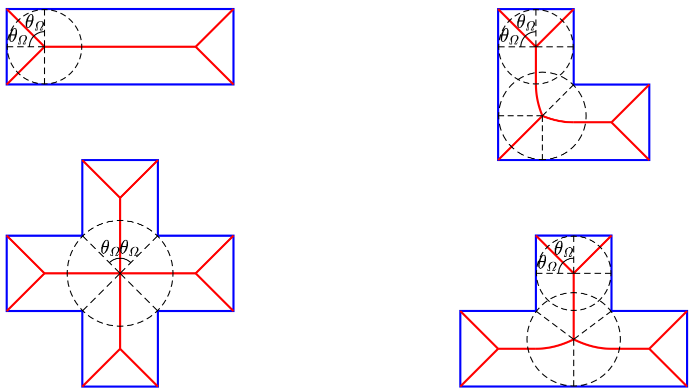

Fig. 5. Rectangular domains with the constant widths; For all cases, we have $\theta_\Omega = \frac{\pi}{4}$ and $g(\theta_\Omega) = 28.089243\ldots$.

5 The Essential Part of the MAT : Relation to Pruning

Theorem 1 together with Example 1 says that the angle θ_Ω is an important quantity reflecting the degree of the "detailed-ness" of a domain Ω. The smaller θ_Ω becomes, the finer approximation, that is, the smaller $\max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}$ is needed for $\mathbf{MAT}(\Omega')$ and $\mathbf{MA}(\Omega')$ of another domain Ω' to contain (approximately) $\mathbf{MAT}(\Omega)$ and $\mathbf{MA}(\Omega)$ respectively.

Suppose we perturb a weakly injective domain with domains which are also weakly injective. In this case, **MAT** and **MA** become stable under the "two-sided" Hausdorff distance. In particular, we have the following corollary:

Corollary 1. (Approximation by Weakly Injective Domains) *Let Ω be a normal domain, and let Ω_1 and Ω_2 be two weakly injective domains such that* $\max\{\mathcal{H}(\Omega_i, \Omega), \mathcal{H}(\partial\Omega_i, \partial\Omega)\} \le \epsilon$ *for $i = 1, 2$. Let $\theta = \min\{\theta_{\Omega_1}, \theta_{\Omega_2}\}$. Then we have* $\mathcal{H}\left(\mathbf{MAT}(\Omega_1), \mathbf{MAT}(\Omega_2)\right), \mathcal{H}\left(\mathbf{MA}(\Omega_1), \mathbf{MA}(\Omega_2)\right) \le 2g(\theta)\cdot\epsilon + o(\epsilon)$.

Thus, the effects on **MAT** and **MA** which arise from the choice of the weakly injective domains to approximate a normal domain is relatively small. So the **MAT** and the **MA** of an approximating weakly injective domain may be considered as a common part among all the other approximations with the same θ_Ω, and hence, an essential part of the original **MAT** and **MA** with the fine details determined by the value of θ_Ω. This suggests that, by approximating a given normal domain with the weakly injective domains, it is possible to extract approximately the most essential part of the original **MAT** and **MA**, which is the main objective of the existing pruning methods.

For example, Let Ω' be the original domain as shown in Figure 6 (a), whose **MA** has much unilluminating parts due to its noisy boundary. We approximate Ω' by a weakly injective domain Ω shown in Figure 6 (b), which has relatively simpler **MA**.

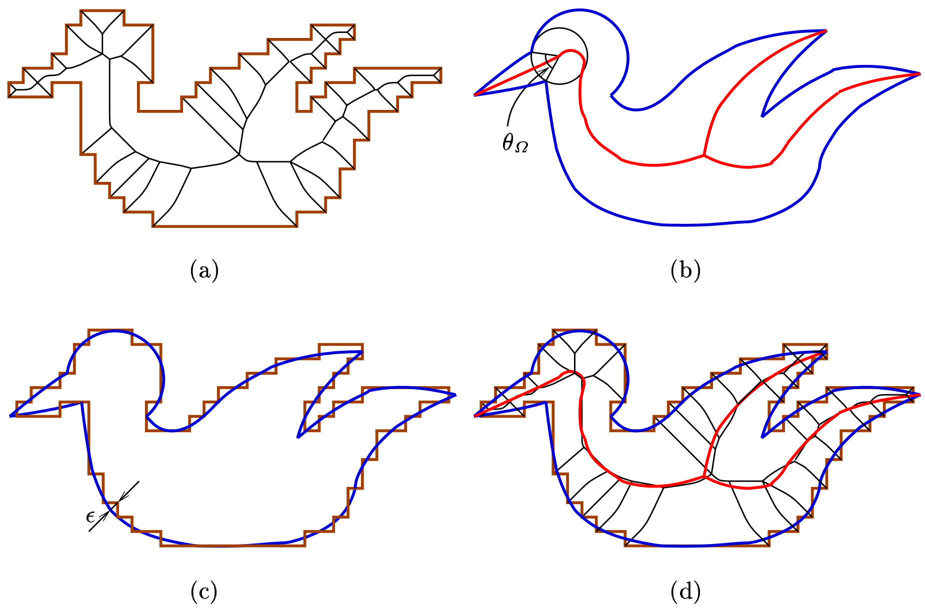

Fig. 6. Pruning **MAT**: (a) The original normal domain Ω' with its **MA**; (b) The approximating weakly injective domain Ω with its **MA**. Note that the sharp corners are allowed; (c) The Hausdorff distance between Ω and Ω', Here, $\epsilon = \max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\}$; (d) Comparison of $\mathbf{MA}(\Omega)$ and $\mathbf{MA}(\Omega')$. Note that $\mathbf{MA}(\Omega)$ captures an essential part of $\mathbf{MA}(\Omega')$, while simplifying $\mathbf{MA}(\Omega')$.

By Theorem 1, we can get a bound on how approximately $\mathbf{MA}(\Omega)$ is contained in $\mathbf{MA}(\Omega')$, or how faithfully $\mathbf{MA}(\Omega)$ approximates parts of $\mathbf{MA}(\Omega')$, from the constant θ_Ω. Moreover, from Corollary 1, we see that $\mathbf{MA}(\Omega)$ is *the* *essential part* of $\mathbf{MA}(\Omega')$ up to the bound in Corollary 1. In overall, by computing the much simpler $\mathbf{MA}(\Omega)$, we can get the essential part (within the bounds) of $\mathbf{MA}(\Omega')$ without ever computing $\mathbf{MA}(\Omega')$ at all. See [4] for the computation of **MAT** for domains with the free-form boundaries.

Of course, there still remains the problem of how to approximate/smooth the original noisy boundary. But we claim that, whatever method is used, our bounds can serve as a theoretical guarantee of the correctness of the approximation/pruning, which is absent in most of the existing pruning schemes.

One notable advance from [5] is that we can now allow the sharp corners for the approximating domain. For the discussion of using the general normal domains for the approximation, See [8].

References

1. J. August, K. Siddiqi and S. W. Zucker, "Ligature instabilities and the perceptual organization of shape," *Computer Vision and Image Understanding*, vol. 76, no. 3, pp. 231–243, Dec. 1999.
2. H. Blum, "A transformation for extracting new descriptors of shape," *Proc. Symp. Models for the Perception of Speech and Visual Form* (W.W. Dunn, ed.), MIT Press, Cambridge, MA, pp. 362–380, 1967.
3. H. I. Choi, S. W. Choi and H. P. Moon, "Mathematical theory of medial axis transform," *Pacific J. Math.*, vol. 181, no. 1, pp. 57–88, Nov. 1997.
4. H. I. Choi, S. W. Choi, H. P. Moon and N.-S. Wee, "New algorithm for medial axis transform of plane domain," *Graphical Models and Image Processing*, vol. 59, no. 6, pp. 463–483, 1997.
5. S. W. Choi and S.-W. Lee, "Stability analysis of medial axis transform," *Proc. 15th ICPR*, (Barcelona, Spain), vol 3, pp. 139–142, Sept. 2000.
6. S. W. Choi and H.-P. Seidel, "Hyperbolic Hausdorff distance for medial axis transform," Research Report, MPI-I-2000-4-003, 2000.
7. S. W. Choi and H.-P. Seidel, "Linear one-sided stability of MAT for weakly injective domain," Research Report, MPI-I-2001-4-004, 2001.
8. S. W. Choi and H.-P. Seidel, "One-sided stability of MAT and its applications," *preprint*, 2001.
9. M. P. Deseilligny, G. Stamon and C. Y. Suen, "Veinerization: A new shape descriptor for flexible skeletonization," *IEEE Trans. PAMI*, vol. 20, no. 5, pp. 505–521, May 1998.
10. P. J. Giblin and B. B. Kimia, "On the local form and transitions of symmetry sets, medial axes, and shocks," *Proc. 7th ICCV*, (Kerkyra, Greece), pp. 385–391, Sept. 1999.
11. F. Mokhtarian and A. K. Mackworth, "A theory of multiscale, curvature-based shape representation for planar curves," *IEEE Trans. PAMI*, vol. 14, no. 8, pp. 789–805, Aug. 1992.
12. U. Montanari, "A method for obtaining skeletons using a quasi-Euclidean distance," *J. of the ACM*, vol. 18, pp. 600–624, 1968.
13. D. Shaked and A. M. Bruckstein, "Pruning medial axes," *Computer Vision and Image Understanding*, vol. 69, no. 2, pp. 156–169, Feb. 1998.