Finite energy global well-posedness of the (3+1)-dimensional Yang-Mills equations using a novel Yang-Mill heat flow gauge

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Abstract

In this thesis, we propose a novel choice of gauge for the *Yang-Mills equations* on the Minkowski space $\mathbb{R}^{1+d}$. A crucial ingredient is the associated *Yang-Mills heat flow*. Unlike the previous approaches (as in [15] and [33]), the new gauge is applicable for large data, while the special analytic structure of the Yang-Mills equations is still manifest.

As the first application of the new approach, we shall give new proofs of H^1_x *local well-posedness* and *finite energy global well-posedness* of the Yang-Mills equations on $\mathbb{R}^{1+3}$. These are classical results first proved by S. Klainerman and M. Machedon [15] using the method of local Coulomb gauges, which had been difficult to extend to other settings. As our approach does not possess its drawbacks (in particular the use of Uhlenbeck's lemma [37] is avoided), it is expected to be more robust and easily applicable to other problems

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Contents

Chapter 1

Introduction

In this thesis, we present a novel approach to the problem of gauge choice for the *Yang-Mills equations*

$$\mathbf{D}^{\mu} F_{\nu\mu} = 0$$

on the Minkowski space $\mathbb{R}^{1+d}$ with a non-abelian structural group $\mathfrak{G}$. An essential ingredient of our approach is the celebrated *Yang-Mills heat flow*

$$\partial_s A_i = \mathbf{D}^{\ell} F_{\ell i},$$

which, first proposed by Donaldson [8], is a well-studied equation in the field of geometric analysis. (See [27], [4] etc.) The idea of using the associated heat flow to deal with the problem of gauge choice had been first put forth by Tao [34], [35] in the context of energy critical wave maps on $\mathbb{R}^{1+2}$, and has been also adapted to the related energy critical Schrödinger maps by [2], [29], [30].

The novel approach using the Yang-Mills heat flow does not possess the drawbacks of the previous choices of gauge; as such, it is expected to be more robust and easily applicable to other problems. As a first application, we shall give in this thesis new proofs of H^1_x *local well-posedness* and *finite energy global well-posedness* of the Yang-Mills equations on $\mathbb{R}^{1+3}$, which have been proved by S. Klainerman and M. Machedon [15] using a different method.

1.1 Background: The Yang-Mills equations on $\mathbb{R}^{1+d}$

Consider the Minkowski space $\mathbb{R}^{1+d}$ with $d \geq 1$, equipped with the Minkowski metric of signature $(- + + \cdots +)$. All tensorial indices will be raised and lowered by using the Minkowski metric.

Moreover, we shall adopt the Einstein summation convention of summing up repeated upper and lower indices. Greek indices, such as μ, ν, λ, will run over $x^0, x^1, x^2, \dots, x^d$, whereas latin indices, such as i, j, k, ℓ, will run *only* over the spatial indices $x^1, x^2, \dots, x^d$. We shall often use t for x^0.

Let $\mathfrak{G}$ be a Lie group with the Lie algebra $\mathfrak{g}$, which is equipped with a bi-invariant inner product[1] $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to [0, \infty)$. The bi-invariant inner produt will be used to define the absolute value of elements in $\mathfrak{g}$, and moreover will be used in turn to define the L^p_x-norm of $\mathfrak{g}$-valued functions.

For simplicity, we shall assume that $\mathfrak{G}$ is a matrix group. An explicit example which is useful to keep in mind is the group of special unitary matrices $\mathfrak{G} = \mathrm{SU}(n)$, in which case $\mathfrak{g} = \mathfrak{su}(n)$ is the set of complex traceless anti-hermitian matrices and the bi-invariant metric is given by $(A, B) := \mathrm{tr}(AB^\star)$.

Consider a $\mathfrak{g}$-valued 1-form A_μ on $\mathbb{R}^{1+d}$, which we shall call a *connection 1-form*, or *connection coefficients*[2]. For any $\mathfrak{g}$-valued tensor field B on $\mathbb{R}^{1+d}$, we define the associated *covariant derivative* $\mathbf{D} = {}^{(A)}\mathbf{D}$ by

$$\mathbf{D}_\mu B := \partial_\mu B + [A_\mu, B], \quad \mu = 0, 1, 2, \dots, d$$

where ∂_μ is the ordinary directional derivative on $\mathbb{R}^{1+d}$.

The commutator of two covariant derivatives gives rise to a $\mathfrak{g}$-valued 2-form $F_{\mu\nu}$, called the *curvature 2-form* associated to A_μ, in the following fashion.

$$\mathbf{D}_\mu \mathbf{D}_\nu B - \mathbf{D}_\nu \mathbf{D}_\mu B = [F_{\mu\nu}, B].$$

Using the definition, it is not difficult to verify that $F_{\mu\nu}$ is expressed directly in terms of A_μ by the formula

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

From the way $F_{\mu\nu}$ arises from A_μ, it follows that the following *Bianchi identity* holds.

$$\mathbf{D}_\mu F_{\nu\lambda} + \mathbf{D}_\nu F_{\lambda\nu} + \mathbf{D}_\lambda F_{\mu\nu} = 0. \tag{Bianchi}$$

A connection 1-form A_μ is said to be a solution to the *Yang-Mills equations* (YM) on $\mathbb{R}^{1+d}$ if

[1] A *bi-invariant inner product* is an inner product on $\mathfrak{g}$ invariant under the adjoint map. A sufficient condition for the existence of such an inner product is that $\mathfrak{G}$ is a product of an abelian and a semi-simple Lie groups.

[2] We take a fairly pragmatic point of view towards the definitions of geometric concepts (such as connection and curvature), for the sake of simplicity. For more information on the geometric background of the concepts introduced here (involving principal bundles, associated vector bundles etc.), we recommend the reader the standard references [3], [21], [22] etc.

the following equation holds for $\nu = 0, 1, 2, \ldots, d$.

$$\mathbf{D}^\mu F_{\mu\nu} = 0. \tag{YM}$$

Note the similarity of (Bianchi) and (YM) with the Maxwell equations $\mathrm{d}F = 0$ and $\partial^\mu F_{\nu\mu} = 0$. In fact, the Maxwell equations are a special case of (YM) in the case $\mathfrak{G} = \mathrm{SU}(1)$.

An essential feature of (YM) is the *gauge structure*, to which we turn now. Let U be a (smooth) $\mathfrak{G}$-valued function on $\mathbb{R}^{1+d}$. This U may act on $A, \mathbf{D}, F$ as a *gauge transform* according to the following rules:

$$\widetilde{A}_\mu = UA_\mu U^{-1} - \partial_\mu U U^{-1}, \qquad \widetilde{\mathbf{D}}_\mu = U\mathbf{D}_\mu U^{-1}, \qquad \widetilde{F}_{\mu\nu} = UF_{\mu\nu}U^{-1}.$$

If a $\mathfrak{g}$-valued tensor transforms in the fashion $\widetilde{B} = UBU^{-1}$, then we say that it is *gauge covariant*, or *covariant under gauge transforms*. Note that the curvature 2-form is gauge covariant. Given a gauge covariant B, its covariant derivative $\mathbf{D}_\mu B$ is also gauge covariant, as the following formula shows:

$$\widetilde{\mathbf{D}}_\mu \widetilde{B} = U\mathbf{D}_\mu B U^{-1}.$$

Due to bi-invariance, we furthermore have $(\widetilde{B}, \widetilde{B}) = (B, B)$.

Note that (YM) is evidently covariant under gauge transforms. It has the implication that a solution to (YM) makes sense only as a class of gauge equivalent connection 1-forms. Accordingly, we make the following definition.

Definition 1.1.1. A *classical solution* to (YM) is a class of gauge equivalent smooth connection 1-forms A satisfying (YM). A *generalized solution* to (YM) is defined to be a class of gauge equivalent connection 1-forms A for which there exists a sufficiently smooth representative A which satisfies (YM) in the sense of distributions.

A choice of a particular representative will be referred to as a *gauge choice*. A gauge is usually chosen by imposing a condition, called a *gauge condition*, on the representative. Some classical examples of gauge conditions are the *temporal gauge* $A_0 = 0$, or the *Coulomb gauge* $\partial^\ell A_\ell = 0$, where ℓ, being a latin index, is summed only over the spatial indices $1, 2, \ldots, d$.

In this thesis, we shall study the Cauchy problem associated to (YM). As in the case of Maxwell equations, the initial data set consists of $(\mathring{A}_i, \mathring{E}_i)$ for $i = 1, 2, \ldots, d$, where $\mathring{A}_i = A_i(t = 0)$ (magnetic potential) and $\mathring{E}_i = F_{0i}(t = 0)$ (electric field). Note that one component of (YM), namely $\nu = 0$, imposes a nontrivial constraint on the possible initial data set $(\mathring{A}_i, \mathring{E}_i)$:

$$\partial^\ell \mathring{E}_\ell + [\mathring{A}^\ell, \mathring{E}_\ell] = 0. \tag{1.1.1}$$

This is called the (Yang-Mills) *constraint equation*.

The system (YM) possesses a positive definite conserved quantity $\mathbf{E}[F_{\mu\nu}](t)$, called the *conserved energy* of $F_{\mu\nu}$ at time t, defined by

$$\mathbf{E}[F_{\mu\nu}(t)] := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{\ell=1,2,\dots,d} (F_{0\ell}(t,x), F_{0\ell}(t,x)) + \sum_{k,\ell=1,2,\dots,d} (F_{k\ell}(t,x), F_{k\ell}(t,x))\,\mathrm{d}x \tag{1.1.2}$$

Note that (YM) remain invariant under the scaling

$$x^\alpha \to \lambda x^\alpha, \quad A \to \lambda^{-1} A, \quad F \to \lambda^{-2} F. \tag{1.1.3}$$

The scaling critical L^2_x Sobolev regularity is $\gamma_c(d) = \frac{d-2}{2}$, i.e. the $\dot{H}^{(d-2)/2}$-norm is invariant under the above scaling. Comparing this with the energy regularity $\gamma_e = 1$, we see that (YM) is *energy sub-critical* for $d \geq 3$, *critical* for $d = 4$ and *super-critical* for $d \geq 5$.

1.2 The problem of gauge choice and previous approaches

We shall begin with a discussion on the importance and difficulty of the problem of choosing an appropriate gauge in the study of the Yang-Mills equations. Our discussion will revolve around the following concrete example, which is a classical result of Klainerman-Machedon [15] in $d = 3$, stated in a simplified form.

Theorem 1.2.1 (Klainerman-Machedon [15])**.** *Consider the Yang-Mills equations* (YM) *on* $\mathbb{R}^{1+3}$*. Let* $(\mathring{A}_i, \mathring{E}_i)$ *be a smooth initial data set satisfying the constraint equation* (1.1.1)*. Consider the Cauchy problem for these data.*

1. (H^1_x local well-posedness) *There exists a classical solution* A_μ *to the Cauchy problem for* (YM) *on a time interval* $(-T^\star, T^\star)$*, where* $T^\star > 0$ *depends only on* $\|\mathring{A}_i\|_{\dot{H}^1_x}, \|\mathring{E}_i\|_{L^2_x}$*. The solution is unique in an appropriate gauge, e.g. in the temporal gauge* $A_0 = 0$*.*

2. (Finite energy global well-posedness) *Furthermore, if the initial data set possesses finite conserved energy* $\mathbf{E}(0) < \infty$*, then the solution* A_μ *extends globally.*

After explaining the importance of gauge choice for proving Theorem 1.2.1, we shall briefly summarize the previous approaches to the problem of gauge choice, namely the *(local) Coulomb gauge* [15] and the *temporal gauge* [28], [9], [33]. It will be seen that each has its own set of drawbacks, which in fact makes Theorem 1.2.1 the best result so far in terms of the regularity condition on the initial data, concerning local and global well-posedness of (YM) for possibly *large*[3] initial data. This will motivate us to propose a novel approach to the problem of gauge choice in §1.3

Importance of gauge choice

There are at least three reasons why a judicious choice of gauge is needed in order to prove Theorem 1.2.1:

A. To reveal the *hyperbolicity*[4] of (YM);

B. To exhibit the 'special structure' (namely, the *null structure*) of (YM);

C. To utilize the conserved energy $\mathbf{E}(t)$ to control $\|\partial_x A_i(t)\|_{L^2_x}$.

In the future, we shall refer to these as Issues A, B and C. Let us discuss each of them further.

Concerning Issue A, observe that the top order terms of (YM) at the level of A_μ have the form

$$\Box A_\nu - \partial^\mu \partial_\nu A_\mu = (\text{lower order terms}).$$

In an arbitrary gauge, due to the presence of the undesirable second order term $-\partial^\mu \partial_\nu A_\mu$, it is even unclear whether the equation for A_μ is hyperbolic (i.e. a wave equation). Therefore, in order to study (YM) as a hyperbolic system of equations, the gauge should be chosen, at the very least, in a way to reveal the hyperbolicity of (YM). We remark that this is analogous to the issue that the Yang-Mills heat flow is only *weakly-parabolic*, to be discussed in §1.4.

Resolution of Issue A suffices to prove local well-posedness of (YM) for sufficiently regular initial data (see [28], [9]). However, it is still insufficient for Theorem 1.2.1, because of Issue B. After an appropriate choice of gauge, which does not have to be precise for the purpose of this heuristic

[3]We remark that there are better results in the case of *small* initial data, for the reasons to be explained below. See [33].

[4]In this work, we shall interpret the notion of *hyperbolicity* in a practical fashion and say that a PDE is *hyperbolic* if its principal part is the wave equation. By 'revealing the hyperbolicity of (YM)', we mean reducing the dynamics of the Yang-Mills system to that of a system of wave equations. As we shall see below, this may involve solving elliptic, parabolic and/or transport equations for some variables.

discussion, the wave equation for the connection 1-form A satisfying (YM) becomes of the form

$$\Box A = \mathcal{O}(A, \partial A) + (\text{cubic and higher}) \tag{1.2.1}$$

where $\mathcal{O}(A, \partial A)$ refers to a linear combination of bilinear terms in A and $\partial_{t,x} A$.

At this point, we encounter an important difficulty of proving Theorem 1.2.1: Strichartz estimates (barely, but in an essential way) fall short of proving H^1_x local well-posedness of (1.2.1), due to the well-known failure of the endpoint $L^2_t L^\infty_x$ estimate on $\mathbb{R}^{1+3}$. In fact, a counterexample, given by Lindblad [23], demonstrates that even local existence may fail at this regularity for a general equation of the form (1.2.1). Such considerations indicate that a proof of Theorem 1.2.1 necessarily has to exploit the 'special structure' of (YM), which distinguishes (YM) from a general system of semi-linear equations of the similar form. As we shall see in sequel, this 'special structure' will go under the name *null form*. Since the precise form of the wave equation for the connection 1-form A is highly dependent on the gauge, it is crucial to make a suitable choice of gauge so as to reveal the structure needed to establish Theorem 1.2.1.

Once Issues A and B are addressed, low regularity local well-posedness of (YM) (in particular, Statement 1 of Theorem 1.2.1) can, in principle, be established. However, yet another difficulty remains in proving Statement 2 of Theorem 1.2.1, namely Issue C. Had the conserved energy $\mathbf{E}(t)$ directly controlled $\|\partial_x A_i(t)\|_{L^2_x}$, finite energy global well-posedness would have followed immediately from H^1_x local well-posedness. However, recalling the expression for the conserved energy

$$\mathbf{E}(t) = \frac{1}{2} \sum_{\mu,\nu} \|\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\|^2_{L^2_x},$$

we see that in an arbitrary gauge, $\mathbf{E}(t)$ can only control a part of the full gradient of A_i: Namely, the curl of A_i, or $\|\partial_i A_j - \partial_j A_i\|_{L^2_x}$. Therefore, in order to prove Statement 2 of Theorem 1.2.1 as well, the chosen gauge must have a structure which allows for utilizing $\mathbf{E}(t)$ to control the L^2_x norm of the full gradient $\partial_x A_i(t)$.

Approach using the (local) Coulomb gauge

We shall now discuss the approach of Klainerman-Machedon [15] using the *local Coulomb gauge*. As we shall see, this approach addresses all of the issues A–C, but possesses the drawback of requiring localization in space-time, causing technical difficulties on the boundaries.

A key observation of Klainerman-Machedon [15] (which in fact goes back to the previous work

[14] of Klainerman-Machedon on the related Maxwell-Klein-Gordon equations) was that under the (global) *Coulomb gauge* $\partial^\ell A_\ell = 0$ imposed everywhere on $\mathbb{R}^{1+3}$, Issues A and B are simultaneously resolved. That is:

- After solving elliptic equations for A_0 and $\partial_0 A_0$, (YM) reduces to a system of wave equations for A_i, and
- The most dangerous quadratic nonlinearities of the wave equations can be shown to be composed of *null forms*.

More precisely, the wave equation for A_i takes the form

$$\Box A_i = \mathcal{Q}(|\partial_x|^{-1} A, A) + |\partial_x|^{-1} \mathcal{Q}(A, A) + (\text{Less dangerous terms}),$$

where each $\mathcal{Q}$ is a linear combination of bilinear forms

$$Q_{jk}(\phi_1, \phi_2) = \partial_j \phi_1 \partial_k \phi_2 - \partial_k \phi_1 \partial_j \phi_2, \quad 1 \leq j < k \leq 3,$$

which are particular examples of a *null form*, introduced by Klainerman [12] and Christodoulou [5] in the context of small data global existence problem for nonlinear wave equations, and first used by Klainerman-Machedon [13] in the context of low regularity well-posedness. Improved estimates are available for such class of bilinear interactions (see [13], [16] etc.), and therefore the desired local well-posedness can be proved.

The Coulomb gauge has an additional benefit that $\|\partial_x A_i(t)\|_{L^2_x}$ may be estimated by $\mathbf{E}(t)$ (provided that A_i is sufficiently regular to start with), as the the Coulomb gauge condition $\partial^\ell A_\ell = 0$ sets the part of $\partial_x A_i$ which is not controlled by $\mathbf{E}(t)$ (namely the *divergence* of A, or $\partial^\ell A_\ell$, according to Hodge decomposition) to be exactly zero. In other words, the Coulomb gauge settles Issue C as well.

Unfortunately, when the structural group $\mathfrak{G}$ is *non-abelian*, there is a fundamental difficulty in imposing the Coulomb gauge globally in space (i.e. on $\mathbb{R}^3$ for each fixed t). Roughly speaking, it is because when $\mathfrak{G}$ is non-abelian, a gauge transform into the Coulomb gauge is given as a solution to a nonlinear elliptic system of PDEs, for which no good regularity theory is available in the large[5]. A closely related phenomenon is *the Gribov ambiguity* [11], which asserts non-uniqueness

[5]In fact, it is possible to show, by a variational argument, that any $A_i \in L^2_x$ may be gauge transformed to a weak solution $\widetilde{A} \in L^2_x$ to the Coulomb gauge equation $\partial^\ell \widetilde{A}_\ell = 0$; see [6]. The problem is that no further regularity of the gauge transform and $\widetilde{A}$ may be inferred, due to the lack of an appropriate regularity theory.

of representative satisfying the Coulomb gauge equation $\partial^\ell A_\ell = 0$ in some equivalence class of connection 1-forms on $\mathbb{R}^3$ when $\mathfrak{G}$ is non-abelian.

At a more technical level, this difficulty manifests in the fact that *Uhlenbeck's lemma* [37], which is a standard result asserting the existence of a gauge transform (possessing sufficient regularity) into the Coulomb gauge, requires the the curvature F to be small in $L^{3/2}_x$. Note that this norm is invariant under the scaling (1.1.3), and therefore cannot be assumed to be small by scaling, unlike the energy $\mathbf{E}[\overline{\mathbf{F}}]$. To get around this problem, the authors of [15] work in what they call *local Coulomb gauges* in small domains of dependence (in which the required norm of F can be assumed small), and glue the local solutions together by exploiting the finite speed of propagation. The execution of this strategy is quite involved due to the presence of the constraint equations (1.1.1). In particular, it requires a delicate boundary condition for $\Box A_i$ in order to mesh the analyses of the elliptic and hyperbolic equations arising from (YM) in the local Coulomb gauge.

Approach using the temporal gauge

A different route to the problem of gauge choice in the context of low regularity well-posedness has been suggested by Tao in his paper [33], where he proved H^s_x local well-posedness for $s > 3/4$ (thus going even below the energy regularity) by working in the *temporal gauge* $A_0 = 0$, under the restriction that the $H^s_x \times H^{s-1}_x$ norm of $(\mathring{A}_i, \mathring{E}_i)$ is *small*. This gauge has the advantage of being easy to impose globally (as gauge transforms into the temporal gauge can be found by solving an ODE), and thus does not have the problem that the Coulomb gauge possesses. Indeed, it had been used by other authors, including Segal [28] and Eardley-Moncrief [9], to prove local and global well-posedness of (YM) for (large) initial data with higher regularity (namely, $s \geq 2$). To reiterate this discussion in our framework, the temporal gauge 'essentially' resolves Issues A and B raised above[6].

However, as indicated earlier, this gauge possesses the drawback that it fails to cope with initial data sets with a large H^s_x norm, when $3/4 < s \leq 1$[7]. Moreover, another drawback is that it is unclear how to deal with Issue C, namely how $\|\partial_x A_i(t)\|_{L^2_x}$ may be controlled for every t using the conserved energy $\mathbf{E}$.

[6]Note that Issue B is not addressed fully in the sense that smallness of the initial data is needed.

[7]One reason is that it still relies on a Uhlenbeck-type lemma to set $\partial^\ell A_\ell = 0$ at $t = 0$, which requires some sort of smallness of the initial data. There is also a technical difficulty in the Picard iteration argument which does not allow one to use the smallness of the length of the time-interval; ultimately, this originates from the presence of a time derivative on the right-hand side of the equation $\partial_t(\partial^\ell A_\ell) = -[A^\ell, \partial_t A_\ell]$ (which is equivalent to the equation $\mathbf{D}^\ell F_{\ell 0} = 0$). See [33] for more details.

1.3 Main idea of the novel approach

The purpose of this thesis is to present a novel approach to the problem of gauge choice which does not possess the drawbacks of the previous methods. As such, this approach does not involve localization in space-time and works well for large initial data. Nevertheless, it is (at the very least) as effective as the previous choices of gauge, as we shall see that it addresses all of the issues A–C discussed above. As the first demonstration of the power of the novel approach, we shall provide a new proof of Theorem 1.2.1. (See the Main Local and Global Well-Posedness Theorems in §1.7.)

Heuristically speaking, the key idea of the novel approach is to *'smooth out'* the problem at hand in a *'geometric fashion'*. The expectation is that the problem of gauge choice for the 'smoothed out problem' would be much easier thanks to the additional regularity. All the difficulties, then, are shifted to the problem of controlling the error generated by the smoothing procedure. That this is possible for a certain choice of smoothing procedure, based on a geometric (weakly-)parabolic PDE called the *Yang-Mills heat flow*, is the main assertion of this thesis.

In the following three sections (§1.4 – §1.6), we shall discuss how the novel approach deals with Issues A–C listed above. After a discussion on the Yang-Mills heat flow in §1.4, we shall summarize the main ideas in the proof of the local well-posedness theorem in §1.5, in which we shall explain how Issues A and B are resolved. Then an overview of the main ideas of behind our proof of the global well-posedness theorem will be given in §1.6, addressing Issue C.

Remark 1.3.1. The present work advances a relatively new idea in the field of hyperbolic PDEs, which is to use a geometric parabolic equation to better understand a hyperbolic equation. To the author's knowledge, the first instance of this idea occurred in the work of Klainerman-Rodnianski [18], in which the linear heat equation on a compact 2-manifold was used to develop an invariant form of Littlewood-Paley theory. This was applied in [17] and [19] to study the causal geometry of solutions to the Einstein's equations under very weak hypotheses.

This idea was carried further by Tao [34], who proposed to use a nonlinear geometric heat flow to deal with the problem of gauge choice in the context of the energy critical wave map problem. This was put into use in a series of preprints [35] to develop a large energy theory of wave maps into a hyperbolic space $\mathbb{H}^n$. In this setting, one begins by solving the associated heat flow, in this case the *harmonic map flow*, starting from a wave map restricted to a fixed t-slice. Then the key idea is that the harmonic map flow converges (under appropriate conditions) to a single point, same for every t, in the target as the heat parameter goes to ∞. For this trivial map at infinity, the canonical choice of gauge is clear[8]; this choice is then parallel-transported back along the harmonic map flow. The resulting gauge is dubbed the *caloric gauge*. This gauge proved to be quite useful, and the use of such gauge has also been successfully extended to the related problem of energy critical Schrödinger maps as well, through the works [2], [29] and [30].

1.4 The Yang-Mills heat flow

Before delving into a more detailed exposition of our approach, let us first introduce the *Yang-Mills Heat Flow* (or (YMHF) in short), which will play an important role in this thesis.

Let us revert to the general setting of $\mathbb{R}^{1+d}$. Consider a spatial connection 1-form $A_i(s)$ ($i = 1, 2, \ldots, d$) on $\mathbb{R}^d$ parametrized by $s \in [0, s_0]$ ($s_0 > 0$). We say that $A_i(s)$ is a *Yang-Mills heat flow* if it satisfies the equation

$$\partial_s A_i = \mathbf{D}^\ell F_{\ell i}, \quad i = 1, 2, \ldots, d. \tag{YMHF}$$

First introduced by Donaldson [8], the Yang-Mills heat flow is the gradient flow for the *Yang-Mills energy* on $\mathbb{R}^d$ (also referred to as the *magnetic energy*)

$$\mathbf{B}[F] := \frac{1}{2} \sum_{1 \leq i < j \leq d} \|F_{ij}\|^2_{L^2_x}, \tag{1.4.1}$$

and plays an important role in differential geometry. It has been a subject of an extensive research by itself; see, for example, [8], [27], [4] etc.

Our intention is to use (YMHF) as a geometric smoothing device for (YM). One must be careful, however, since (YMHF) is *not* strictly parabolic as it stands at the level of A_i. Indeed, expanding (YMHF) in terms of A_i, the top order terms look like

$$\partial_s A_i = \triangle A_i - \partial^\ell \partial_i A_\ell + \text{(lower order terms)},$$

where $\triangle A_i - \partial_i \partial^\ell A_\ell$ possesses a non-trivial kernel (any $A_i = \partial_i \phi$, for ϕ a $\mathfrak{g}$-valued function). Due to this fact, the Yang-Mills heat flow is said to be only *weakly-parabolic*.

The culprit of the non-parabolicity of (YMHF) turns out to be the gauge covariance of the term $\mathbf{D}^\ell F_{\ell i}$, which suggests that it can be remedied by studying the gauge structure of the Yang-Mills heat flow in detail. Upon inspection, we see that the gauge structure of the equations (YMHF) is somewhat restrained, as it is covariant only under gauge transforms that are *independent* of s. To

[8]Namely, one chooses the same orthonormal frame at each point on the domain.

deal with the problem of non-parabolicity, we shall begin by fixing this issue, i.e. reformulating the Yang-Mills heat flow in a way that is covariant under gauge transforms which may as well depend on the s-variable.

Along with A_i, let us also add a component A_s and consider A_a ($a = x^1, x^2, \dots, x^d, s$), which is a connection 1-form on the product manifold $\mathbb{R}^d \times [0, s_0]$. Corresponding to A_s, we also introduce the *covariant derivative* along the ∂_s-direction

$$\mathbf{D}_s := \partial_s + [A_s, \cdot].$$

A *covariant Yang-Mills Heat Flow* is a solution A_a to the system of equations

$$F_{si} = \mathbf{D}^\ell F_{\ell i}, \quad i = 1, 2, \dots, d, \tag{cYMHF}$$

where F_{si} is the commutator between $\mathbf{D}_s$ and $\mathbf{D}_i$, given by the formula

$$F_{si} = \partial_s A_i - \partial_i A_s + [A_s, A_i]. \tag{1.4.2}$$

The system (cYMHF) is underdetermined for A_a, and therefore requires an additional gauge condition (typically on A_s) in order to be solved. Note that the original Yang-Mills heat flow (YMHF) is a special case of (cYMHF), namely when $A_s = 0$. On the other hand, choosing $A_s = \partial^\ell A_\ell$, the top order terms of (cYMHF) becomes

$$\partial_s A_i - \partial_i \partial^\ell A_\ell = \triangle A_i - \partial^\ell \partial_i A_\ell + (\text{lower order terms}).$$

The term $\partial^\ell \partial_i A_\ell$ on each side are cancelled, and we are consequently left with a *strictly parabolic* system of equations for A_i. In other words, the weakly-parabolic system (YMHF) is equivalent to a strictly parabolic system of equations, connected via gauge transforms for (cYMHF) back and forth $A_s = 0$ and $A_s = \partial^\ell A_\ell$.

Henceforth, the gauge condition $A_s = 0$ will be referred to as the *caloric gauge*, in deference to the term introduced by Tao in his work [34]. The condition $A_s = \partial^\ell A_\ell$ will be dubbed the *DeTurck gauge*, as the procedure outlined above may be viewed as a geometric reformulation of the standard DeTurck's trick, introduced first by DeTurck [7] in the context of the Ricci flow and adapted to the Yang-Mills heat flow by Donaldson [8].

We remark that the scaling critical L^2_x Sobolev regularity is again $\gamma_c(d) = \frac{d-2}{2}$. Moreover, being the gradient flow, the magnetic energy $\mathbf{B}[F_{ij}(s)]$ is monotonically non-increasing along the flow, provided that the solution is sufficiently smooth.

1.5 Overview of the proof of local-wellposedness

Acquainted with the covariant formulation of the Yang-Mills heat flow, we are ready to return to the task of describing our approach in more detail. In what follows, we shall restrict ourselves to the case $d = 3$.

We shall begin by providing a short overview of the proof of local well-posedness for initial data sets with $\dot{H}^1_x$ regularity. In particular, we shall explain how Issues A, B raised in §1.2 are resolved in the novel approach.

To avoid too much technical details, we shall treat here the simpler problem of proving an *a priori* bound of a solution to (YM) in the temporal gauge. That is, for some interval $I := (-T_0, T_0) \subset \mathbb{R}$, we shall presuppose the existence of a solution $A^\dagger_\mu$ to (YM) in the temporal gauge on $I \times \mathbb{R}^3$ and aim to establish an estimate of the form

$$\|\partial_{t,x} A^\dagger_\mu\|_{C_t(I, L^2_x)} \leq C \sum_{i=1,2,3} \|(\mathring{A}_i, \mathring{E}_i)\|_{\dot{H}^1_x \times L^2_x}$$

where $A^\dagger_i(t=0) = \mathring{A}_i, \partial_t A^\dagger_i(t=0) = \mathring{E}_i$.

Step 1: Geometric smoothing of $A^\dagger_\mu$ by the (dynamic) Yang-Mills heat flow

The first step of the proof is to smooth out the solution $A^\dagger_\mu$, essentially using the covariant Yang-Mills heat flow. Let us introduce a new variable $s \in [0, s_0]$, and extend $A^\dagger_\mu = A^\dagger_\mu(t, x)$ to $A_{\mathbf{a}} = A_{\mathbf{a}}(t, x, s)$ (where $\mathbf{a} = x^0, x^1, x^2, x^3, s$) on $I \times \mathbb{R}^3 \times (0, s_0]$ by solving the equations

$$F_{s\mu} = \mathbf{D}^\ell F_{\ell\mu}, \quad \mu = 0, 1, 2, 3 \qquad \text{(dYMHF)}$$

with an appropriate choice of A_s, starting with $A_\mu(s = 0) = A^\dagger_\mu$. Note that that this system is (cYMHF) appended with the equation $F_{s0} = \mathbf{D}^\ell F_{\ell 0}$ for A_0; it will be referred to as the *dynamic Yang-Mills heat flow* or, in short, (dYMHF). Using Picard iteration, these equations can be solved provided that $s_0 > 0$ is small enough.

As a result, we arrive at a connection 1-form $A_{\mathbf{a}}$ (where $\mathbf{a} = x^0, x^1, x^2, x^3, s$) on $I \times \mathbb{R}^3 \times [0, s_0]$

which solves the following system of equations.

$$\begin{cases} F_{s\mu} = \mathbf{D}^\ell F_{\ell\mu} & \text{on} \quad I \times \mathbb{R}^3 \times [0, s_0], \\ \mathbf{D}^\mu F_{\mu\nu} = 0 & \text{along } I \times \mathbb{R}^3 \times \{0\}. \end{cases} \tag{HPYM}$$

We shall refer to this as the *Hyperbolic Parabolic Yang-Mills system* or, in short, (HPYM). This will be the system of equations that we shall mainly work with in place of (YM). Accordingly, instead of $A^\dagger_\mu$, we shall estimate $\underline{A}_\mu := A_\mu(s = s_0)$, which may be viewed as a smoothed-out version of $A^\dagger_\mu$, and the error $\partial_s A_\mu(s)$ (for $s \in (0, s_0)$) in between.

Step 2: Gauge choices for (HPYM): DeTurck and caloric-temporal gauges

The next step consists of estimating $\partial_s A_\mu$ and $\underline{A}_\mu$ by using the equations arising from (HPYM). Basically, the strategy is to first use the parabolic (in the s-direction) equations to estimate the new variables $\partial_s A_\mu, \underline{A}_\mu$ at $t = 0$, and then to use the hyperbolic (in the t-directions) equations to estimate their evolution in t. As (HPYM) is manifestly gauge covariant (under gauge transforms fully dependent on all the variables x^0, x^1, x^2, x^3, s), we need to fix a gauge in order to carry out such analyses.

As it turns out, a different gauge choice is needed to achieve each goal. For the purpose of deriving estimates at $t = 0$, it is essential to exploit the smoothing property of (dYMHF). As such, the gauge of choice here is the *DeTurck gauge* $A_s = \partial^\ell A_\ell$. On the other hand, completely different considerations are required for estimating the t-evolution, and here the gauge condition we impose is

$$\begin{cases} A_s = 0 & \text{on } I \times \mathbb{R}^3 \times (0, s_0), \\ \underline{A}_0 = 0 & \text{on } I \times \mathbb{R}^3 \times \{s_0\}. \end{cases}$$

which will be referred to as the *caloric-temporal gauge*. In practice, the DeTurck gauge will be first used to obtain estimates at $t = 0$, and then we shall perform a gauge transformation into the caloric-temporal gauge to carry out the analysis of the evolution in t. We remark that finding such gauge transform is always possible, as it amounts to simply solving a hierarchy of ODEs.

A brief discussion on the motivation behind our choice of the *caloric-temporal gauge* is in order. For $\partial_s A_\mu$ on $I \times \mathbb{R}^3 \times (0, s_0)$, let us begin by considering the following rearrangement of the formula (1.4.2) for F_{si}:

$$\partial_s A_i = F_{si} + \mathbf{D}_i A_s. \tag{1.5.1}$$

A simple computation[9] shows that F_{si} is *covariant-divergence-free*, i.e. $\mathbf{D}^{\ell} F_{s\ell} = 0$. This suggests that (1.5.1) may be viewed (heuristically) as a *covariant Hodge decomposition* of $\partial_s A_i$, where F_{si} is the covariant-divergence-free part and $\mathbf{D}_i A_s$, being a pure covariant-gradient term, may be regarded as the 'covariant-curl-free part' (although, strictly speaking, the covariant-curl does not vanish but is only of lower order for this term). Recall that the Coulomb gauge condition, which had a plenty of good properties as discussed earlier, is equivalent to having zero curl-free part. Therefore, to imitate the Coulomb gauge as closely as possible, we are motivated to set $A_s = 0$ on $I \times \mathbb{R}^3 \times (0, s_0)$; incidentally, this turns out to be the *caloric* gauge condition discussed earlier.

On the other hand, at $s = s_0$, the idea is that $\underline{A}_\mu$ possesses *smooth* initial data $(\underline{A}_i, \underline{F}_{0i})(t=0)$, thanks to the smoothing property of (dYMHF). Therefore, we expect that the problem of gauge choice for $\underline{A}_\mu$ is not as delicate as the original problem; as such, we choose the temporal gauge condition $\underline{A}_0 = 0$, which is easy to impose yet sufficient for the analogous problem with smoother initial data, as the previous works [28], [9] had shown.

Remark 1.5.1. Performing a gauge transformation $U = U(t, x, s)$ from the DeTurck gauge to the caloric gauge with $U(t=0, s=0) = \mathrm{Id}$ corresponds exactly to carrying out the standard DeTurck trick [8]. However, this will be inappropriate for our purposes, as we shall see that the resulting gauge transform U does *not* retain the smoothing estimates proved in the DeTurck gauge. Instead, we shall use the gauge transform for which $U(t=0, s=s_0) = \mathrm{Id}$. Under such gauge transform, $\underline{A}_i(t=0)$ remains the same, and thus smooth, at the cost of introducing a non-trivial gauge transform for the initial data at $t = 0, s = 0$. In some sense, this procedure is an analogue of the Uhlenbeck's lemma [37] in our approach. See §3.5 for further discussion.

Step 3: Analysis of the time evolution - Resolution of Issues A and B

With the caloric-temporal gauge, we are finally ready to describe how Issues A and B are resolved in the novel approach. Let us begin by introducing the *Yang-Mills tension field* $w_\nu(s) := \mathbf{D}^\mu F_{\nu\mu}(s)$, which measures the extent of failure of $A_\mu(s)$ to satisfy (YM). Then we may derive the following system of equations (See Chapter 2):

$$\mathbf{D}_s w_\nu = \mathbf{D}^\ell \mathbf{D}_\ell w_\nu + 2[F_\nu{}^\ell, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}], \tag{1.5.2}$$

$$\mathbf{D}^\mu \mathbf{D}_\mu F_{si} = 2[F_s{}^\mu, F_{i\mu}] - 2[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell} + \mathbf{D}_\ell F_{i\mu}] - \mathbf{D}^\ell \mathbf{D}_\ell w_i + \mathbf{D}_i \mathbf{D}^\ell w_\ell - 2[F_i{}^\ell, w_\ell], \tag{1.5.3}$$

$$\underline{\mathbf{D}}^\mu \underline{F}_{\nu\mu} = \underline{w}_\nu. \tag{1.5.4}$$

[9]The identity $\mathbf{D}^\ell F_{s\ell} = 0$ follows from (cYMHF) and $\mathbf{D}^\ell \mathbf{D}^k F_{\ell k} = 0$, which is proved simply by anti-symmetrizing the indices ℓ, k.

The underlines of (1.5.4) signify that each variable is restricted to $\{s = s_0\}$. Furthermore, $w_\nu \equiv 0$ at $s = 0$, for all $\nu = 0, 1, 2, 3$.

The parabolic equation (1.5.2) can be used to derive estimates for the Yang-Mills tension field w_μ. It is important to note that its data at $s = 0$ is *zero*, thanks to the fact that $A_\mu(s = 0)$ satisfies (YM). Moreover, note that $w_0 = -F_{s0}$, which is equal to $-\partial_s A_0$ thanks to the caloric gauge condition $A_s = 0$. In conclusion, after solving the parabolic equation (1.5.2), the dynamics of (HPYM) is reduced to that of the variables $F_{si} = \partial_s A_i$ (again due to $A_s = 0$) and $\underline{A}_i$. These are, in turn, estimated by (1.5.3), which is a wave equation for F_{si}, and (1.5.4), which is the Yang-Mills equation with a source $\underline{w}_\nu$ for $\underline{A}_\mu$ under the temporal gauge $\underline{A}_0 = 0$. This shows the hyperbolicity of (YM), which takes care of Issue A.

Next, let us address the issue of exhibiting null forms (i.e. Issue B). Let us begin by observing that for the system (1.5.4) for $\underline{A}_i$, *no null form is needed* to close the estimates; this is because $(\underline{A}_i, \underline{F}_{0i})(t = 0)$ has been smoothed out by (dYMHF) as mentioned earlier. For the wave equation (1.5.3) for F_{si}, on the other hand, we need to reveal the null structure of the quadratic terms in order to prove low regularity local well-posedness. Indeed, despite the superficial complexity, (1.5.3) in the caloric-temporal gauge has the miraculous structure that *all quadratic terms can be written in terms of null forms*, modulo essentially cubic and higher order terms. We shall sketch this procedure in §2.2. This settles Issue B.

In the case of $d = 3$ and H^1_x regularity, however, it turns out that the full null structure of the Yang-Mills equations is not necessary[10]. More precisely, there turns out to be only a single term which cannot be dealt with simply by Strichartz estimates, which is

$$2[A_\ell - \underline{A}_\ell, \partial^\ell F_{si}].$$

If $A_\ell - \underline{A}_\ell$ were divergence-free, i.e. $\partial^\ell(A_\ell - \underline{A}_\ell) = 0$, then an argument of Klainerman-Machedon [14], [15] would show that this nonlinearity may be rewritten in as a linear combination of null forms $Q_{jk}(|\partial_x|^{-1}(A - \underline{A}), F_{si})$. Although this is not strictly true, we have

$$A_\ell - \underline{A}_\ell = -\int_0^{s_0} F_{s\ell}(s)\, \mathrm{d}s$$

thanks to the condition $A_s = 0$, where $F_{s\ell}$ is *covariant-divergence-free*, i.e. $\mathbf{D}^\ell F_{s\ell} = 0$. This suffices for a variant of the argument of Klainerman-Machedon to work, revealing the null structure of the

[10]It is amusing to compare this with the analysis in the Coulomb gauge, in which null structure is needed for every quadratic term involving only A_i.

above term.

Provided that $s_0, |I|$ are sufficiently small[11], an analysis of (HPYM) using the gauge conditions indicated above leads to estimates for $\partial_s A_i, \underline{A}_i$ in the caloric-temporal gauge, such as

$$\begin{cases} \sup_{0<s<s_0} s^{-(m+1)/2} \|\partial_x^{(m-1)} \partial_{t,x}(\partial_s A_i)(s)\|_{C_t(I,L^2_x)} \leq C_m \sum_{j=1,2,3} \|(\mathring{A}_j, \mathring{E}_j)\|_{\dot{H}^1_x \times L^2_x} \\ \Big(\int_0^{s_0} s^{-(m+1)} \|\partial_x^{(m-1)} \partial_{t,x}(\partial_s A_i)(s)\|^2_{C_t(I,L^2_x)} \frac{\mathrm{d}s}{s} \Big)^{1/2} \leq C_m \sum_{j=1,2,3} \|(\mathring{A}_j, \mathring{E}_j)\|_{\dot{H}^1_x \times L^2_x} \\ s_0^{-(k-1)/2} \|\partial_x^{(k-1)} \partial_{t,x} \underline{A}_i\|_{C_t(I,L^2_x)} \leq C_k \sum_{j=1,2,3} \|(\mathring{A}_j, \mathring{E}_j)\|_{\dot{H}^1_x \times L^2_x} \end{cases} \tag{1.5.5}$$

up to some integers $m_0, k_0 > 1$, i.e. $1 \leq m \leq m_0$, $1 \leq k \leq k_0$. We remark that the weights of s are dictated by scaling.

Step 4: Returning to $A^\dagger_\mu$

The only remaining step is to translate (1.5.5) to the desired estimate for $\|\partial_{t,x} A^\dagger_\mu\|_{C_t(I,L^2_x)}$. The first issue arising in this step is that the naive approach of integrating the estimates (1.5.5) in s *fails* to bound $\|\partial_{t,x} A^\dagger_\mu\|_{C_t(I,L^2_x)}$, albeit only by a logarithm. To resolve this issue, we take the weakly-parabolic equations

$$\partial_s A_i = \triangle A_i - \partial^\ell \partial_i A_\ell + (\text{lower order terms}).$$

differentiate by $\partial_{t,x}$, multiply by $\partial_{t,x} A_i$ and then integrate the highest order terms by parts over $\mathbb{R}^3 \times [0, s_0]$. This procedure, combined with the $L^2_{\mathrm{d}s/s}$-type estimates of (1.5.5), overcomes the logarithmic divergence.

Another issue is that (1.5.5), being in the caloric-temporal gauge, is in a different gauge from the temporal gauge along $s = 0$. Therefore, we are required to control the gauge transform back to the temporal gauge along $s = 0$, for which appropriate estimates for $A_0(s = 0)$ in the caloric-temporal gauge are needed. These are obtained ultimately as a consequence of the analysis of the hyperbolic equations of (HPYM) (Strichartz estimates, in particular, are used).

Remark 1.5.2. Although we have assumed $d = 3$ and H^1_x regularity throughout this section, most part of the scheme described above can be easily applied to the Yang-Mills equations in other

[11]In the Coulomb gauge, the equation for A_0 is elliptic and therefore smallness of the time interval I cannot be utilized to solve for A_0 using perturbation; in [15], the authors exploits the spatial localization to overcome this issue. For (HPYM) in the caloric-temporal gauge, A_0 estimated by integrating $F_{s0} = \partial_s A_0$, where the latter variable satisfies a parabolic equation. For this, smallness of s_0 can be used, and thus the estimates are still global on $\mathbb{R}^3$.

dimensions $d \geq 2$ and/or for different scaling sub-critical regularity H^γ_x, $\gamma > \frac{d-2}{2}$. Indeed, note that the results in Chapter 3 are valid in all such cases. The bottleneck is the wave equation (1.5.3) for F_{si}, and whether certain H^γ local well-posedness can be established will generally depend crucially on whether (1.5.3) can be analyzed at the corresponding regularity. See the remark at the end of §2.2 for more discussion.

1.6 Overview of the proof of global well-posedness

In the work of Klainerman-Machedon [15], as pointed out earlier, finite energy global well-posedness was a rather easy corollary of the H^1_x local well-posedness proof thanks to the fact that in the (local) Coulomb gauge, the conserved energy $\mathbf{E}(t)$ essentially controls $\|(A_i, F_{0i})(t)\|_{H^1_x \times L^2_x}$. However, in the temporal gauge, making use of the conserved energy $\mathbf{E}(t)$ is not as straightforward since $\mathbf{E}(t)$ only controls certain components (namely, the curl) of the full gradient of $A_i(t)$. We remind the reader that this was referred to as Issue C in §1.2.

Nevertheless, it is another remarkable property of the novel approach that Issue C can also be resolved, and therefore finite energy global well-posedness of (YM) can be proved. Our proof proceeds roughly in three steps, each of which uses the conserved energy $\mathbf{E}(t)$ in a crucial way.

Step 1: Transformation to caloric-temporal gauge, improved version

Let us start with a solution $A^\dagger_\mu$ to (YM) in the temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3$. As in the proof of local well-posedness, the first step is to solve (dYMHF) to extend $A^\dagger_\mu$ to a solution $A_{\mathbf{a}}$ to (HPYM). *A priori*, however, it is not clear whether this is possible when T_0 is large.

To illustrate, suppose that $A^\dagger_\mu$ does not extend past the time T_0. Then from the local well-posedness statement, it is necessary that

$$\|\partial_{t,x} A^\dagger_\mu(t)\|_{L^2_x} \to \infty \quad \text{ as } t \to T_0.$$

Because of this, the size of the s-interval on which (dYMHF) can be solved by perturbative methods shrinks as $t \to T_0$. As a consequence, there might not exist a non-trivial interval $[0, s_0]$ on which (dYMHF) can be solved for every $t \in (-T_0, T_0)$.

However, such a scenario is ruled out, thanks to the conserved energy $\mathbf{E}(t)$, and (dYMHF) can be solved in a uniform manner globally in time[12]. More precisely, it is possible to show that there exists

[12] In fact, along our proof we shall discover that this is essentially the (dYMHF)-analogue of *finite energy global well-posedness* of (YMHF) (Corollary 5.4.3), which was first established by Råde [27].

$s_0 > 0$ depending only on $\mathbf{E}(t)$ such that (dYMHF) on a fixed t-slice can be solved on an interval $[0, s_0]$. As $\mathbf{E}(t)$ is conserved, this shows that $A^{\dagger}_{\mu}$ can be extended to a solution $A_{\mathbf{a}}$ to (HPYM) on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$. As before, with a solution $A_{\mathbf{a}}$ of (HPYM) in hand, we shall impose the caloric-temporal gauge condition via an appropriate gauge transform.

Step 2: Fixed-time control by $\mathbf{E}(t)$ in the caloric-temporal gauge

We wish to demonstrate that the conserved energy[13] $\mathbf{E}(t)$ controls the appropriate fixed-time norms of the dynamic variables in the caloric-temporal gauge, i.e. $\underline{A}_i$ and $F_{si} = \partial_s A_i$.

The key observation is that $\|\mathbf{D}_x^{(k)} F_{\mu\nu}(t, s)\|_{L^2_x}$ is estimated (with an appropriate weight of s) by $\mathbf{E}(t)$, thanks to covariant parabolic estimates. In particular, $\|\mathbf{D}_x^{(k)} \underline{F}_{0i}(t)\|_{L^2_x}$ is under control, where $\underline{F}_{\mu\nu}$ is the connection 2-form restricted to $\{s = s_0\}$. As the temporal gauge condition $\underline{A}_0 = 0$ is enforced, we have $\underline{F}_{0i} = \partial_t \underline{A}_i$; therefore, the preceding norm may be integrated in t to control the size of $\|\partial_x^{(k)} \underline{A}_i(t)\|_{L^2_x}$ for $t \in (-T_0, T_0)$. On the other hand, as $F_{si} = \mathbf{D}^{\ell} F_{\ell i}$ is already of the form $\mathbf{D}_x F_{\mu\nu}$, we can use the conserved energy $\mathbf{E}(t)$ to control the appropriate (fixed-time) norms of $F_{si}(t)$ as well, for each $t \in (-T_0, T_0)$.

Step 3: Short time estimates for (HPYM) in the caloric-temporal gauge

Finally, we must unwind all the gauge transformations which have been done and return to $A^{\dagger}_{\mu}$. As in the last step of the proof of local well-posedness, this requires estimating A_0 along $s = 0$ in the caloric-temporal gauge, where an important ingredient for the latter is the estimates obtained from the hyperbolic equations of (HPYM). Iterating the techniques developed for proving local well-posedness on a short time interval, coupled with some new estimates arising from the conserved energy $\mathbf{E}$, we shall arrive at the desired estimates.

1.7 Statement of the Main Theorems

We shall now give the precise statements of our main theorems. Let us begin by defining the class of initial data sets of interest.

Definition 1.7.1 (Admissible H^1_x initial data set)**.** We say that a pair $(\mathring{A}_i, \mathring{E}_i)$ of 1-forms on $\mathbb{R}^3$ is an *admissible* H^1_x *initial data set* for the Yang-Mills equations if the following conditions hold:

1. $\mathring{A}_i \in \dot{H}^1_x \cap L^3_x$ and $\mathring{E}_i \in L^2$,

[13] For a solution $A_{\mathbf{a}}$ to (HPYM), $\mathbf{E}(t)$ is defined to be the conserved energy of A_{μ} at $(t, s = 0)$. We remark that this is a gauge-invariant quantity.

2. The *constraint equation*

$$\partial^\ell \mathring{E}_\ell + [\mathring{A}^\ell, \mathring{E}_\ell] = 0,$$

holds in the distributional sense.

Let us also define the notion of *admissible solutions*.

Definition 1.7.2 (Admissible solutions)**.** Let $I \subset \mathbb{R}$. We say that a generalized solution A_μ to the Yang-Mills equations (YM) in the temporal gauge $A_0 = 0$ defined on $I \times \mathbb{R}^3$ is *admissible* if

$$A_\mu \in C_t(I, \dot{H}^1_x \cap L^3_x), \quad \partial_t A_\mu \in C_t(I, L^2_x)$$

and A_μ can be approximated by classical solutions in the temporal gauge in the above topology.

We begin with a H^1_x local well-posedness theorem, which will be called the *Main Local Well-posedness (LWP) Theorem.*

Main LWP Theorem. *Let $(\mathring{A}_i, \mathring{E}_i)$ be an admissible H^1_x initial data set, and define $\mathring{\mathcal{I}} := \|\mathring{A}\|_{\dot{H}^1_x} + \|\mathring{E}\|_{L^2_x}$. Consider the initial value problem (IVP) for* (YM) *with $(\mathring{A}_i, \mathring{E}_i)$ as the initial data. Then the following statements hold.*

1. *There exists $T^\star = T^\star(\mathring{\mathcal{I}}) > 0$, which is non-increasing in $\mathring{\mathcal{I}}$, such that a unique admissible solution $A_\mu = A_\mu(t,x)$ to the IVP in the temporal gauge $A_0 = 0$ exists on the t-interval $I := (-T^\star, T^\star)$. Furthermore, the following estimates hold.*

$$\|\partial_{t,x} A\|_{C_t(I, L^2_x)} \leq C \mathring{\mathcal{I}}, \qquad \|A\|_{C_t(I, L^3_x)} \leq \|\mathring{A}\|_{L^3_x} + T^{1/2} C_{\mathring{\mathcal{I}}} \mathring{\mathcal{I}}. \tag{1.7.1}$$

2. *Let $(\mathring{A}'_i, \mathring{E}'_i)$ be another admissible H^1_x initial data set such that $\|\mathring{A}'\|_{\dot{H}^1_x} + \|\mathring{E}'\|_{L^2_x} \leq \mathring{\mathcal{I}}$, and let A'_μ be the corresponding solution to the IVP in the temporal gauge, given by Statement 1. Then the following estimates for the difference hold.*

$$\|\partial_{t,x} \delta A\|_{C_t(I, L^2_x)} \leq C_{\mathring{\mathcal{I}}} (\|\delta \mathring{A}\|_{\dot{H}^1_x} + \|\delta \mathring{E}\|_{L^2_x}), \tag{1.7.2}$$

$$\|\delta A\|_{C_t(I, L^3_x)} \leq C_{\mathring{\mathcal{I}}} \|\delta \mathring{A}\|_{L^3_x} + T^{1/2} C_{\mathring{\mathcal{I}}} (\|\delta \mathring{A}\|_{\dot{H}^1_x} + \|\delta \mathring{E}\|_{L^2_x}). \tag{1.7.3}$$

3. *Finally, the following version of* persistence of regularity *holds: if $\partial_x \mathring{A}_i, \mathring{E}_i \in H^m_x$ for an integer $m \geq 0$, then the corresponding solution given by Statement 1 satisfies*

$$\partial_{t,x} A_i \in C^{k_1}_t((-T^\star, T^\star), H^{k_2}_x)$$

for every pair (k_1, k_2) *of nonnegative integers such that* $k_1 + k_2 \leq m$*.*

Our second main theorem is a global well-posedness statement, which (in essence) says that the solution given by the Main LWP Theorem can be extended globally in time. It uses crucially the fact that an admissible initial data set always possesses finite *conserved energy*, whose precise definition is as follows: Given a space-time 2-form $\mathbf{F} = F_{\mu\nu}$, we define its *conserved energy* to be

$$\mathbf{E}[\mathbf{F}] := \frac{1}{2} \sum_{\mu<\nu} \|F_{\mu\nu}\|^2_{L^2_x}.$$

We are ready to state the *Main Global Well-posedness (GWP) Theorem.*

Main GWP Theorem. *Let* $(\mathring{A}_i, \mathring{E}_i)$ *be an admissible* H^1_x *initial data set, and consider the initial value problem (IVP) for* (YM) *with* $(\mathring{A}_i, \mathring{E}_i)$ *as the initial data. (Note that by admissibility,* $(\mathring{A}_i, \mathring{E}_i)$ *always possesses finite conserved energy, i.e.* $\mathbf{E}[\overline{\mathbf{F}}] < \infty$*.) Then the following statements hold.*

1. *The admissible solution given by the Main LWP Theorem extends globally in time, uniquely as an admissible solution in the temporal gauge* $A_0 = 0$*.*

2. *Moreover, if* $\partial_x \mathring{A}_i, \mathring{E}_i \in H^m_x$ *for an integer* $m \geq 0$*, then the corresponding solution given by Statement 1 satisfies*

$$\partial_{t,x} A_i \in C^{k_1}_t(\mathbb{R}, H^{k_2}_x)$$

for every pair (k_1, k_2) *of nonnegative integers such that* $k_1 + k_2 \leq m$*.*

Remark 1.7.3. We remark that quantitative estimates (as in Statements 1, 2 of the Main LWP Theorem) can be obtained by applying the Main LWP Theorem repeatedly. We have omitted these statements for the sake of brevity.

Remark 1.7.4. The temporal gauge condition in both theorems above will plays rather an auxiliary role, and most of the analysis will take place in the caloric-temporal gauge, as discussed earlier. Indeed, our very method of proof of the above theorems is essentially to first establish their analogues in the caloric-temporal gauge, and then pass to the temporal gauge. It is mainly due to the difficulty of stating the precise gauge condition in a concise algebraic fashion that we have omitted these statements here. On the other hand, it may be of interest that sufficient control on the gauge transform can be achieved so as to allow for such a transition.

1.8 Outline of the thesis

In addition to the present introductory chapter, this thesis consists of 4 chapters and 1 appendix. Below, a brief description of their contents will be given. For an outline of each section, we refer the reader to the beginning of each chapter.

- In Chapter 2, we shall present the basic structural properties of the *hyperbolic parabolic Yang-Mills system* (HPYM) which will play a central role in this thesis. In §2.1, we shall derive the covariant equations of motion for the curvature components, as well as the Yang-Mills tension field, of solutions to (HPYM). In §2.2, we shall exhibit the null structure of the wave equation for $F_{si} = \partial_s A_i$ in the caloric-temporal gauge.

- In Chapter 3, the covariant and dynamic Yang-Mills heat flows (cYMHF) and (dYMHF) will be studied under the DeTurck and caloric gauge conditions. Our analysis will culminate in the last two sections, namely §3.6 and 3.7. In the former, local well-posedness for both (cYMHF) and (dYMHF) will be established in the caloric gauge. In the latter, we shall make precise the ideas outlined in Steps 1 and 2 of §1.5.

  Our basic strategy for studying both (cYMHF) and(dYMHF) will be to first establish parabolic smoothing estimates for the connection 1-form in the DeTurck gauge $A_s = \partial^\ell A_\ell$ (in which the flows are genuinely parabolic), then pass to the caloric gauge $A_s = 0$ by an appropriate gauge transform. We shall consider general dimension $d \geq 2$ and any sub-critical regularity $\gamma > \frac{d-2}{2}$.

In Chapters 4 and 5, we shall restrict to $d = 3$ and $\gamma = 1$.

- In Chapter 4, the Main LWP Theorem will be proved, following the ideas outlined in §1.5. The Main LWP Theorem will be reduced to Theorems A (Transformation to caloric-temporal gauge) and B (Time dynamics of (HPYM) in the caloric-temporal gauge), which correspond to Steps 1 & 2 and 3 & 4 in §1.5, respectively. Theorem A will follow from the results in §3.7, and the remainder of Chapter 4 will be concerned with a proof of Theorem B. The main idea will be to use the parabolic equations of (HPYM) to reduce the time dynamics of the full solution to (HPYM) to that of $F_{si} = \partial_s A_i$ and $\underline{A}_i$ in the caloric-temporal gauge, and then to analyze the wave equations satisfied by F_{si} and $\underline{A}_i$.

- In Chapter 5, the Main GWP Theorem will be proved, following the ideas outlined in §1.6. As in Chapter 4, we shall begin by reducing the Main GWP Theorem to Theorems E (Transformation to the caloric-temporal gauge, improved version), F (Fixed-time control by $\mathbf{E}$ in the caloric-temporal gauge) and G (Short time estimates for (HPYM) in the caloric-temporal gauge), which correspond to Steps 1, 2 and 3 in §1.6. The key common ingredient of the proofs of the latter three theorems will be a *covariant parabolic estimates* for $F_{\mu\nu}$, which will be proved in §5.3.

- Finally, in Appendix A, we shall give proofs of the results concerning gauge transforms, namely Propositions 3.5.1, 3.5.2 and Lemma 4.3.6, which were deferred in the main body of the thesis.

Guide for the reader

To assist the reader's navigation through this thesis, we shall give a list of dependencies of each chapter on the earlier materials.

- Basic to the rest of the thesis are §1.9 (Notations and conventions) and §2.1 (Equations of motion), in which many conventions and notations will be set.

- We remark that §2.2 (Null structure of (HPYM) in the caloric-temporal gauge) is not used in any major way in the rest of the thesis, and thus may be skipped by the hurried reader.

- For Chapter 3: There are no prerequisites (other than §1.9 and §2.1).

- For Chapter 4: The reader will need §3.1.1 (Basic estimates) – §3.1.4 (Correspondence Principle for p-normalized norms) for basic estimates, notations concerning p-normalized norms, abstract parabolic theory and the Correspondence Principle. Moreover, Theorems 3.7.1, 3.7.2 proved in §3.7 (Transformation to the caloric-temporal gauge) will also be needed.

- For Chapter 5: The reader should consult §3.1.1, §3.1.2 and §3.1.4 for basic estimates, p-normalized norms and the Correspondence Principle; on the other hand, the abstract parabolic theory developed in §3.1.3 (Abstract parabolic theory) will note be needed. Further prerequisites for this chapter are: Propositions 3.6.1, 3.6.4 and Lemmas 3.6.3, 3.6.6 in §3.6 (Yang-Mills heat flows in the caloric gauge); Theorem A in §4.3 (Reduction of the Main LWP Theorem to Theorems A and B); and Propositions 4.4.1 - 4.4.4, Theorems C, D in §4.4 (Definition of norms and reduction of Theorem B).

1.9 Notations and conventions

1.9.1 Indices

Throughout the thesis, greek indices (e.g. μ, ν, ...) will run over $x^0, x^1, \ldots, x^d$, whereas plain latin indices (e.g. i, j, ...) will run only over the *spatial* indices $x^1, \ldots, x^d$. In addition, we shall utilize bold latin indices, such as $\mathbf{a}, \mathbf{b}$, to refer to all possible indices $x^0, x^1, \ldots, x^d, s$ on $\mathbb{R}^{1+d} \times [0, \infty)$. Indices will be raised and lowered using the Minkowski metric, and we shall assume the Einstein convention of summing up repeated upper and lower indices.

We shall use bold kernel letters to refer to all space-time components; more precisely, $\mathbf{F}$ denotes any component of $F_{\mu\nu}$, and $\mathbf{A}$, $\mathbf{F}_s$ denote any component of A_ν, $F_{s\nu}$, respectively. On the other hand, plain kernel letters will refer to *only* spatial components, i.e. $F = F_{ij}$, $A = A_i$, and $F_s = F_{si}$ for $i, j = 1, 2, \ldots, d$. A norm of such an expression, such as $\|\mathbf{A}\|$ or $\|A\|$, is to be understood as the maximum over the respective range of indices, i.e. $\|\mathbf{A}\| = \sup_{\mu=0,1,\ldots,d} \|A_\mu\|$, $\|A\| = \sup_{i=1,\ldots,d} \|A_i\|$ etc.

1.9.2 Schematic notations

We shall use the notation $\mathcal{O}(\phi_1, \ldots, \phi_k)$ to denote a k-linear expression in the *values* of $\phi_1, \ldots, \phi_k$, or equivalently, *translation-invariant* k-linear map. For example, when ϕ_i and the expression itself are scalar-valued, then $\mathcal{O}(\phi_1, \ldots, \phi_k) = C\phi_1\phi_2 \cdots \phi_k$ for some constant C. In many cases, however, each ϕ_i and the expression $\mathcal{O}(\phi_1, \ldots, \phi_k)$ will actually be matrix-valued. In such case, $\mathcal{O}(\phi_1, \ldots, \phi_k)$ will be a matrix, whose each entry is a k-linear functional of the matrices ϕ_i. Note that the Leibniz rule holds for $\mathcal{O}$.

Similarly, for $\mathfrak{G}$-covariant tensors σ_i, we shall use the notation $\mathbb{O}(\sigma_1, \ldots, \sigma_k)$ to denote *covariant, translation-invariant* k-linear map. By covariance, it follows that the *covariant Leibniz rule* holds for $\mathbb{O}$, e.g. $\mathbf{D}_i\mathbb{O}(\sigma_1, \sigma_2) = \mathbb{O}(\mathbf{D}_i\sigma_1, \sigma_2) + \mathbb{O}(\sigma_1, \mathbf{D}_i\sigma_2)$.

In addition to estimating a single solution to various equations, we shall also be estimating the difference between two nearby solution. We shall often distinguish the second solution from the first by putting a prime, e.g. $A'_{\mathbf{a}}$, $F'_{s\mu}$, w'_i etc. The corresponding differences will be denoted by a δ, i.e. $\delta A_{\mathbf{a}} := A_{\mathbf{a}} - A'_{\mathbf{a}}$, $\delta F_{s\mu} := F_{s\mu} - F'_{s\mu}$ and $\delta w_i = w_i - w'_i$ etc.

We shall also use equations for differences, which are derived by taking the difference of the equations for the original variables. In writing such equations schematically using the $\mathcal{O}$-notation, the primed and unprimed variable will *not* be distinguished. For example, the expression $\mathcal{O}(A, \partial_x(\delta A))$

refers to a sum of bilinear expressions, of which the first factor could be any of A_i, A_i', and the second is one of $\partial_i(\delta A_j)$.

The following rule, which we call the *formal Leibniz's rule for* δ, is quite useful:

$$\delta\mathcal{O}(\phi_1, \phi_2, \dots, \phi_k) = \mathcal{O}(\delta\phi_1, \phi_2, \dots, \phi_k) + \mathcal{O}(\phi_1, \delta\phi_2, \dots, \phi_k) + \cdots + \mathcal{O}(\phi_1, \phi_2, \dots, \delta\phi_k).$$

1.9.3 Convention for implicit constants

In stating various estimates, we shall adopt the standard convention of denoting finite positive constants which are different, possibly line to line, by the same letter C. Dependence of C on other parameters will be made explicit by subscripts. Furthermore, we shall adopt the convention that C *always* depends in a non-decreasing manner with respect to each of its parameters, in its respective range, unless otherwise specified. For example, $C_{\mathbf{E}, {}^{(\underline{A})}\mathcal{I}}$, where $\mathbf{E}$, ${}^{(\underline{A})}\mathcal{I}$ range over positive real numbers, is a positive, non-decreasing function of both $\mathbf{E}$ and ${}^{(\underline{A})}\mathcal{I}$.

1.9.4 Small parameters

The following global small parameters will be used in this thesis.

$$0 < \delta_H \ll \delta_E, \delta_C \ll \delta_P \ll \delta_A \ll 1.$$

Often, we shall need an auxiliary small parameter which is used only within a certain part. For such parameters, we reserve the letter ϵ, and the variants thereof.

1.9.5 Miscellaneous notations

For $\gamma \in \mathbb{R}$, we shall define $|\partial_x|^\gamma := (-\triangle)^{\gamma/2}$. The $\dot{H}^\gamma_x$-(semi-)norm will defined by $\| |\partial_x|^\gamma (\cdot) \|_{L^2_x}$.

Chapter 2

Hyperbolic-Parabolic Yang-Mills system

In this short chapter, we shall collect some important properties of the system (HPYM).

In §2.1, we shall give a systematic exposition of the basic properties of the system (HPYM). The central result will be the derivation of the *covariant equations of motions* for (HPYM). The key heuristic point is that this system is *parabolic* in the s-direction and *hyperbolic* in the t-direction. We remark that much of the computation of this section will apply to (cYMHF) and (dYMHF) as well; see Remark 2.1.2

In §2.2, a discussion on the *null structure* of the wave equations in the caloric-temporal gauge will be given. We shall reveal (at least heuristically) the null structure of *all the main quadratic terms* in the wave equation for F_{si}.

2.1 Equations of motion

Let $I \subset \mathbb{R}$ be an open interval, $s_0 > 0$, and $A_{\mathbf{a}}$ a connection 1-form on $I \times \mathbb{R}^d \times [0, s_0]$ with coordinates $(x^0 = t, x^1, x^2, \dots, x^d, s)$. Recall that a bold-faced latin index $\mathbf{a}$ runs over all the indices corresponding to $x^0, x^1, x^2, \dots, x^d, s$. As in the Introduction, we may define the *covariant derivative* $\mathbf{D}_{\mathbf{a}}$ associated to $A_{\mathbf{a}}$. The commutator of the covariant derivatives in turn defines the *curvature 2-form* $F_{\mathbf{ab}}$. Note that the *Bianchi identity* holds automatically:

$$\mathbf{D}_{\mathbf{a}} F_{\mathbf{bc}} + \mathbf{D}_{\mathbf{b}} F_{\mathbf{ca}} + \mathbf{D}_{\mathbf{c}} F_{\mathbf{ab}} = 0. \tag{2.1.1}$$

In this section, we shall consider a solution $A_{\mathbf{a}}$ to the *hyperbolic parabolic Yang-Mills system*, which have been introduced in the special case $d = 3$ in the Introduction. The general hyperbolic parabolic Yang-Mills system for any $d \geq 2$ is similarly defined as follows:

$$\begin{cases} F_{s\mu} = \mathbf{D}^{\ell} F_{\ell\mu} & \text{on} \quad I \times \mathbb{R}^d \times [0, s_0], \\ \mathbf{D}^{\mu} F_{\mu\nu} = 0 & \text{along} \quad I \times \mathbb{R}^d \times \{0\}. \end{cases} \tag{HPYM}$$

The *Yang-Mills tension field* w_ν is defined as before by the formula

$$w_\nu := \mathbf{D}^{\mu} F_{\nu\mu}.$$

We shall use the convention of using an over- or underline to signify the variable being evaluated at $s = 0$ and $s = s_0$, respectively. For example, $\overline{A}_\mu = A_\mu(s = 0)$, $\overline{w}_\mu = w_\mu(s = 0)$, $\underline{A}_\mu = A_\mu(s = s_0)$, $\underline{\mathbf{D}}_\mu B = \partial_\mu B + [A_\mu(s = s_0), B]$, $\underline{w}_\mu = w_\mu(s = s_0)$ etc. We remark that s_0 will usually set to be equal to 1 by scaling.

Theorem 2.1.1 (Covariant equations of the hyperbolic-parabolic Yang-Mills system). *Let $A_{\mathbf{a}}$ be a smooth solution to the hyperbolic parabolic Yang-Mills system* (HPYM). *Then the following covariant equations hold.*

$$\mathbf{D}^{\ell} F_{s\ell} = 0, \tag{2.1.2}$$

$$\mathbf{D}_0 F_{s0} = - \mathbf{D}_0 w_0 = -\mathbf{D}^{\ell} w_\ell, \tag{2.1.3}$$

$$\mathbf{D}_s F_{\mathbf{ab}} = \mathbf{D}^{\ell} \mathbf{D}_\ell F_{\mathbf{ab}} - 2[F_{\mathbf{a}}{}^{\ell}, F_{\mathbf{b}\ell}], \tag{2.1.4}$$

$$\mathbf{D}_s w_\nu = \mathbf{D}^{\ell} \mathbf{D}_\ell w_\nu + 2[F_\nu{}^{\ell}, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}], \tag{2.1.5}$$

$$\mathbf{D}^{\mu} \mathbf{D}_\mu F_{s\nu} = 2[F_s{}^{\mu}, F_{\nu\mu}] - 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}] - \mathbf{D}^{\ell} \mathbf{D}_\ell w_\nu + \mathbf{D}_\nu \mathbf{D}^{\ell} w_\ell - 2[F_\nu{}^{\ell}, w_\ell]. \tag{2.1.6}$$

$$\underline{\mathbf{D}}^{\mu} \underline{F}_{\nu\mu} = \underline{w}_\nu \tag{2.1.7}$$

Moreover, we have $\overline{w}_\mu = 0$.

Remark 2.1.2. An inspection of the proof shows that (2.1.2)–(2.1.7) hold under the weaker hypothesis that $A_{\mathbf{a}}$ satisfies only (dYMHF). Moreover, these statements for non-temporal indices $\mathbf{a}, \mathbf{b} = x^1, \dots, x^d, s$ hold for a solution (A_i, A_s) to (cYMHF) as well. However, the last statement of the theorem is equivalent to (YM) along $\{s = 0\}$.

Proof. Let us begin with (2.1.2). This is a consequence of the following simple computation:

$$\begin{aligned}\mathbf{D}^\ell F_{s\ell} = \mathbf{D}^\ell \mathbf{D}^k F_{k\ell} =& \frac{1}{2}\mathbf{D}^\ell \mathbf{D}^k F_{k\ell} + \frac{1}{2}\mathbf{D}^k \mathbf{D}^\ell F_{k\ell} + \frac{1}{2}[F^{\ell k}, F_{k\ell}] \\ =& \frac{1}{2}(\mathbf{D}^\ell \mathbf{D}^k + \mathbf{D}^k \mathbf{D}^\ell) F_{k\ell} = 0,\end{aligned}$$

where we have used anti-symmetry of $F_{k\ell}$ for the last equality.

Next, in order to derive (2.1.3), we first compute

$$\mathbf{D}^\mu w_\mu = \mathbf{D}^\mu \mathbf{D}^\nu F_{\nu\mu} = 0,$$

by a computation similar to the preceding one. This give the second equality of (2.1.3). In order to prove the first equality, we compute

$$\mathbf{D}^\mu F_{s\mu} = \mathbf{D}^\mu \mathbf{D}^\ell F_{\ell\mu} = \mathbf{D}^\ell \mathbf{D}^\mu F_{\ell\mu} + [F^{\mu\ell}, F_{\ell\mu}] = \mathbf{D}^\ell w_\ell \tag{2.1.8}$$

and note that $\mathbf{D}^\mu F_{s\mu} = \mathbf{D}^0 F_{s0} = -\mathbf{D}_0 F_{s0}$ by (2.1.2).

Next, let us derive (2.1.4). We begin by noting that the equation $F_{s\mathbf{a}} = \mathbf{D}^\ell F_{\ell\mathbf{a}}$ holds for $\mathbf{a} = t, x, s$; for the last case, we use (2.1.2). Using this and the Bianchi identity (2.1.1), we compute

$$\begin{aligned}\mathbf{D}_s F_{\mathbf{ab}} =& \mathbf{D}_\mathbf{a} F_{s\mathbf{b}} - \mathbf{D}_\mathbf{b} F_{s\mathbf{a}} = \mathbf{D}_\mathbf{a} \mathbf{D}^\ell F_{\ell\mathbf{b}} - \mathbf{D}_\mathbf{b} \mathbf{D}^\ell F_{\ell\mathbf{a}} \\ =& \mathbf{D}^\ell(\mathbf{D}_\mathbf{a} F_{\ell\mathbf{b}} - \mathbf{D}_\mathbf{b} F_{\ell\mathbf{a}}) + [F_{\mathbf{a}}{}^{\ell}, F_{\ell\mathbf{b}}] - [F_{\mathbf{b}}{}^{\ell}, F_{\ell\mathbf{a}}] \\ =& \mathbf{D}^\ell \mathbf{D}_\ell F_{\mathbf{ab}} - 2[F_{\mathbf{a}}{}^{\ell}, F_{\mathbf{b}\ell}].\end{aligned}$$

In order to prove (2.1.5), we shall use (2.1.4). We compute as follows.

$$\begin{aligned}\mathbf{D}_s w_\nu =& \mathbf{D}_s \mathbf{D}^\mu F_{\nu\mu} \\ =& \mathbf{D}^\mu \Big(\mathbf{D}^\ell \mathbf{D}_\ell F_{\nu\mu} - 2[F_{\nu}{}^{\ell}, F_{\mu\ell}]\Big) + [F_{s}{}^{\mu}, F_{\nu\mu}] \\ =& \mathbf{D}^\ell \mathbf{D}_\ell \Big(\mathbf{D}^\mu F_{\nu\mu}\Big) + [F^{\mu\ell}, \mathbf{D}_\ell F_{\nu\mu}] + \mathbf{D}^\ell [F^{\mu}{}_{\ell}, F_{\nu\mu}] - 2\mathbf{D}^\mu [F_{\nu}{}^{\ell}, F_{\mu\ell}] + [F_{s}{}^{\mu}, F_{\nu\mu}] \\ =& \mathbf{D}^\ell \mathbf{D}_\ell w_\nu + 2[F_{\nu}{}^{\ell}, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}].\end{aligned}$$

Note that (2.1.7) is exactly the definition of w_ν at $s = s_0$. We are therefore only left to prove (2.1.6).

Here, the idea is to start with the Bianchi identity $0 = \mathbf{D}_\mu F_{s\nu} + \mathbf{D}_s F_{\nu\mu} + \mathbf{D}_\nu F_{\mu s}$ and to take

$\mathbf{D}^{\mu}$ of both sides. The first term on the right-hand side gives the desired term $\mathbf{D}^{\mu}\mathbf{D}_{\mu}F_{s\nu}$. For the second term, we compute

$$\mathbf{D}^{\mu}\mathbf{D}_{s}F_{\nu\mu} = \mathbf{D}_{s}\mathbf{D}^{\mu}F_{\nu\mu} + [F^{\mu}{}_{s}, F_{\nu\mu}] = \mathbf{D}_{s}w_{\nu} - [F_{s}{}^{\mu}, F_{\nu\mu}],$$

and for the third term, we compute, using (2.1.8),

$$\mathbf{D}^{\mu}\mathbf{D}_{\nu}F_{\mu s} = \mathbf{D}_{\nu}\mathbf{D}^{\mu}F_{\mu s} + [F^{\mu}{}_{\nu}, F_{\mu s}] = -\mathbf{D}_{\nu}\mathbf{D}^{\ell}w_{\ell} - [F_{s}{}^{\mu}, F_{\nu\mu}]$$

Combining these with (2.1.5), we obtain (2.1.6). □

2.2 Null structure of (HPYM) in the caloric-temporal gauge

The purpose of this section is to reveal the null structure of the system (HPYM) in the caloric-temporal gauge

$$\begin{cases} A_s = 0, & \text{everywhere,} \\ \underline{A}_0 = 0, & \text{along } s = s_0. \end{cases}$$

As discussed in §1.5, for the purpose of proving the Main LWP Theorem, we do not need to reveal the null structure of every quadratic term in the wave equations of (HPYM). Recall that there is only one place where the null structure is needed, which is the term

$$2[A^{\ell} - \underline{A}^{\ell}, \partial_{\ell}F_{si}].$$

arising from expanding $\mathbf{D}^{\mu}\mathbf{D}_{\mu}F_{si}$. We refer the reader to §4.7.2 for the precise derivation of the null structure of this term, as well as its rigorous analysis.

Despite the apparent complexity, miraculously, it turns out that the remaining quadratic terms in the wave equation for F_{si} in the caloric-temporal gauge may also be expressed in terms of null forms, modulo less dangerous terms. The structure is very close to that of the wave equation for A_i in the Coulomb gauge[1]. Below, we shall give a brief account of the full null structure of the wave equation for F_{si}. To simplify the presentation, let us make a few heuristic assumptions which are easily justifiable in application.

- Any variable at $s = s_0$, e.g. $\underline{A}_i$, will be ignored, as they cannot contribute to dangerous

[1] Recall, in fact, that this analogy had been our motivation for the choice of the caloric(-temporal) gauge condition in §1.5.

interaction for F_{si} at $0 < s < s_0$ nor $\underline{A}_i$ at s_0.

- The variables F_{s0}, w_i, $\partial_0 A_0$, A_0, $\partial^\ell A_\ell - \partial^\ell \underline{A}_\ell$ and $(A^{\mathrm{cf}} - \underline{A}^{\mathrm{cf}})_i := -(-\triangle)^{-1}\partial_i(\partial^\ell A_\ell - \partial^\ell \underline{A}_\ell)$ will be considered *(at least) quadratic*. The justifications are as follows:

  – For $F_{s0} = -w_0$ and w_i, this is because they satisfy a semi-linear parabolic equation (2.1.5) with zero data at $s = 0$.

  – For $\partial_0 A_0$, we use the equation $\mathbf{D}^\mu w_\mu = 0$, which, by the caloric-temporal gauge condition, implies

$$\partial_s(\partial_0 A_0) = [A_0, F_{s0}] + \mathbf{D}^\ell w_\ell.$$

  – For A_0, we use $\partial_s A_0 = F_{s0}$ and $\underline{A}_0$, which hold thanks to the caloric-temporal gauge condition.

  – For $\partial^\ell A_\ell - \partial^\ell \underline{A}_\ell$, note that $\partial_s \partial^\ell A_\ell = -[A^\ell, F_{si}]$ since $\mathbf{D}^\ell F_{s\ell} = 0$ by the caloric-temporal gauge condition.

  – Finally, for $A^{\mathrm{cf}} - \underline{A}^{\mathrm{cf}}$, we use the preceding point regarding $\partial^\ell A_\ell - \partial^\ell \underline{A}_\ell$.

- We shall decompose everything in terms of A_i, for which we have $A_i = -\int_s^1 F_{si}(s')\,\mathrm{d}s' + \underline{A}_i$ by the caloric-temporal gauge condition. In terms of scaling, $F_{si} \sim \partial_x^{(2)} A$. As the wave equation would gain F_{si} one derivative, up to three derivatives (one may be a time derivative) are allowed to fall on A.

- Finally, we shall assume that all variables are in $\mathcal{S}_{t,x}$, and furthermore that all Riesz and Hodge projections (such as $\mathbb{P}^{\mathrm{df}}$, defined below) can be ignored, in view of requiring that the norm of our function space depends only on the size of the Fourier transform.

For the reader's convenience, let us recall the wave equation (2.1.6) for F_{si}:

$$\mathbf{D}^\mu \mathbf{D}_\mu F_{s\nu} = 2[F_s{}^\mu, F_{\nu\mu}] - 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}] - \mathbf{D}^\ell \mathbf{D}_\ell w_\nu + \mathbf{D}_\nu \mathbf{D}^\ell w_\ell - 2[F_\nu{}^\ell, w_\ell]. \quad (2.1.6)$$

Let us first deal with the bilinear terms arising form $\mathbf{D}^\mu \mathbf{D}_\mu F_{si}$.

Lemma 2.2.1. *We have*

$$\begin{aligned}\mathbf{D}^\mu \mathbf{D}_\mu F_{si} =& \Box F_{si} + 2[A^\ell, \partial_\ell F_{si}] - 2[A_0, \partial_0 F_{si}] \\ &+ [\partial^\ell A_\ell, F_{si}] - [\partial_0 A_0, F_{si}] + [A^\mu, [A_\mu, F_{si}]] \\ =& \Box F_{si} + \mathcal{O}(A^\ell, \partial_\ell F_{si}) + \textit{(Cubic and higher)}.\end{aligned} \quad (2.2.1)$$

Moreover, the following (schematic) identity holds for any function ϕ and constant coefficient bilinear form $\mathcal{O}$:

$$\mathcal{O}(A^\ell, \partial_\ell \phi) = Q_x(|\partial_x|^{-1} A, \phi) + \textit{(Cubic and higher)}. \tag{2.2.2}$$

where Q_x is a linear combination (modulo Riesz projections) of null forms of the type

$$Q_{ij}(\phi_1, \phi_2) = \partial_i \phi_1 \partial_j \phi_2 - \partial_j \phi_1 \partial_i \phi_2.$$

Proof. The first identity (2.2.1) is a simple computation using the above heuristics. To prove (2.2.2), we shall use some simple facts regarding the *Hodge decomposition*. Given a sufficiently nice (say Schwartz) 1-form B_i on $\mathbb{R}^d$, its Hodge decomposition is defined by

$$B_i := (\mathbb{P}^{\mathrm{df}} B)_i + (\mathbb{P}^{\mathrm{cf}} B)_i,$$

where

$$(\mathbb{P}^{\mathrm{df}} B)_i := (-\triangle)^{-1} \partial^\ell (\partial_i B_\ell - \partial_\ell B_i), \quad (\mathbb{P}^{\mathrm{cf}} B)_i := -(-\triangle)^{-1} \partial_i \partial^\ell B_\ell.$$

We shall often use the shorthand $B^{\mathrm{df}} := \mathbb{P}^{\mathrm{df}} B$ and $B^{\mathrm{cf}} := \mathbb{P}^{\mathrm{cf}} B$. Applying this to A^ℓ and discarding A^{cf} according to the heuristic above, we may compute

$$\begin{aligned}
\mathcal{O}(A^\ell, \partial_\ell \phi) =& \mathcal{O}((A^{\mathrm{df}})^\ell, \partial_\ell \phi) + \text{(Cubic and higher)} \\
=& \mathcal{O}\Big((-\triangle)^{-1} \partial_k (\partial^\ell A^k - \partial^k A^\ell), \partial_\ell \phi\Big) + \text{(Cubic and higher)} \\
=& \mathcal{O}\Big((-\triangle)^{-1} \partial_k \partial^\ell A^k, \partial_\ell \phi\Big) - \mathcal{O}\Big((-\triangle)^{-1} \partial_\ell \partial^\ell A^k, \partial_k \phi\Big) + \text{(Cubic and higher)} \\
=& Q_{ij}(|\partial_x|^{-1} A, \phi) + \text{(Cubic and higher)}.
\end{aligned}$$

□

Next, we shall consider the quadratic terms on the right-hand side of (2.1.6).

Lemma 2.2.2. *The following (schematic) identity holds:*

$$\begin{aligned}
&2[F_s{}^\mu, F_{i\mu}] - 2[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell} + \mathbf{D}_\ell F_{i\mu}] \\
=& \partial_x Q_x(A, A) + \partial_x Q_0(A, A) + Q_0(A, \partial_x A) + Q_x(A, \partial_x A) \\
&+ \textit{(Cubic or higher)},
\end{aligned} \tag{2.2.3}$$

where Q_x is as before and Q_0 is a linear combination (up to Riesz projections) of null forms of the *type*

$$Q_0(\phi_1, \phi_2) = \partial^\mu \phi_1 \partial_\mu \phi_2.$$

Proof. Substitute ${F_s}^\mu = \mathbf{D}_\ell F^{\ell\mu}$ and rewrite the above expression as follows.

$$\begin{aligned}
&-2[{F_s}^\mu, F_{i\mu}] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell} + \mathbf{D}_\ell F_{i\mu}] \\
&= -2[\mathbf{D}_\ell F^{\ell\mu}, F_{i\mu}] - 2[F^{\ell\mu}, \mathbf{D}_\ell F_{i\mu}] - 2[F^{\ell\mu}, \mathbf{D}_\mu F_{i\ell}] \\
&= -2\mathbf{D}^\ell[{F_\ell}^\mu, F_{i\mu}] - 2[F^{\ell\mu}, \mathbf{D}_\mu F_{i\ell}].
\end{aligned}$$

For the first term, it suffices to show that the bilinear terms in $[{F_\ell}^\mu, F_{i\mu}]$ can be expressed as a null form. We proceed as follows:

$$[{F_\ell}^\mu, F_{i\mu}] = [\partial_\ell A^\mu, \partial_i A_\mu] - [\partial^\mu A_\ell, \partial_i A_\mu] - [\partial_\ell A^\mu, \partial_\mu A_i] + [\partial^\mu A_\ell, \partial_\mu A_i] + (\text{Cubic and higher}).$$

To begin with, discard all terms involving A_0, following the heuristic above. The fourth term on the right hand side is a Q_0-type null form, whereas the second and third terms are of the type $\mathcal{O}(\partial_x A^\ell, \partial_\ell A)$, to which we apply (2.2.2). Finally, for the first term, we may use the anti-symmetry of the Lie bracket to write

$$[\partial_\ell A^\ell, \partial_i A_\ell] = \frac{1}{2}\Big([\partial_\ell A^\ell, \partial_i A_\ell] - [\partial_i A^\ell, \partial_\ell A_\ell]\Big)$$

which is a $Q_{\ell i}$-type null form.

On the other hand, we may also compute

$$[F^{\ell\mu}, \mathbf{D}_\mu F_{i\ell}] = [\partial^\ell A^\mu, \partial_\mu F_{i\ell}] - [\partial^\mu A^\ell, \partial_\mu F_{i\ell}] + (\text{Cubic and higher}).$$

Discard also the terms involving A_0. The second term on the right-hand side is a Q_0-type null form, whereas the first term is of the type $\mathcal{O}(\partial_x A^\ell, \partial_\ell F)$. Expanding $F_{i\ell}$ in terms of A and applying (2.2.2), we obtain (2.2.3). □

Finally, we also need to address the linear term in w_i, which is heuristically also quadratic. Note that the only such term in (2.1.6) is

$$\partial^\ell(\partial_\ell w_i - \partial_i w_\ell),$$

which is exactly $\triangle(\mathbb{P}^{\mathrm{df}} w)_i$. After an application of $\mathbb{P}^{\mathrm{df}}$ to the parabolic equation (2.1.5) and inversion, we are left to reveal the null structure of the main term, namely

$$\triangle \mathbb{P}^{\mathrm{df}} w \sim \mathbb{P}^{\mathrm{df}}([F^{\mu\ell}, \mathbf{D}_\mu F_{\cdot\ell} + \mathbf{D}_\ell F_{\cdot\mu}]).$$

Let us consider the expression under $\mathbb{P}^{\mathrm{df}}$. We have already taken care of the term

$$[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell}].$$

in the proof of Lemma 2.2.2. Therefore, we are left to consider $[F^{\mu\ell}, \mathbf{D}_\ell F_{i\mu}]$. Expanding $F, \mathbf{D}$ in terms of A, ∂, we see that all bilinear terms are acceptable, except

$$\mathbb{P}^{\mathrm{df}}[\partial^\ell A^\mu, \partial . \partial_\ell A_\mu].$$

A simple computation using the definition of $\mathbb{P}^{\mathrm{df}}$ gives

$$\begin{aligned}(\mathbb{P}^{\mathrm{df}}[\partial^\ell A^\mu, \partial . \partial_\ell A_\mu])_i =& (-\triangle)^{-1}\partial^k(\partial_i[\partial^\ell A^\mu, \partial_k\partial_\ell A_\mu] - \partial_k[\partial^\ell A^\mu, \partial_i\partial_\ell A_\mu]) \\ =& (-\triangle)^{-1}\partial^k([\partial_i\partial^\ell A^\mu, \partial_k\partial_\ell A_\mu] - [\partial_k\partial^\ell A^\mu, \partial_i\partial_\ell A_\mu]) \\ =& |\partial_x|^{-1}Q_x(\partial^\ell A^k, \partial_\ell A_k) + (\text{Cubic and higher}),\end{aligned}$$

as desired.

Remark 2.2.3. The null structure uncovered above is a key ingredient for dealing with low regularity problems in the caloric-temporal gauge. Thanks to the close analogy, it seems to be often the case that the Fourier analytic methods for (YM) in the Coulomb gauge can be applied to (HPYM) in the caloric-temporal gauge. Note that, as remarked at the end of §1.5, other parts of the proof of LWP are often easily adaptable to any scaling sub-critical regularity for $d \geq 2$.

Following such ideas, in a forthcoming work [26] we shall establish *almost optimal local well-posedness* of the $(1+4)$-dimensional (YM) for arbitrarily large initial data. The necessary Fourier analytic tools are provided by [20].

Chapter 3

Analysis of the Yang-Mills heat flows

In this chapter, we shall analyze the *covariant Yang-Mills heat flow* on $\mathbb{R}^d \times [0, \infty)$ $(d \geq 2)$

$$F_{si} = \mathbf{D}^\ell F_{\ell i}, \quad (i = 1, \dots, d) \tag{cYMHF}$$

and *dynamic Yang-Mills heat flow* on $I \times \mathbb{R}^d \times [0, \infty)$ ($I \subset \mathbb{R}$ is an interval)

$$F_{s\nu} = \mathbf{D}^\ell F_{\ell \nu}, \qquad (\nu = 0, 1, \dots, d) \tag{dYMHF}$$

under two gauge conditions, namely the *DeTurck gauge* $A_s = \partial^\ell A_\ell$ and *caloric gauge* $A_s = 0$. The goal of this chapter is two-fold: First to establish local well-posedness of (cYMHF) and (dYMHF) in the caloric gauge, and second to rigorously carry out the ideas outlined in Steps 1 and 2 in §1.5 concerning transformation of a solution to (YM) to solution to (HPYM) in the caloric-temporal gauge, with appropriate estimates at $t = 0$. These goals will be achieved in §3.6 and §3.7, respectively.

In §3.1, we shall carry out some preliminary work which will be useful throughout the rest of the thesis. In particular, a simple machinery called *abstract parabolic theory* will be developed, which would allow us to handle various parabolic equations arising in this thesis in a consistent, unified manner. The material of this section will be used throughout the rest of this thesis.

From §3.2, we shall begin our analysis of the Yang-Mills heat flows. The first key idea for our study is that both (cYMHF) and (dYMHF) are *genuinely parabolic* at the level of A in the DeTurck gauge $A_s = \partial^\ell A_\ell$. As such, in §3.2 – §3.4, we shall establish basic properties (e.g. local well-posedness, infinite instantaneous smoothing) of both flows in the DeTurck gauge, using the standard theory of semi-linear parabolic equations. More precisely, in §3.2, we shall first derive local well-posedness and smoothing of (cYMHF) in the DeTurck gauge. Then, in §3.3, we shall establish smoothing estimates for a general linear parabolic equation, with assumptions on the coefficients compatible with the estimates proved for A_i in §3.2. These will be applied in §3.4 to establish local well-posedness and smoothing of (dYMHF) in the DeTurck gauge.

In the rest of this chapter, we shall analyze (cYMHF) and (dYMHF) in the caloric gauge $A_s = 0$. In §3.5, we shall establish estimates for the gauge transform U from the DeTurck to caloric gauge. As an application of these results, in §3.6 we shall prove local well-posedness of (cYMHF) and (dYMHF) in the caloric gauge. We remark that the former system is in fact the original Yang-Mills heat flow, and our strategy for proving its local well-posedness is essentially the classical DeTurck trick [7], [8]. The theory that we develop here will be useful in our proof of the global well-posedness theorem in Chapter 5.

The classical DeTurck trick, however, leads to a loss of smoothing estimates for A_i. In §3.5, we shall see that this is due to requiring that the initial gauge transform $U(s = 0)$ is equal to the identity, and if we instead require $U(s = s_1) = \mathrm{Id}$, then the smoothing estimates for A_i are preserved (at the expense of introducing a non-trivial gauge transform at $s = 0$). This is the other key idea of this chapter; this will be put into use in §3.7 to prove (morally) the following statements: Given a connection 1-form[1] $A^\dagger_\mu$ on $I \times \mathbb{R}^d$ in $L^\infty_t \dot{H}^\gamma_x(I)$ $(\gamma > \frac{d-2}{2})$, there exists a gauge transform V on $I \times \mathbb{R}^d$ and solution $A_{\mathbf{a}}$ to (HPYM) on $I \times \mathbb{R}^d \times [0, s_0]$ for s_0 sufficiently small[2] such that $A_\mu(s = 0)$ is the gauge transformation of $A^\dagger_\mu$ by V, i.e.

$$\overline{A}_\mu := A_\mu(s = 0) = V(A^\dagger_\mu)V^{-1} - \partial_\mu V V^{-1},$$

and $A_{\mathbf{a}}$ is in the caloric-temporal gauge, i.e. $A_s = 0$ and $\underline{A}_0 := A_0(s = s_0) = 0$. Moreover, $F_{si} = \partial_s A_i$ and $\underline{A}_i$ will obey smoothing estimates at $t = 0$. Finally, estimates for $\mathring{V} := V(t = 0)$ will be obtained as well. We refer the reader to Theorems 3.7.1 and 3.7.2 for more details. These results will be useful in our proof of local well-posedness in Chapter 4.

The results in this chapter hold for any dimension $d \geq 2$ and sub-critical regularity $\gamma > \frac{d-2}{2}$. These are generalization of those in [25, §3 – §6] and [24, §6], in which only the case $d = 3$ and $\gamma = 1$ had been considered.

[1]In practice, this will be a solution to (YM) in the temporal gauge; see Theorem A in §4.3. In this case, the resulting solution $A_{\mathbf{a}}$ will be a solution to (HPYM).

[2]In practice, s_0 will be set to 1 by scaling.

3.1 Preliminaries

After stating some basic estimates in §3.1.1, we shall develop in the rest of this section what we call an *abstract parabolic theory*, which is essentially a book-keeping scheme which allows for a unified treatment of a diverse array of parabolic equations. In §3.1.2, we shall introduce the notion of *p-normalization* of norms, which is a formalization of the simple heuristics $\partial_x \sim s^{-1/2}$, $\mathrm{d}x \sim s^{1/2}$, etc. for solutions to the heat equation $(\partial_s - \triangle)\psi = 0$. Then in §3.1.3, we shall prove the *main theorem of abstract parabolic theory* (Theorem 3.1.10), which is simply the theory of energy integrals for the linear heat equation recast in the language of p-normalized norms. Finally in §3.1.4, we shall formulate the *Correspondence Principle*, which will allow to easily transfer estimates concerning homogeneous norms to their counterparts for p-normalized norms.

3.1.1 Basic estimates

We collect here some basic estimates that will be frequently used throughout the thesis. Let us begin with some inequalities involving Sobolev norms for $\phi \in \mathcal{S}_x(\mathbb{R}^d)$, where $\mathcal{S}_x(\mathbb{R}^d)$ refers to the space of Schwartz functions on $\mathbb{R}^d$.

Lemma 3.1.1 (Inequalities for Sobolev norms)**.** *Let $d \geq 1$. Then for $\phi \in \mathcal{S}_x(\mathbb{R}^d)$, the following statements hold.*

- **(Sobolev inequality)** *For $1 < r \leq q$, $k \geq 0$ such that $\frac{d}{q} = \frac{d}{r} - k$, we have*

$$\|\phi\|_{L^q_x} \leq C \|\phi\|_{\dot{W}^{k,r}_x}, \tag{3.1.1}$$

  where $\dot{W}^{k,r}_x$ is the L^r-based homogeneous Sobolev norm of order k.

- **(Interpolation inequality)** *For $1 < q < \infty$, $k_1 \leq k_0 \leq k_2$, $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$ and $k_0 = \theta_1 k_1 + \theta_2 k_2$, we have*

$$\|\phi\|_{\dot{W}^{k_0,q}_x} \leq C \|\phi\|^{\theta_1}_{\dot{W}^{k_1,q}_x} \|\phi\|^{\theta_2}_{\dot{W}^{k_2,q}_x}. \tag{3.1.2}$$

- **(Gagliardo-Nirenberg inequality)** *For $1 \leq q_1, q_2, r \leq \infty$ and $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$ and $\frac{d}{r} = \theta_1 \cdot \frac{d}{q_1} + \theta_2(\frac{d}{q_2} - 1)$, we have*

$$\|\phi\|_{L^r_x} \leq C \|\phi\|^{\theta_1}_{L^{q_1}_x} \|\partial_x \phi\|^{\theta_2}_{L^{q_2}_x}. \tag{3.1.3}$$

Remark 3.1.2. This lemma is applicable for $\phi_1, \phi_2 \in H_x^\infty$, by a simple approximation argument.

Proof. These inequalities are standard; we refer the reader to [1]. □

The following two lemmas are standard results concerning product estimates with respect to homogeneous Sobolev norms.

Lemma 3.1.3 (Homogeneous Sobolev product estimates, non-endpoint case)**.** *Let $d \geq 1$, and γ_0, γ_1, γ_2 real numbers satisfying*

$$\gamma_0 + \gamma_1 + \gamma_2 = d/2, \quad \gamma_0 + \gamma_1 + \gamma_2 > \max(\gamma_0, \gamma_1, \gamma_2). \tag{3.1.4}$$

Then for any $\phi_1, \phi_2 \in \mathcal{S}_x(\mathbb{R}^d)$, we have

$$\|\phi_1 \phi_2\|_{\dot{H}_x^{-\gamma_0}} \leq C_{d,\gamma_1,\gamma_2,\gamma_3} \|\phi_1\|_{\dot{H}_x^{\gamma_1}} \|\phi_2\|_{\dot{H}_x^{\gamma_2}}. \tag{3.1.5}$$

Lemma 3.1.4 (Homogeneous Sobolev product estimate, an endpoint case)**.** *Let $d \geq 1$ and $-\frac{d}{2} < \gamma < \frac{d}{2}$. Then for any $\phi_1, \phi_2 \in \mathcal{S}_x(\mathbb{R}^d)$, we have*

$$\|\phi_1 \phi_2\|_{\dot{H}_x^{\gamma}} \leq C_{d,\gamma} \|\phi_1\|_{\dot{H}_x^{d/2} \cap L^\infty} \|\phi_2\|_{\dot{H}_x^{\gamma}}. \tag{3.1.6}$$

In the 'double-endpoint' case $\gamma = d/2$, the following variant of the preceding estimate holds.

$$\|\phi_1 \phi_2\|_{\dot{H}_x^{d/2} \cap L_x^\infty} \leq C_d \|\phi_1\|_{\dot{H}_x^{d/2} \cap L_x^\infty} \|\phi_2\|_{\dot{H}_x^{d/2} \cap L_x^\infty}. \tag{3.1.7}$$

Remark 3.1.5. Again, by a simple approximation argument, both lemmas are applicable for $\phi_1, \phi_2 \in H_x^\infty$.

Finally, we state Gronwall's inequality, which will be useful in several places below.

Lemma 3.1.6 (Gronwall's inequality)**.** *Let $(s_0, s_1) \subset \mathbb{R}$ be an interval, $D \geq 0$, and $f(s)$, $r(s)$ non-negative measurable functions on J. Suppose that for all $s \in (s_0, s_1)$, the following inequality holds:*

$$\sup_{\overline{s} \in (s_0, s]} f(\overline{s}) \leq \int_{s_0}^{s} r(\overline{s}) f(\overline{s}) \, \mathrm{d}\overline{s} + D.$$

Then for all $s \in (s_0, s_1)$, we have

$$\sup_{\overline{s} \in (s_0, s]} f(\overline{s}) \leq D \exp\Big(\int_{s_0}^{s} r(\overline{s}) \, \mathrm{d}\overline{s} \Big).$$

Proof. See [31, Lemma 3.3]. □

3.1.2 P-normalized norms

Given a function ϕ on $\mathbb{R}^d$, we consider the operation of *scaling by* $\lambda > 0$, defined by

$$\phi \to \phi_\lambda(x) := \phi(x/\lambda).$$

We say that a norm $\|\cdot\|_X$ is *homogeneous* if it is covariant with respect to scaling, i.e. there exists a real number ℓ such that

$$\|\phi_\lambda\|_X = \lambda^\ell \|\phi\|_X.$$

The number ℓ is called the *degree of homogeneity* of the norm $\|\cdot\|_X$.

Let ϕ be a solution to the heat equation $\partial_s\phi - \triangle\phi = 0$ on $\mathbb{R}^d \times [0,\infty)$. Note that this equation 'respects' the scaling $\phi_\lambda(x,s) := \phi(x/\lambda, s/\lambda^2)$, in the sense that any scaled solution to the linear heat equation remains a solution. Moreover, one has *smoothing estimates* of the form $\|\partial_x^{(k)}\phi(s)\|_{L^p_x} \leq s^{-\frac{k}{2}+(\frac{d}{2p}-\frac{d}{2q})}\|\phi(0)\|_{L^q_x}$ (for $q \leq p$, $k \geq 0$) which are invariant under this scaling. The norms $\|\partial_x \cdot\|_{L^p_x}$ and $\|\cdot\|_{L^q_x}$ are homogeneous, and the above estimate can be rewritten as

$$s^{-\ell_1/2}\|\partial_x\phi(s)\|_{L^p_x} \leq s^{-\ell_2/2}\|\phi(0)\|_{L^q_x}$$

where ℓ_1, ℓ_2 are the degrees of homogeneity of the norms $\|\partial_x \cdot\|_{L^p_x}$ and $\|\cdot\|_{L^q_x}$, respectively.

Motivated by this example, we shall define the notion of *parabolic-normalized*, or *p-normalized*, norms and derivatives. These are designed to facilitate the analysis of parabolic equations by capturing their scaling properties.

Consider a homogeneous norm $\|\cdot\|_X$ of degree 2ℓ, which is well-defined for smooth functions ϕ on $\mathbb{R}^d$. (i.e. for every smooth ϕ, $\|\phi\|_X$ is defined uniquely as either a non-negative real number or ∞.) We shall define its *p-normalized* analogue $\|\cdot\|_{\mathcal{X}(s)}$ for each $s > 0$ by

$$\|\cdot\|_{\mathcal{X}(s)} := s^{-\ell}\|\cdot\|_X.$$

We shall also define the p-normalization of space-time norms. As we shall be concerned with functions restricted to a time interval, we shall adjust the notion of homogeneity of norms as follows. For $I \subset \mathbb{R}$, consider a *family* of norms $X(I)$ defined for functions ϕ defined on $I \times \mathbb{R}^d$. For $\lambda > 0$, consider the scaling $\phi_\lambda(t,x) := \phi(t/\lambda, x/\lambda)$. We shall say that $X(I)$ is *homogeneous of degree* ℓ if

$$\|\phi_\lambda\|_{X(I)} = \lambda^\ell \|\phi\|_{X(\lambda I)}.$$

As before, we define its *p-normalized* analogue $\|\cdot\|_{\mathcal{X}(I,s)}$ as $\|\cdot\|_{\mathcal{X}(I,s)} := s^{-\ell}\|\cdot\|_{X(T)}$.

Let us furthermore define the *parabolic-normalized derivative* $\nabla_\mu(s)$ by $s^{1/2}\partial_\mu$. Accordingly, for $k > 0$ we define the *homogeneous k-th derivative norm* $\|\cdot\|_{\dot{\mathcal{X}}^k(s)}$ by

$$\|\cdot\|_{\dot{\mathcal{X}}^k(s)} := \|\nabla_x^{(k)}(s)\cdot\|_{\mathcal{X}(s)}.$$

We shall also define the *parabolic-normalized covariant derivative* $\mathcal{D}_\mu(s) := s^{1/2}\mathbf{D}_\mu$.

We shall adopt the convention $\dot{\mathcal{X}}^0 := \mathcal{X}$. For $m > 0$ an integer, we define *inhomogeneous m-th derivative norm* $\|\cdot\|_{\mathcal{X}^k(s)}$ by

$$\|\cdot\|_{\dot{\mathcal{X}}^m(s)} := \sum_{k=0}^{m} \|\cdot\|_{\dot{\mathcal{X}}^k(s)}.$$

We shall often omit the s-dependence of $\mathcal{X}(s)$, $\dot{\mathcal{X}}(s)$ and $\nabla_\mu(s)$ by simply writing $\mathcal{X}$, $\dot{\mathcal{X}}$ and ∇_μ, where the value of s should be clear from the context.

Example 3.1.7. A few examples of homogeneous norms and their p-normalized versions are in order. We shall also take this opportunity to fix the notations for the p-normalized norms which will be used in the rest of the thesis.

1. $X = L^p_x$, in which case the degree of homogeneity is $2\ell = d/p$. We shall define $\mathcal{X} = \mathcal{L}^p_x$ and $\dot{\mathcal{X}}^\gamma := \dot{\mathcal{W}}^{k,p}_x$ as follows.

$$\|\cdot\|_{\mathcal{L}^p_x(s)} := s^{-d/(2p)}\|\cdot\|_{L^p_x}, \qquad \|\cdot\|_{\dot{\mathcal{W}}^{k,p}_x(s)} := s^{(k-d/p)/2}\|\cdot\|_{\dot{W}^{k,p}_x}.$$

The norm $\mathcal{X}^m := \mathcal{W}^{m,p}_x$ will be defined as the sum of $\dot{\mathcal{W}}^{k,p}_x$ norms for $k = 0, \dots, m$.

2. The case $p = 2$ is the most frequently used in this thesis, and merits a special mention. For $m \geq 0$ an integer, we shall use the notation $\dot{\mathcal{X}}^m = \dot{\mathcal{H}}^m_x$ and $\mathcal{X}^m = \mathcal{H}^m_x$. That is,

$$\|\cdot\|_{\dot{\mathcal{H}}^m_x} := s^{(k-d/2)/2}\|\cdot\|_{\dot{H}^m_x}, \quad \|\cdot\|_{\mathcal{H}^m_x} := \sum_{k=0}^{m}\|\cdot\|_{\dot{\mathcal{H}}^k_x}.$$

By interpolation, the summands for $k = 1, \dots, m-1$ may be omitted in the definition of $\mathcal{H}^m_x$.

Using this observation, we shall extend k to real numbers as follows: For $\gamma \in \mathbb{R}$ we define

$$\| \cdot \|_{\dot{\mathcal{H}}^\gamma_x} := s^{(\gamma - d/2)/2} \| \cdot \|_{\dot{H}^\gamma_x}, \quad \| \cdot \|_{\mathcal{H}^\gamma_x} := \| \cdot \|_{\dot{\mathcal{H}}^\gamma_x} + \| \cdot \|_{\mathcal{L}^2_x}.$$

Finally, throughout this chapter, for $\gamma \in \mathbb{R}$, the notation

$$\ell_\gamma := \frac{1}{2}\Big(\frac{d}{2} - \gamma\Big),$$

will also be used, which is simply the degree of homogeneity of the space $\dot{H}^\gamma_x$.

3. Consider a time interval $I \subset \mathbb{R}$. For $X = L^q_t L^p_x(I \times \mathbb{R}^d)$, note that $2\ell = 1/q + d/p$. We shall write

$$\begin{aligned} \| \cdot \|_{\mathcal{L}^q_t \mathcal{L}^p_x(I,s)} :=& s^{-1/(2q) - d/(2p)} \| \cdot \|_{L^q_t L^p_x}, \\ \| \cdot \|_{\mathcal{L}^q_t \dot{\mathcal{W}}^{k,p}_x(I,s)} :=& s^{k/2 - 1/(2q) - d/(2p)} \| \cdot \|_{L^q_t \dot{W}^{k,p}_x}. \end{aligned}$$

   The norms $\mathcal{L}^q_t \mathcal{W}^{k,p}_x(I,s)$, $\mathcal{L}^q_t \dot{\mathcal{H}}^\gamma_x(I,s)$ and $\mathcal{L}^q_t \mathcal{H}^\gamma_x(I,s)$ are defined accordingly.

For $f = f(s)$ a measurable function defined on an s-interval $J \subset (0,\infty)$, we define its p-normalized Lebesgue norm $\|f\|_{\mathcal{L}^p_s(J)}$ by $\|f\|^p_{\mathcal{L}^p_s(J)} := \int_J |f(s)|^p \frac{\mathrm{d}s}{s}$ for $1 \le p < \infty$, and $\|f\|_{\mathcal{L}^\infty_s(J)} := \|f\|_{L^\infty_s(J)}$. Given $\ell \ge 0$, we shall define the weighted norm $\|f\|_{\mathcal{L}^{\ell,p}_s}(J)$ by

$$\|f\|_{\mathcal{L}^{\ell,p}_s(J)} := \|s^\ell f(s)\|_{\mathcal{L}^p_s(J)}.$$

Let us consider the case $J = (0, s_0)$ or $J = (0, s_0]$ for some $s_0 > 0$. For $\ell > 0$ and $1 \le p \le \infty$, note the obvious computation $\|s^\ell\|_{\mathcal{L}^p_s(0,s_0)} = C_{\ell,p} s_0^\ell$. Combining this with the Hölder inequality

$$\|fg\|_{\mathcal{L}^p_s} \le \|f\|_{\mathcal{L}^{p_1}_s} \|g\|_{\mathcal{L}^{p_2}_s} \text{ for } \tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2},$$

we obtain the following simple lemma.

Lemma 3.1.8 (Hölder for $\mathcal{L}^{\ell,p}_s$)**.** *Let $\ell, \ell_1, \ell_2 \ge 0$, $1 \le p, p_1, p_2 \le \infty$ and f, g functions on $J = (0, s_0)$ (or $J = (0, s_0]$) such that $\|f\|_{\mathcal{L}^{\ell_1,p_1}_s}, \|g\|_{\mathcal{L}^{\ell_2,p_2}_s} < \infty$. Then we have*

$$\|fg\|_{\mathcal{L}^{\ell,p}_s(J)} \le C s_0^{\ell - \ell_1 - \ell_2} \|f\|_{\mathcal{L}^{\ell_1,p_1}_s(J)} \|g\|_{\mathcal{L}^{\ell_2,p_2}_s(J)}$$

provided that either $\ell = \ell_1 + \ell_2$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*, or* $\ell > \ell_1 + \ell_2$ *and* $\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}$*. In the former case,* $C = 1$*, while in the latter case,* C *depends on* $\ell - \ell_1 - \ell_2$ *and* $\frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2}$*.*

We shall often used the mixed norm $\|\psi\|_{\mathcal{L}^{\ell,p}_s \mathcal{X}(J)} := \| \|\psi(s)\|_{\mathcal{X}(s)} \|_{\mathcal{L}^{\ell,p}_s(J)}$ for $\psi = \psi(x, s)$ such that $s \to \|\psi(s)\|_{\mathcal{X}(s)}$ is measurable. The norms $\mathcal{L}^{\ell,p}_s \dot{\mathcal{X}}^k(J)$ and $\mathcal{L}^{\ell,p}_s \mathcal{X}^k(J)$ are defined analogously.

3.1.3 Abstract parabolic theory

Let $J \subset (0, \infty)$ be an s-interval. Given a homogeneous norm X and $k \geq 1$ an integer, let us define the (semi-)norm $\mathcal{P}^\ell \dot{\mathcal{X}}^k(J)$ for a smooth function ψ by

$$\|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^k(J)} := \|\psi\|_{\mathcal{L}^{\ell,\infty}_s \dot{\mathcal{X}}^{k-1}(J)} + \|\psi\|_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{X}}^k(J)}.$$

For $m_0 < m_1$, we shall also define the (semi-)norm $\mathcal{P}^\ell \mathcal{X}^{m_1}_{m_0}(J)$ by

$$\|\psi\|_{\mathcal{P}^\ell \mathcal{X}^{m_1}_{m_0}(J)} := \sum_{k=m_0+1}^{m_1} \|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^k(J)}.$$

We shall omit m_0 when $m_0 = 0$, i.e. $\mathcal{X}^m := \mathcal{X}^m_0$.

We remark that despite the notation $\mathcal{P}^\ell \dot{\mathcal{X}}^k$, this norms controls both the $\dot{\mathcal{X}}^{k-1}$ as well as the $\dot{\mathcal{X}}^k$ norm of ψ. Note furthermore that $\|\psi\|_{\mathcal{P}^\ell \mathcal{X}^{m_1}_{m_0}}$ controls the derivatives of ψ of order from m_0 to m_1.

Definition 3.1.9. Let X be a homogeneous norm of degree $2\ell_0$. We shall say that X satisfies the *parabolic energy estimate* if there exists $C_X > 0$ such that for all $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0, \infty)$ and smooth ψ satisfying $\|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^1(s_1,s_2]} < \infty$, the following estimate holds.

$$\begin{aligned} \|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^1(s_1,s_2]} \leq & C_X s_1^\ell \|\psi(s_1)\|_{\mathcal{X}(s_1)} + C_X(\ell - \ell_0)\|\psi\|_{\mathcal{L}^{\ell,2}_s \mathcal{X}(s_1,s_2]} \\ & + C_X \|(\partial_s - \triangle)\psi\|_{\mathcal{L}^{\ell+1,1}_s \mathcal{X}(s_1,s_2]}. \end{aligned} \tag{3.1.8}$$

We shall say that the norm X satisfies the *parabolic smoothing estimate* if there exists $C_X > 0$ such that for all $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0, \infty)$ and smooth ψ satisfying $\|\psi\|_{\mathcal{P}^\ell \mathcal{X}^2(s_1,s_2]} < \infty$, the following estimate holds:

$$\begin{aligned} \|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^2(s_1,s_2]} \leq & C_X s_1^\ell \|\psi(s_1)\|_{\dot{\mathcal{X}}^1(s_1)} + C_X(\ell + 1/2 - \ell_0)\|\psi\|_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{X}}^1(s_1,s_2]} \\ & + C_X \|(\partial_s - \triangle)\psi\|_{\mathcal{L}^{\ell+1,2}_s \mathcal{X}(s_1,s_2]}. \end{aligned} \tag{3.1.9}$$

For the purpose of application, we shall consider vector-valued solutions ψ to an inhomogeneous heat equation. The norms X, $\mathcal{X}$, $\mathcal{P}^\ell \mathcal{X}$, etc. of a vector-valued function ψ are defined in the obvious manner.

Theorem 3.1.10 (Main theorem of abstract parabolic theory)**.** *Let $d \geq 1$, X a homogeneous norm of degree $2\ell_0$, and ψ a vector-valued smooth solution to $\partial_s \psi - \triangle \psi = \mathcal{N}$ on $[0, s_0]$. (The function ψ is defined on $\mathbb{R}^d \times [0, s_0]$ or $I \times \mathbb{R}^d \times [0, s_0]$ depending on whether X is for functions on the space or the space-time, respectively.)*

1. *Let X satisfy the parabolic energy and smoothing estimates (3.1.8), (3.1.9), and ψ satisfy $\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} < \infty$. Let $1 \leq p < \infty$, $\epsilon > 0$, $D > 0$ and $C(s)$ a function defined on $(0, s_0]$ which satisfies*

$$\int_0^{s_0} C(s)^p \frac{\mathrm{d}s}{s} < \infty$$

   for some $1 \leq p < \infty$, and

$$\|\mathcal{N}\|_{\mathcal{L}^{\ell_0+1,1}_s \mathcal{X}(0,\underline{s}]} + \|\mathcal{N}\|_{\mathcal{L}^{\ell_0+1,2}_s \mathcal{X}(0,\underline{s}]} \leq \|C(s)\psi\|_{\mathcal{L}^{\ell_0,p}_s \mathcal{X}^1(0,\underline{s}]} + \epsilon \|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D, \tag{3.1.10}$$

   for every $\underline{s} \in (0, s_0]$.

   Then there exists a constant $\delta_A = \delta_A(C_X, \int_0^{s_0} C(s)^p \frac{\mathrm{d}s}{s}, p) > 0$ such that if $0 < \epsilon < \delta_A$, then the following a priori estimate holds.

$$\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} \leq C e^{C \int_0^{s_0} C(s)^p \frac{\mathrm{d}s}{s}} (\|\psi(s=0)\|_X + D), \tag{3.1.11}$$

   where C depends only on C_X and p.

2. *Suppose that X satisfies the the parabolic smoothing estimate (3.1.9), and that for some $\ell \geq \ell_0 - 1/2$ and $0 \leq m_0 \leq m_1$ (where m_0, m_1 are integers) we have $\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m_1+2}_{m_0}(0,s_0]} < \infty$. Suppose furthermore that for $m_0 \leq m \leq m_1$, there exists $\epsilon > 0$ and a non-negative non-decreasing function $\mathcal{B}_m(\cdot)$ such that*

$$\|\mathcal{N}\|_{\mathcal{L}^{\ell+1,2}_s \dot{\mathcal{X}}^m(0,s_0]} \leq \epsilon \|\psi\|_{\mathcal{P}^{\ell+1}\dot{\mathcal{X}}^{m+2}(0,s_0]} + \mathcal{B}_m(\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+1}_{m_0}(0,s_0]}). \tag{3.1.12}$$

   Then for $0 < \epsilon < 1/(2C_X)$, the following smoothing estimate *holds:*

$$\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m_1+2}_{m_0}(0,s_0]} \leq C \tag{3.1.13}$$

   where C is determined from C_X, $\mathcal{B}_{m_0}, \ldots, \mathcal{B}_{m_1}$ and $\|\psi\|_{\mathcal{P}^{\ell}\dot{\mathcal{X}}^{m_0+1}(0,s_0]}$.

More precisely, consider the non-decreasing function $\widetilde{\mathcal{B}_m}(r) := (2C_X(\ell - \ell_0 + 1/2) + 1)r + 2C_X\mathcal{B}_m(r)$. *Then* C *in* (3.1.13) *is given by the composition*

$$C = \widetilde{\mathcal{B}}_{m_1} \circ \widetilde{\mathcal{B}}_{m_1-1} \circ \cdots \circ \widetilde{\mathcal{B}}_{m_0}(\|\psi\|_{\mathcal{P}^\ell \dot{\mathcal{X}}^{m_0+1}(0,s_0]}). \tag{3.1.14}$$

Proof. Step 1: Proof of Statement 1. Without loss of generality, assume that $C_X \geq 2$. Thanks to the hypothesis on ψ and (3.1.10), we can apply the parabolic energy estimate (3.1.8) to obtain

$$\|\psi\|_{\mathcal{P}^{\ell_0}\dot{\mathcal{X}}^1(0,\underline{s}]} \leq C_X\|\psi(0)\|_X + C_X(\|C(s)\psi\|_{\mathcal{L}^{\ell_0,p}_s\mathcal{X}^1(0,\underline{s}]} + \epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D). \tag{3.1.15}$$

where we have used the fact that $\liminf_{s_1\to 0} s_1^{\ell_0}\|\psi(s_1)\|_{\mathcal{X}(s_1)} = \|\psi(0)\|_X$. Using again the hypothesis on ψ and (3.1.10), we can apply the parabolic smoothing estimate (3.1.9) and get

$$\|\psi\|_{\mathcal{P}^{\ell_0}\dot{\mathcal{X}}^2(0,\underline{s}]} \leq \frac{C_X}{2}\|\psi\|_{\mathcal{L}^{\ell_0,2}_s\dot{\mathcal{X}}^1(0,\underline{s}]} + C_X(\|C(s)\psi\|_{\mathcal{L}^{\ell_0,p}_s\mathcal{X}^1(0,\underline{s}]} + \epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D), \tag{3.1.16}$$

where we used $\liminf_{s_1\to 0} s_1^{\ell_0}\|\psi(s_1)\|_{\dot{\mathcal{X}}^1(s_1)} = 0$, which holds as $\|\psi\|_{\mathcal{L}^{\ell_0,2}_s\dot{\mathcal{X}}^1} < \infty$. Using (3.1.15) to bound the first term on the right-hand of (3.1.16), we arrive at

$$\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,\underline{s}]} \leq C_X^2\|\psi(0)\|_X + C_X(1+C_X)(\|C(s)\psi\|_{\mathcal{L}^{\ell_0,p}_s\mathcal{X}^1(0,\underline{s}]} + \epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D),$$

for every $0 < \underline{s} \leq s_0$.

We shall apply Gronwall's inequality to deal with the term involving $C(s)\psi$. For convenience, let us make the definition

$$D' = C_X^2\|\psi(0)\|_X + C_X(1+C_X)(\epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D).$$

Recalling the definition of $\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,\underline{s}]}$ and unravelling the definition of $\mathcal{L}^{\ell_0,p}_s\mathcal{X}^1$, we see in particular that

$$\sup_{0<s\leq\underline{s}} s^{\ell_0}\|\psi(s)\|_{\mathcal{X}^1} \leq C_X(1+C_X)\Big(\int_0^{\underline{s}} C(s)^p(s^{\ell_0}\|\psi(s)\|_{\mathcal{X}^1})^p\frac{\mathrm{d}s}{s}\Big)^{1/p} + D',$$

for every $0 < \underline{s} \leq s_0$. Taking the p-th power, using Gronwall's inequality and then taking the p-th root back, we arrive at

$$\sup_{0<s\leq \underline{s}} s^{\ell_0}\|\psi(s)\|_{\mathcal{X}^1} \leq 2^{1/p} D' \exp\Big(\frac{2C_X^p(1+C_X)^p}{p}\int_0^{\underline{s}} C(s)^p \frac{\mathrm{d}s}{s}\Big).$$

Iterating this bound into $\|C(s)\psi\|_{\mathcal{L}^{\ell_0,p}_s \mathcal{X}^1(0,s_0]}$ and expanding D' out, we obtain

$$\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} \leq C_0\Big(C_X^2\|\psi(0)\|_X + C_X(1+C_X)(\epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]} + D)\Big).$$

where $C_0 = \exp\Big(\frac{2C_X^p(1+C_X)^p}{p}\int_0^{s_0} C(s)^p \frac{\mathrm{d}s}{s}\Big)$.

Let us define $\delta_A := (2C_0C_X(1+C_X))^{-1}$. Then from the hypothesis $0<\epsilon<\delta_A$, we can absorb the term $C_0C_X(1+C_X)\epsilon\|\psi\|_{\mathcal{P}^{\ell_0}\mathcal{X}^2(0,s_0]}$ into the left-hand side. The desired estimate (3.1.11) follows.

Step 2: Proof of Statement 2. In this step, we shall always work on the whole s-interval $(0,s_0]$.

We claim that under the assumptions of (2), the following inequality holds for $m_0 \leq m \leq m_1$:

$$\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+2}_{m_0}} \leq \widetilde{\mathcal{B}}_m(\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+1}_{m_0}}). \tag{3.1.17}$$

Assuming the claim, we can start from $\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m_0+1}_{m_0}} = \|\psi\|_{\mathcal{P}^{\ell}\dot{\mathcal{X}}^{m_0+1}}$ and iterate (3.1.17) for $m = m_0, m_0+1, \dots, m_1$ (using the fact that each $\widetilde{\mathcal{B}}_m$ is non-decreasing) to conclude the proof.

To prove the claim, we use the hypothesis on ψ and (3.1.12) to apply the parabolic smoothing estimate (3.1.9), which gives

$$\|\psi\|_{\mathcal{P}^{\ell}\dot{\mathcal{X}}^{m+2}} \leq C_X(\ell-\ell_0+1/2)\|\psi\|_{\mathcal{L}^{\ell,2}_s\dot{\mathcal{X}}^{m+1}} + C_X(\epsilon\|\psi\|_{\mathcal{P}^{\ell}\dot{\mathcal{X}}^{m+2}} + \mathcal{B}_m(\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+1}_{m_0}})),$$

where we have used $\liminf_{s_1\to 0} s_1^{\ell}\|\psi(s_1)\|_{\dot{\mathcal{X}}^{m+1}(s_1)} = 0$, which holds as $\|\psi\|_{\mathcal{L}^{\ell,2}_s\dot{\mathcal{X}}^{m+1}} < \infty$.

Using the smallness of $\epsilon>0$, we can absorb $C_X\epsilon\|\psi\|_{\mathcal{P}^{\ell}\dot{\mathcal{X}}^{m+2}}$ into the left-hand side. Then adding $\|\psi\|_{\mathcal{L}^{\ell,2}_s\mathcal{X}^{m+1}_{m_0}}$ to both sides, we easily obtain

$$\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+2}_{m_0}} \leq (2C_X(\ell-\ell_0+1/2)+1)\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+1}_{m_0}} + 2C_X\mathcal{B}_m(\|\psi\|_{\mathcal{P}^{\ell}\mathcal{X}^{m+1}_{m_0}}).$$

Recalling the definition of $\widetilde{\mathcal{B}}_m$, this is exactly (3.1.17). □

The following proposition allows us to use Theorem 3.1.10 in the situations of interest for us.

Proposition 3.1.11. *Let $d \geq 1$. Then the following statements hold.*

1. *Let ψ a function in $C^{\infty}_s(J, H^{\infty}(\mathbb{R}^d))$ (resp. in $C^{\infty}_{t,s} I \times J, H^{\infty}(\mathbb{R}^d)$ with $I \subset \mathbb{R}$ an interval),*

where J is a finite interval. Then for $X = L^2_x$ (resp. $X = L^2_{t,x}$), we have

$$\|\psi\|_{\mathcal{L}^{\ell,p}_s \dot{\mathcal{X}}^k(J)} < \infty \tag{3.1.18}$$

if either $1 \leq p \leq \infty$ and $\ell - \ell_0 + k/2 > 0$, or $p = \infty$ and $\ell - \ell_0 + k/2 = 0$.

2. *Furthermore, the norms L^2_x and $L^2_{t,x}$ satisfy the parabolic energy and smoothing estimates (3.1.8), (3.1.9).*

Proof. By definition, we have

$$\|\psi\|_{\mathcal{L}^{\ell,p}\dot{\mathcal{X}}^k} = \|s^{\ell-\ell_0+k/2}\|\partial_x^{(k)}\psi(s)\|_X\|_{\mathcal{L}^p_s}.$$

Since $\sup_{s\in J}\|\partial_x^{(k)}\psi(s)\|_X < \infty$ for each X under consideration when $\psi \in C^\infty_s(J, H^\infty_x)$, the first statement follows.

To prove the second statement, let us begin by proving that the norm L^2_x satisfies the parabolic energy estimate (3.1.8). In this case, $\ell_0 = d/4$. Let $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0,\infty)$ and ψ a smooth (complex-valued) function such that $\|\psi\|_{\mathcal{P}^\ell\dot{\mathcal{H}}^1_x} < \infty$. We may assume that $\|(\partial_s - \triangle)\psi\|_{\mathcal{L}^{\ell+1,1}_s\mathcal{L}^2_x} < \infty$, as the other case is trivial. Multiplying the equation $(\partial_s - \triangle)\psi$ by $s^{2(\ell-\ell_0)}\overline{\psi}$ and integrating by parts over $[s_1, \underline{s}]$ (where $s_1 \leq \underline{s} \leq s_2$), we obtain

$$\begin{aligned}
&\frac{1}{2}\underline{s}^{2(\ell-\ell_0)}\int |\psi(\underline{s})|^2 \,\mathrm{d}x + \int_{s_1}^{\underline{s}}\int s^{2(\ell-\ell_0)+1}|\partial_x\psi|^2 \,\mathrm{d}x \frac{\mathrm{d}s}{s} \\
&\quad= \frac{1}{2}s_1^{2(\ell-\ell_0)}\int |\psi(s_1)|^2 \,\mathrm{d}x + (\ell-\ell_0)\int_{s_1}^{\underline{s}}\int s^{2(\ell-\ell_0)}|\psi|^2 \,\mathrm{d}x \frac{\mathrm{d}s}{s} \\
&\qquad + \int_{s_1}^{\underline{s}}\int s^{2(\ell-\ell_0)+1}(\partial_s - \triangle)\psi\cdot\overline{\psi}\,\mathrm{d}x\frac{\mathrm{d}s}{s}.
\end{aligned} \tag{3.1.19}$$

Taking the supremum over $s_1 \leq \underline{s} \leq s_2$ and rewriting in terms of p-normalized norms, we obtain

$$\begin{aligned}
\frac{1}{2}\|\psi\|^2_{\mathcal{L}^{\ell,\infty}_s\mathcal{L}^2_x(s_1,s_2]} + \|\psi\|^2_{\mathcal{L}^{\ell,2}_s\dot{\mathcal{H}}^1_x(s_1,s_2]} \leq& \frac{1}{2}s_1^{2\ell}\|\psi(s_1)\|^2_{\mathcal{L}^2_x(s_1)} + (\ell-\ell_0)\|\psi\|^2_{\mathcal{L}^{\ell,2}_s\mathcal{L}^2_x(s_1,s_2]} \\
&+ \|(\partial_s - \triangle)\psi\cdot\overline{\psi}\|_{\mathcal{L}^{2\ell+1,1}_s\mathcal{L}^1_x(s_1,s_2]}.
\end{aligned}$$

By Hölder and Lemma 3.1.8, we can estimate the last term by $\|(\partial_s - \triangle)\psi\|^2_{\mathcal{L}^{\ell+1,1}_s\mathcal{L}^2_x(s_1,s_2]} + (1/4)\|\psi\|^2_{\mathcal{L}^{\ell,\infty}_s\mathcal{L}^2_x(s_1,s_2]}$, where the latter can be absorbed into the left-hand side. Taking the square root of both sides, we obtain (3.1.8) for L^2_x.

Next, let us prove that the norm L^2_x satisfies the parabolic smoothing estimate (3.1.9). Let $\ell \in \mathbb{R}$, $[s_1, s_2] \subset (0,\infty)$ and ψ a smooth (complex-valued) function such that $\|\psi\|_{\mathcal{P}^\ell\dot{\mathcal{H}}^2_x} < \infty$. As before, we assume that $\|(\partial_s - \triangle)\psi\|_{\mathcal{L}^{\ell+1,2}_s \mathcal{L}^2_x} < \infty$. Multiplying the equation $(\partial_s - \triangle)\psi$ by $s^{2(\ell-\ell_0)+1}\triangle\overline{\psi}$ and integrating by parts over $[s_1, \underline{s}]$ (where $s_1 \leq \underline{s} \leq s_2$), we obtain

$$\begin{aligned}
&\frac{1}{2}\underline{s}^{2(\ell-\ell_0)} \int |\partial_x \psi(\underline{s})|^2 \, \mathrm{d}x + \int_{s_1}^{\underline{s}} \int s^{2(\ell-\ell_0)+2} |\triangle \psi|^2 \, \mathrm{d}x \, \frac{\mathrm{d}s}{s} \\
&\quad = \frac{1}{2} s_1^{2(\ell-\ell_0)+1} \int |\partial_x \psi(s_1)|^2 \, \mathrm{d}x + (\ell - \ell_0 + \frac{1}{2}) \int_{s_1}^{\underline{s}} \int s^{2(\ell-\ell_0)+1} |\partial_x \psi|^2 \, \mathrm{d}x \frac{\mathrm{d}s}{s} \\
&\qquad + \int_{s_1}^{\underline{s}} \int s^{2(\ell-\ell_0)+2} (\partial_s - \triangle)\psi \cdot \triangle\overline{\psi} \, \mathrm{d}x \, \frac{\mathrm{d}s}{s}.
\end{aligned} \tag{3.1.20}$$

By a further integration by parts, the second term on the left-hand side is equal to $\|\psi\|^2_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{H}}^2_x(s_1,s_2]}$. Taking the supremum over $s_1 \leq \underline{s} \leq s_2$ and rewriting in terms of p-normalized norms, we obtain

$$\begin{aligned}
\frac{1}{2}\|\psi\|^2_{\mathcal{L}^{\ell,\infty}_s \dot{\mathcal{H}}^1_x(s_1,s_2]} + \|\psi\|^2_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{H}}^2_x(s_1,s_2]} \leq & \frac{1}{2} s_1^{2\ell} \|\psi(s_1)\|^2_{\dot{\mathcal{H}}^1_x(s_1)} + (\ell - \ell_0 + \frac{1}{2}) \|\psi\|^2_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{H}}^1_x(s_1,s_2]} \\
& + \|(\partial_s - \triangle)\psi \cdot \nabla^k \nabla_k \overline{\psi}\|_{\mathcal{L}^{2\ell+1,1}_s \mathcal{L}^1_x(s_1,s_2]}.
\end{aligned}$$

By Cauchy-Schwarz and Lemma 3.1.8, we can estimate the last term by

$$(1/2)\|(\partial_s - \triangle)\psi\|^2_{\mathcal{L}^{\ell+1,2}_s \mathcal{L}^2_x(s_1,s_2]} + (1/2)\|\psi\|^2_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{H}}^2_x(s_1,s_2]},$$

where the latter can be absorbed into the left-hand side. Taking the square root of both sides, we obtain (3.1.9) for L^2_x.

For the norm $L^2_{t,x}$, in which case $\ell_0 = (1+d)/4$, it simply suffices to repeat the above proof with the new value of ℓ_0, and integrate further in time. □

Remark 3.1.12. A point that the reader should keep in mind is that, despite the heavy notations and abstract concepts developed in this subsection, the analytic heart of the *abstract parabolic theory* is simply the standard L^2-energy integral estimates for the linear heat equation, as we have seen in Proposition 3.1.11.

3.1.4 Correspondence Principle for p-normalized norms

In this subsection, we shall develop a systematic method of obtaining linear and multi-linear estimates in terms of p-normalized norms, which will be very useful to us later. The idea is to start with an estimate involving the norms of functions independent of the s-variable, and arrive at the corresponding estimate for s-dependent functions in terms of the corresponding p-normalized norms by putting appropriate weights of s.

Throughout this subsection, we shall denote by $J \subset (0, \infty)$ an s-interval, $\phi_i = \phi_i(x)$ a smooth function independent of s, and $\psi_i = \psi_i(s, x)$ a smooth function of both $s \in J$ and x. All norms below will be assumed *a priori* to be finite. In application, ϕ_i may be usually taken to be H^∞_x and ψ would be in $C^\infty_s(J, H^\infty_x)$. The discussion to follow holds also for functions which depend additionally on t.

It is rather cumbersome to give a precise formulation of the Correspondence Principle. We shall instead adopt a more pragmatic approach and be satisfied with the following 'cookbook-recipe' type statement.

Correspondence Principle. *Suppose that we are given an estimate (i.e. an inequality) in terms of the norms X_i of functions $\phi_i = \phi_i(x)$, all of which are homogeneous. Suppose furthermore that the estimate is scale-invariant, in the sense that both sides transform the same under scaling.*

Starting from the usual estimate, make the following substitutions on both sides:

$$\phi_i \to \phi_i(s), \quad \partial_x \to \nabla_x(s), \quad \mathbf{D}_x \to \mathcal{D}_x(s), \quad X_i \to \mathcal{X}_i(s).$$

Then the resulting estimate still holds, with the same constant, for every $s \in J$.

In other words, given an s-independent, scale-invariant estimate which involve only homogeneous norms, we obtain its p-normalized analogue by replacing the norms and the derivatives by their respective p-normalizations. The 'proof' of this principle is very simple: For each fixed s, the substitution procedure above amounts to applying the usual estimate to $\psi_i(s)$ and multiplying each side by an appropriate weight of s. The point is that the same weight works for both sides, thanks to scale-invariance of the estimate that we started with.

Example 3.1.13. Some examples are in order to clarify the use of the principle. We remark that all the estimates below will be used freely in what follows.

1. **(Sobolev)** We begin with the Sobolev inequality (3.1.1) from Lemma 3.1.1. Applying the Correspondence Principle, for every $1 < q \leq r$, $k \geq 0$ such that $\frac{d}{q} = \frac{d}{r} - k$, we obtain

$$\|\psi(s)\|_{\mathcal{L}^r_x(s)} \leq C \|\psi(s)\|_{\mathcal{L}^{\ell,p}_s \dot{\mathcal{W}}^{k,q}_x(s)},$$

for every $s \in J$.

2. **(Interpolation)** Recall the interpolation inequality (3.1.2) from Lemma 3.1.1. Applying the Correspondence Principle, for $1 < q < \infty$, $k_1 \leq k_0 \leq k_2$, $0 < \theta_1, \theta_2 < 1$ such that $\theta_1 + \theta_2 = 1$

and $k_0 = \theta_1 k_1 + \theta_2 k_2$, we obtain

$$\|\psi(s)\|_{\dot{\mathcal{W}}_x^{k_0,q}(s)} \le C\|\psi(s)\|^{\theta_1}_{\dot{\mathcal{W}}_x^{k_1,q}(s)}\|\psi(s)\|^{\theta_2}_{\dot{\mathcal{W}}_x^{k_2,q}(s)},$$

for every $s \in J$.

3. **(Gagliardo-Nirenberg)** Let us apply the Correspondence Principle to the Gagliardo-Nirenberg inequality (3.1.3) from Lemma 3.1.1. Then for $q \le q_1, q_2, r \le \infty$, $0 < \theta_1, \theta_2 < 1$ such that $\frac{d}{r} = \theta_1 \cdot \frac{d}{q_1} + \theta_2(\frac{d}{q_2} - 1)$, we obtain

$$\|\psi(s)\|_{\mathcal{L}_x^r(s)} \le C\|\psi(s)\|^{\theta_1}_{\mathcal{L}_x^{q_1}(s)}\|\nabla_x\psi(s)\|^{\theta_2}_{\mathcal{L}_x^{q_2}}$$

for every $s \in J$.

4. **(Hölder)** Let us start with $\|\phi_1\phi_2\|_{L_x^r} \le \|\phi_1\|_{L_x^{q_1}}\|\phi_2\|_{L_x^{q_2}}$, where $1 \le q_1, q_2, r \le \infty$ and $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Applying the Correspondence Principle, for every $s \in J$, we obtain

$$\|\psi_1\psi_2(s)\|_{\mathcal{L}_x^r(s)} \le \|\psi_1(s)\|_{\mathcal{L}_x^{q_1}(s)}\|\psi_2(s)\|_{\mathcal{L}_x^{q_2}(s)}.$$

All the estimates above extend to functions on $I \times \mathbb{R}^d$ with $I \subset \mathbb{R}$ in the obvious way. In this case, we have the following analogue of the Hölder inequality:

$$\|\psi(s)\|_{\mathcal{L}_t^{q_1}\mathcal{L}_x^p(s)} \le s^{-\left(\frac{1}{2q_1}-\frac{1}{2q_2}\right)}|I|^{\frac{1}{q_1}-\frac{1}{q_2}}\|\psi(s)\|_{\mathcal{L}_t^{q_2}\mathcal{L}_x^p(s)} \text{ for } q_1 \le q_2.$$

The following consequence of the Gagliardo-Nirenberg and Sobolev inequalities is useful enough to be separated as a lemma on its own. It provides a substitute for the incorrect $\dot{H}_x^{d/2} \subset L_x^\infty$ Sobolev embedding, and has the benefit of being scale-invariant. We shall refer to this simply as *Gagliardo-Nirenberg* for p-normalized norms.

Lemma 3.1.14 (Gagliardo-Nirenberg)**.** *For every $s \in J$, the following estimate holds.*

$$\begin{aligned}\|\nabla_x^{(k)}\psi(s)\|_{\dot{\mathcal{H}}_x^{d/2}\cap\mathcal{L}_x^\infty(s)} \le& C_k\|\psi(s)\|^{1/2}_{\dot{\mathcal{H}}_x^{k+2}(s)}\|\psi(s)\|^{1/2}_{\dot{\mathcal{H}}_x^{k+1}(s)} \\ \le& \frac{C_k}{2}(\|\psi(s)\|_{\dot{\mathcal{H}}_x^{k+2}(s)} + \|\psi(s)\|_{\dot{\mathcal{H}}_x^{k+1}(s)}).\end{aligned} \tag{3.1.21}$$

Proof. Without loss of generality, assume $k = 0$. To prove the first inequality, by Gagliardo-Nirenberg, interpolation and the Correspondence Principle, it suffices to prove $\|\phi\|_{L_x^6} \le C\|\phi\|_{\dot{H}_x^1}$

and $\|\partial_x \phi\|_{L^6_x} \leq C\|\phi\|_{\dot{H}^2_x}$; the latter two are simple consequences of Sobolev. Next, the second inequality follows from the first by Cauchy-Schwarz. $\square$

We remark that in practice, the Correspondence Principle, after multiplying by an appropriate weight of s and integrating over J, will often be used in conjunction with Hölder's inequality for the spaces $\mathcal{L}^{\ell,p}_s$ (Lemma 3.1.8).

Finally, recall that the notation $\mathcal{O}(\psi_1, \psi_2, \dots, \psi_k)$ refers to a linear combination of expressions in *the values of* the arguments $\psi_1, \psi_2, \dots, \psi_k$, where they could in general be vector-valued. It therefore follows immediately that any multi-linear estimate for the usual product $\|\psi_1 \cdot \psi_2 \cdots \psi_k\|$ for scalar-valued functions $\psi_1, \psi_2, \dots, \psi_k$ implies the corresponding estimate for $\|\mathcal{O}(\psi_1, \psi_2, \dots, \psi_k)\|$, where $\psi_1, \psi_2, \dots, \psi_k$ may now be vector-valued, at the cost of some absolute constant depending on $\mathcal{O}$. This remark will be used repeatedly in the sequel.

3.2 Covariant Yang-Mills heat flow in the DeTurck gauge

The subject of this section is the *covariant Yang-Mills heat flow* (cYMHF) on $\mathbb{R}^d \times [0, \infty)$ ($d \geq 2$) under the *DeTurck gauge* condition $A_s = \partial^\ell A_\ell$. Such a choice of gauge, as discussed in the Introduction, gives rise to a strictly parabolic system of equations for A_i which may be analyzed using the standard theory of semi-linear parabolic equations. Among the important consequences are the (sub-critical) *local well-posedness* and *infinite instantaneous smoothing property* of (cYMHF), which we shall describe below in more detail.

We begin by deriving the equations satisfied by a solution A_i to (cYMHF) in the DeTurck gauge. Writing out (cYMHF) in terms of A_i, A_s, we obtain

$$\begin{aligned} \partial_s A_i =& \triangle A_i + 2[A^\ell, \partial_\ell A_i] - [A^\ell, \partial_i A_\ell] + [A^\ell, [A_\ell, A_i]] \\ &+ \partial_i (A_s - \partial^\ell A_\ell) + [A_i, A_s - \partial^\ell A_\ell]. \end{aligned} \tag{3.2.1}$$

Using the DeTurck gauge condition $A_s = \partial^\ell A_\ell$, we obtain the semi-linear parabolic system

$$(\partial_s - \triangle) A_i = 2[A^\ell, \partial_\ell A_i] - [A^\ell, \partial_i A_\ell] + [A^\ell, [A_\ell, A_i]]. \tag{3.2.2}$$

Conversely, any distributional solution A_i to (3.2.2) with enough regularity (say $A_i \in C_s \dot{H}^\gamma_x$ with $\frac{d-2}{2} \leq \gamma < \frac{d}{2}$), along with $A_s = \partial^\ell A_\ell$, is also a distributional solution to (cYMHF). Henceforth, we shall concentrate on the reduced system (3.2.2).

We are ready to formulate a *local well-posedness* statement for the system (3.2.2). It is *subcritical* with respect to scaling, in the sense that the initial data is assumed to have a regularity higher than the scaling critical one ($\dot{H}_x^{(d-2)/2}$ in the present case).

Proposition 3.2.1. *Let $d \geq 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ and define $q \geq 2$ by $\frac{d}{q} = \frac{d}{2} - \gamma$. Then the following statements hold.*

1. *There exists a number $\delta_P = \delta_P(d, \gamma) > 0$ such that for any initial data $\overline{A}_i \in \dot{H}_x^\gamma \cap L_x^q$ with*

$$\|\overline{A}\|_{\dot{H}_x^\gamma} \leq \delta_P, \tag{3.2.3}$$

there exists a unique solution $A_i = A_i(x, s) \in C_s([0,1], \dot{H}_x^\gamma \cap L_x^q) \cap L_s^2((0,1], \dot{H}_x^{\gamma+1})$ to the system (3.2.2) on $s \in [0,1]$ which satisfies

$$\|A\|_{C_s([0,1],\dot{H}_x^\gamma \cap L_x^q)} + \|A\|_{L_s^2((0,1],\dot{H}_x^{\gamma+1})} \leq C_{d,\gamma} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.4}$$

2. *Consider an additional initial data $\overline{A}_i' \in \dot{H}_x^\gamma \cap L_x^q$ satisfying (3.2.3), and let A_i' be the corresponding solution to (3.2.2) given by Statement 1. Then the difference δA between the two solutions obeys the estimate*

$$\|\delta A\|_{C_s([0,1],\dot{H}_x^\gamma \cap L_x^q)} + \|\delta A\|_{L_s^2((0,1],\dot{H}_x^{\gamma+1})} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.5}$$

3. *We have* persistence of regularity*; in particular, the following statement holds:*

   Suppose that $\overline{A}_i(t) \in \dot{H}_x^\gamma \cap L_x^q$ is an initial data set satisfying (3.2.3) and also $\overline{A}_i \in H_x^\infty$. Then the corresponding solution A_i satisfies $A_i \in C_s^\infty([0,1], H_x^\infty)$.

4. *We have* smooth dependence on the initial data*; in particular, the following statement holds:*

   Suppose that $\overline{A}_i(t) \in \dot{H}_x^\gamma \cap L_x^q$ is a family of initial data satisfying (3.2.3), which is parametrized by $t \in I$ ($I \subset \mathbb{R}$ is an interval) and $\overline{A}_i \in C_t^\infty(I, H_x^\infty)$. Then the corresponding solution $A_i(t)$ satisfies $A_i \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$.

Remark 3.2.2. The space $\dot{H}_x^\gamma \cap L_x^q$ is the closure of $\mathcal{S}_x$ or H_x^∞ with respect to the $\dot{H}_x^\gamma$-norm

Remark 3.2.3. The above proposition is applicable to an arbitrary (possibly large) data $\overline{A}_i \in \dot{H}_x^\gamma$ by scaling. More precisely, note first that the system (3.2.2) is invariant under the scaling

$$x^\alpha \to \lambda x^\alpha, \quad s \to \lambda^2 s, \quad A \to \lambda^{-1} A \qquad (\lambda > 0).$$

Then for $\gamma > \frac{d-2}{2}$ (note that the $\dot{H}_x^{(d-2)/2}$-norm is scaling critical), the $\dot{H}_x^\gamma$ norm of any $\overline{A}_i \in \dot{H}_x^\gamma$ may be made arbitrarily small by taking $\lambda \to 0$, after which Proposition 3.2.1 may be applied. Undoing the scaling, we finally obtain local well-posedness on an interval $[0, s_0]$, where $s_0 = s_0(\delta_P, \|\overline{A}\|_{\dot{H}_x^\gamma}) > 0$. In particular, s_0 depends on $\|\overline{A}\|_{\dot{H}_x^\gamma}$ in a non-increasing manner.

Remark 3.2.4. Recall the notations for p-normalized norms and

$$\ell_\gamma = \frac{1}{2}\Big(\frac{d}{2} - \gamma\Big),$$

from §3.1.2. Then the estimates (3.2.4) and (3.2.5) may be rewritten respectively as follows:

$$\|A\|_{\mathcal{L}_s^{\ell_\gamma,\infty}(\dot{\mathcal{H}}_x^\gamma \cap \mathcal{L}_x^q)(0,1]} + \|A\|_{\mathcal{L}_s^{\ell_\gamma,2}\dot{\mathcal{H}}_x^{\gamma+1}(0,1]} \leq C_{d,\delta}\|\overline{A}\|_{\dot{H}_x^\gamma}, \tag{3.2.4$'$}$$

$$\|\delta A\|_{\mathcal{L}_s^{\ell_\gamma,\infty}(\dot{\mathcal{H}}_x^\gamma \cap \mathcal{L}_x^q)(0,1]} + \|\delta A\|_{\mathcal{L}_s^{\ell_\gamma,2}\dot{\mathcal{H}}_x^{\gamma+1}(0,1]} \leq C_{d,\delta,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}}\|\delta\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.5$'$}$$

In fact, we shall mostly use this notation for stating parabolic estimates in the remainder of this chapter.

Sketch of proof. The idea is to set up a Picard iteration scheme in a bounded subset of

$$C_s([0,1], \dot{H}_x^\gamma \cap L_x^q) \cap L_x^2((0,1], \dot{H}_x^{\gamma+1}),$$

using the energy inequality for the heat equation. As this is standard, we shall only sketch the main ideas by showing how to prove the *a priori* estimate (3.2.4), given a solution $A_i \in C_s^\infty([0,1], H_x^\infty)$ to (3.2.2), which satisfies (3.2.3) for sufficiently small $\delta_P > 0$.

Instead of A_i, let us work with $\Psi_i := |\partial_x|^\gamma A_i$. Applying $|\partial_x|^\gamma$ to (3.2.2), we obtain

$$(\partial_s - \triangle)\Psi_i = {}^{(\Psi_i)}\mathcal{N}, \tag{3.2.6}$$

where

$${}^{(\Psi_i)}\mathcal{N} := s^{-(1+\gamma)/2}|\nabla_x|^\gamma \mathcal{O}(A, \nabla_x A) + s^{-\gamma/2}|\nabla_x|^\gamma \mathcal{O}(A, A, A).$$

Consider a subinterval $(0, \underline{s}] \subset (0,1]$, assuming the bootstrap assumption

$$\|\Psi\|_{\mathcal{L}_s^{d/4,\infty}\mathcal{L}_x^2(0,\underline{s}]} + \|\nabla_x\Psi\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \leq B\|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.7}$$

for some $B > 0$ to be fixed later.

Let $\theta \in (0,1)$ be defined by $\frac{d}{2} = (1-\theta)\gamma + \theta(\gamma+1)$. By Lemma 3.1.4, Gagliardo-Nirenberg and the Correspondence Principle, along with the fact that $\Psi_i = s^{-\gamma/2} |\nabla_x|^\gamma A_i$, we have

$$\begin{aligned} s^{-(1+\gamma)/2} \| |\nabla_x|^\gamma \mathcal{O}(A, \nabla_x A) \|_{\mathcal{L}^2_x} \leq & C s^{-(1+\gamma)/2} \|A\|_{\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x} \| |\nabla|^\gamma_x \nabla_x A \|_{\mathcal{L}^2_x} \\ \leq & C s^{-(1-\gamma)/2} \|\Psi\|^{1-\theta}_{\mathcal{L}^2_x} \|\nabla_x \Psi\|^{1+\theta}_{\mathcal{L}^2_x}. \end{aligned}$$

Thus multiplying by $s^{d/4+1}$ and integrating over $(0, \underline{s}]$ with respect to $\mathrm{d}s/s$, we obtain

$$\begin{aligned} \|s^{-(1+\gamma)/2} |\nabla_x|^\gamma \mathcal{O}(A, \nabla_x A)\|_{\mathcal{L}^{d/4+1,1}_s \mathcal{L}^2_x} \leq & C \| \|\Psi\|^{1-\theta}_{\mathcal{L}^2_x} \|\nabla_x \Psi\|^{1+\theta}_{\mathcal{L}^2_x} \|_{\mathcal{L}^{(d/2+1+\gamma)/2,1}_s} \\ \leq & C \underline{s}^{(\gamma-(d-2)/2)/2} \|\Psi\|^{1-\theta}_{\mathcal{L}^{d/4,\infty}_s \mathcal{L}^2_x} \|\nabla_x \Psi\|^{1+\theta}_{\mathcal{L}^{d/4,2}_s \mathcal{L}^2_x} \\ \leq & C B^2 \|\overline{A}\|^2_{\dot{H}^\gamma_x}, \end{aligned}$$

where on the last line, we have used $\gamma > \frac{d-2}{2}$, $0 < \underline{s} \leq 1$ and the bootstrap assumption.

Similarly, by Lemma 3.1.4, Gagliardo-Nirenberg and the Correspondence Principle, we have

$$\begin{aligned} s^{-\gamma/2} \| |\nabla_x|^\gamma \mathcal{O}(A, A, A) \|_{\mathcal{L}^2_x} \leq & C s^{-\gamma/2} \|A\|_{\dot{\mathcal{H}}^\gamma_x} \|A\|^2_{\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x} \\ \leq & C s^\gamma \|\Psi\|^{3-2\theta}_{\mathcal{L}^2_x} \|\nabla_x \Psi\|^{2\theta}_{\mathcal{L}^2_x}. \end{aligned}$$

Multiplying by $s^{d/4+1}$ and integrating over $(0, \underline{s}]$ with respect to $\mathrm{d}s/s$, we obtain

$$\begin{aligned} \|s^{-\gamma/2} |\nabla_x|^\gamma \mathcal{O}(A, A, A)\|_{\mathcal{L}^{d/4+1,1}_s \mathcal{L}^2_x} \leq & C \| \|\Psi\|^{3-2\theta}_{\mathcal{L}^2_x} \|\nabla_x \Psi\|^{2\theta}_{\mathcal{L}^2_x} \|_{\mathcal{L}^{(d/2+2+2\gamma)/2,1}_s} \\ \leq & C \underline{s}^{\gamma-(d-2)/2} \|\Psi\|^{3-2\theta}_{\mathcal{L}^{d/4,\infty}_s \mathcal{L}^2_x} \|\nabla_x \Psi\|^{2\theta}_{\mathcal{L}^{d/4,2}_s \mathcal{L}^2_x} \\ \leq & C B^3 \|\overline{A}\|^3_{\dot{H}^\gamma_x}. \end{aligned}$$

In sum, we have proved

$$\|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}^{d/4+1,1}_s \mathcal{L}^2_x} \leq C B^2 \|\overline{A}\|^2_{\dot{H}^\gamma_x} + C B^3 \|\overline{A}\|^3_{\dot{H}^\gamma_x}. \tag{3.2.8}$$

Applying the parabolic energy estimate (3.1.8), we arrive at the inequality

$$\|\Psi\|_{\mathcal{L}^{d/4,\infty}_s \mathcal{L}^2_x(0,\underline{s}]} + \|\nabla_x \Psi\|_{\mathcal{L}^{d/4,2}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C \|\overline{A}\|_{\dot{H}^\gamma_x} + C B^2 \|\overline{A}\|^2_{\dot{H}^\gamma_x} + C B^3 \|\overline{A}\|^3_{\dot{H}^\gamma_x}.$$

Taking $B > 2C$, say, and $\delta_P > 0$ sufficiently small, we retrieve the bootstrap assumption. As

$\Psi_i = |\partial_x|^\gamma A_i$, the desired *a priori* estimate for A_i is now evident. $\square$

Next, we shall establish the *(infinite, instantaneous) smoothing property* of the system (3.2.2).

Proposition 3.2.5. *Let $d \geq 2$ and $\frac{d-2}{2} < \gamma < \frac{d}{2}$. Consider $\dot{H}^\gamma_x$ initial data sets $\overline{A}_i$, $\overline{A}'_i$ satisfying* (3.2.3)*. Denote by A_i, A'_i the corresponding solutions to* (3.2.2) *given by Proposition 3.2.1, respectively. Then the following statements hold.*

1. *For every integer $m \geq 1$, the following estimates for A_i holds.*

$$\sum_{k=1}^{m} \Big(\|\nabla_x^{(k)} A\|_{\mathcal{L}^{\ell\gamma,\infty}_s \dot{\mathcal{H}}^\gamma_x(0,1]} + \|\nabla_x^{(k+1)} A\|_{\mathcal{L}^{\ell\gamma,2}_s \dot{\mathcal{H}}^\gamma_x(0,1]} \Big) \leq C_{d,\delta,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}. \tag{3.2.9}$$

2. *For every integer $m \geq 1$, the following estimates for δA_i holds.*

$$\sum_{k=1}^{m} \Big(\|\nabla_x^{(k)} (\delta A)\|_{\mathcal{L}^{\ell\gamma,\infty}_s \dot{\mathcal{H}}^\gamma_x(0,1]} + \|\nabla_x^{(k+1)} (\delta A)\|_{\mathcal{L}^{\ell\gamma,2}_s \dot{\mathcal{H}}^\gamma_x(0,1]} \Big) \leq C_{d,\delta,m,\|\overline{A}\|_{\dot{H}^\gamma_x},\|\overline{A}'\|_{\dot{H}^\gamma_x}} \|\delta\overline{A}\|_{\dot{H}^\gamma_x}. \tag{3.2.10}$$

Sketch of proof. The idea is to differentiate the system (3.2.6) and apply the second part of Theorem 3.1.10 with $\ell = d/4$. Below, we shall give a proof of the non-difference estimate (3.2.9), as the difference analogue (3.2.10) may be proved in a similar manner.

In what follows, unless stated otherwise, all norms with respect to s will be taken over $(0,1]$. By approximation, it suffices to consider $A_i \in C^\infty_s([0,1], H^\infty_x)$. Recall the definition $\Psi_i := |\partial_x|^\gamma A_i$. We claim that for integers $m \geq 3$, we have

$$\begin{aligned} \|\nabla_x^{(m-1)}({}^{(\Psi)}\mathcal{N})\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x} \leq & C_m \|\Psi\|_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^{m-1}_x} \|\nabla_x \Psi\|_{\mathcal{L}^{d/4,2}_s \mathcal{H}^{m-1}_x} \\ & + C_m \|\Psi\|^2_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^{m-1}_x} \|\nabla_x \Psi\|_{\mathcal{L}^{d/4,2}_s \mathcal{H}^{m-1}_x}, \end{aligned} \tag{3.2.11}$$

whereas for the exceptional cases $m = 1, 2$, we shall use the following statement: For all $\overline{s} \in (0,1)$, we have

$$\begin{aligned} &\|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}^{d/4+1,1}_s \mathcal{L}^2_x(0,\overline{s}]} + \|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x(0,\overline{s}]} + \|\nabla_x({}^{(\Psi)}\mathcal{N})\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x(0,\overline{s}]} \\ &\leq C\|\Psi\|^2_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^2_x(0,\overline{s}]} + C\|\Psi\|^3_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^2_x(0,\overline{s}]}. \end{aligned} \tag{3.2.12}$$

Proceeding as in the proof of Proposition 3.2.1, using the parabolic energy and smoothing estimates (3.1.8), (3.1.9), we first obtain (3.2.9) for $m = 1, 2$. Applying the second part of Theorem 3.1.10, the smoothing estimate (3.2.9) would follow.

We shall begin with (3.2.11). As before, let $\theta \in (0,1)$ be defined by $\frac{d}{2} = (1-\theta)\gamma + \theta(\gamma+1)$. By the Leibniz rule, Lemma 3.1.4, Gagliardo-Nirenberg and the Correspondence Principle, we have

$$\begin{aligned} & s^{-(1+\gamma)/2} \|\nabla_x^{(m-1)} |\nabla_x|^\gamma \mathcal{O}(A, \nabla_x A)\|_{\mathcal{L}^2_x} \\ & \quad \leq C s^{-(1-\gamma)/2} \sum_{k_1+k_2=m;\, 0\leq k_1 \leq k_2} \|\nabla_x^{(k_1)}\Psi\|^{1-\theta}_{\mathcal{L}^2_x} \|\nabla_x^{(k_1+1)}\Psi\|^{\theta}_{\mathcal{L}^2_x} \|\nabla_x^{(k_2)}\Psi\|_{\mathcal{L}^2_x}. \end{aligned}$$

For $m \geq 3$, note that $2 \leq \lceil \frac{m}{2} \rceil \leq k_2$. Therefore, we have $0 \leq k_1 \leq k_1+1 \leq m-1$ and $1 \leq k_2 \leq m$. Multiplying by $s^{d/4+1}$ and taking the square integral over $(0,1]$ with respect to $\mathrm{d}s/s$, we obtain

$$\|s^{-(1+\gamma)/2} \nabla_x^{(m-1)} |\nabla_x|^\gamma \mathcal{O}(A, \nabla_x, A)\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x} \leq C_m \|\Psi\|_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^{m-1}_x} \|\nabla_x \Psi\|_{\mathcal{L}^{d/4,2}_s \mathcal{H}^{m-1}_x}.$$

Next, consider the cubic term in ${}^{(\Psi)}\mathcal{N}$. By the Leibniz rule, Lemma 3.1.4, Gagliardo-Nirenberg and the Correspondence Principle, we have

$$\begin{aligned} & s^{-\gamma/2} \|\nabla_x^{(m-1)} |\nabla_x|^\gamma \mathcal{O}(A, A, A)\|_{\mathcal{L}^2_x} \\ & \quad \leq C s^{-(1-\gamma)/2} \sum \|\nabla_x^{(k_1)}\Psi\|_{\mathcal{L}^2_x} \|\nabla_x^{(k_2)}\Psi\|^{1-\theta}_{\mathcal{L}^2_x} \|\nabla_x^{(k_2+1)}\Psi\|^{\theta}_{\mathcal{L}^2_x} \|\nabla_x^{(k_3)}\Psi\|^{1-\theta}_{\mathcal{L}^2_x} \|\nabla_x^{(k_3+1)}\Psi\|^{\theta}_{\mathcal{L}^2_x} \end{aligned}$$

where the summation is over $\{(k_1, k_2, k_3) \in \mathbb{N}^3 : k_1 + k_2 + k_3 = m-1,\, 0 \leq k_1 \leq k_2 \leq k_3\}$. For $m \geq 2$, observe that we have $k_3 \geq 1$ and $0 \leq k_1 \leq k_2 \leq m-2$. Thus, multiplying by $s^{d/4+1}$ and taking the square integral over $(0,1]$ with respect to $\mathrm{d}s/s$, we may estimate

$$\|s^{-\gamma/2} \nabla_x^{(m-1)} |\nabla_x|^\gamma \mathcal{O}(A, A, A)\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x} \leq C_m \|\Psi\|^2_{\mathcal{L}^{d/4,\infty}_s \mathcal{H}^{m-1}_x} \|\nabla_x \Psi\|_{\mathcal{L}^{d/4,2}_s \mathcal{H}^{m-1}_x},$$

which proves (3.2.11) for $m \geq 3$. On the other hand, the proof of the exceptional cases, namely (3.2.12), is routine after unravelling the definitions of p-normalized norms, and thus is left to the reader. (We remark that the term $\|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}^{d/4+1}_s \mathcal{L}^2_x(0,\overline{s}]}$ has already been estimated in the proof of Proposition 3.2.1.) □

Our final proposition in this section concerns estimates for A_s, which would be used to derive estimates for the gauge transform from the DeTurck to caloric gauge.

Proposition 3.2.6. *Let $d \geq 2$ and $\frac{d-2}{2} < \gamma < \frac{d}{2}$. Consider $\dot{H}^\gamma_x$ initial data sets $\overline{A}_i$, $\overline{A}'_i$ satisfying* (3.2.3)*. Denote by A_i, A'_i the corresponding solutions to* (3.2.2) *given by Proposition 3.2.1, respectively, and $A_s = \partial^\ell A_\ell$, $A'_s = \partial^\ell A'_\ell$. Then:*

1. For every integer $m \geq 0$, the following estimates for A_s hold.

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} A_s\|_{\mathcal{L}_s^{\ell_\gamma+1/2,\infty} \dot{\mathcal{H}}_x^{\gamma}} + \|\nabla_x^{(k)} A_s\|_{\mathcal{L}_s^{\ell_\gamma+1/2,2} \dot{\mathcal{H}}_x^{\gamma+1}} \Big) \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.13}$$

$$\sup_{0<s\leq 1} \| \int_s^1 A_s(s') \, \mathrm{d}s' \|_{\dot{H}_x^{\gamma+1}} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.14}$$

2. Furthermore, for every integer $m \geq 0$, the following estimates for δA_s hold.

$$\begin{aligned} &\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} (\delta A_s)\|_{\mathcal{L}_s^{\ell_\gamma+1/2,\infty} \dot{\mathcal{H}}_x^{\gamma}} + \|\nabla_x^{(k)} (\delta A_s)\|_{\mathcal{L}_s^{\ell_\gamma+1/2,2} \dot{\mathcal{H}}_x^{\gamma+1}} \Big) \\ &\qquad \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}_x^\gamma}, \|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta\overline{A}\|_{\dot{H}_x^\gamma}, \end{aligned} \tag{3.2.15}$$

$$\sup_{0<s\leq 1} \| \int_s^1 \delta A_s(s') \, \mathrm{d}s' \|_{\dot{H}_x^{\gamma+1}} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma}, \|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.2.16}$$

Proof. The estimates (3.2.13) and (3.2.15) are direct consequences of Proposition 3.2.5 and the DeTurck gauge condition $A_s = \partial^\ell A_\ell$. Below, we shall give a proof of (3.2.14); the proof of (3.2.16) is similar. Again, it suffices to consider $A_i \in C_s^\infty([0,1], H_x^\infty)$.

Taking $|\partial_x|^{\gamma-1}\partial^\ell$ of (3.2.2), we get a parabolic equation for $|\partial_x|^{\gamma-1} A_s$ of the form

$$(\partial_s - \triangle)|\partial_x|^{\gamma-1} A_s = \sum_{\ell=1}^{d} |\partial_x|^{\gamma-1} \partial_\ell ({}^{(A_i)}\mathcal{N}),$$

where ${}^{(A_i)}\mathcal{N}$ refers to the right-hand side of (3.2.2). Integrating this equation from s to 1, we obtain

$$\int_s^1 \triangle |\partial_x|^{\gamma-1} A_s(s') \, \mathrm{d}s' = |\partial_x|^{\gamma-1} A_s(1) - |\partial_x|^{\gamma-1} A_s(s) + \sum_{\ell=1}^{d} \int_s^1 |\partial_x|^{\gamma-1} \partial_\ell ({}^{(A_i)}\mathcal{N})(s') \, \mathrm{d}s'.$$

Take the L_x^2-norm of both sides, and take furthermore the supremum over $s \in (0,1]$. Then we see that

$$\sup_{s\in(0,1]} \| \int_s^1 A_s(s') \, \mathrm{d}s' \|_{\dot{H}_x^{\gamma+1}} \leq C \|A_s\|_{L_s^\infty \dot{H}_x^{\gamma-1}} + C \int_0^1 \|{}^{(A)}\mathcal{N}(s')\|_{\dot{H}_x^\gamma} \, \mathrm{d}s'$$

The first term on the right-hand side is bounded by $C\|\overline{A}\|_{\dot{H}_x^\gamma}$ by (3.2.4). The second term, on the other hand, is equivalent to $\|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}_s^{d/4+1,1} \mathcal{L}_x^2(0,1]}$. Then by (3.2.8) with $\underline{s} = 1$ (from the proof of Proposition 3.2.1), where we use (3.2.4) instead of (3.2.7), we have

$$\|{}^{(\Psi)}\mathcal{N}\|_{\mathcal{L}_s^{d/4+1,1} \mathcal{L}_x^2(0,1]} \leq C\|\overline{A}\|_{\dot{H}_x^\gamma}^2 + C\|\overline{A}\|_{\dot{H}_x^\gamma}^3.$$

This proves (3.2.14). □

3.3 Linear parabolic estimates

As we have seen in §2.1, curvature components $F_{\mu\nu}$ of (cYMHF) or (dYMHF) satisfy the covariant parabolic equations

$$\mathbf{D}_s F_{\mu\nu} - \mathbf{D}^\ell \mathbf{D}_\ell F_{\mu\nu} = -2[F_\mu{}^\ell, F_{\nu\ell}].$$

Once the estimates for A_i have been established, as we have done in §3.2 in the DeTurck gauge, by expanding out the covariant derivatives and spatial curvature components, such an equation may be viewed as a system of *linear parabolic equation* for $F_{\mu\nu}$. Below, we shall present a general lemma (Lemma 3.3.1) for deriving estimates for such linear parabolic equations. As we shall see in §3.4, this lemma may be used to solve the dynamic Yang-Mills heat flow (dYMHF), building on the theory of (cYMHF) we have developed in §3.2.

For simplicity, we shall restrict to initial data and coefficients in H^∞_x in the statement of the following lemma.

Lemma 3.3.1 (Linear parabolic estimates)**.** *Let $d \geq 2$, X a finite-dimensional normed space and $\frac{d-2}{2} < \gamma < \frac{d}{2}$. For any $\gamma' \in \mathbb{R}$, let $\ell_{\gamma'} = \frac{1}{2}(\frac{d}{2} - \gamma')$. Let $\overline{\Psi}$ be an X-valued function in $H^\infty_x(\mathbb{R}^d)$; Ω^i_1 an $L(X)$-valued vector, Ω^i_1, Ω_0 an $L(X)$-valued function and N an X-valued function in $C^\infty_s([0,1], H^\infty_x)$. Consider the following initial value problem for the linear parabolic equation*

$$\begin{cases} \partial_s \Psi - \triangle \Psi = \Omega^\ell_1(\partial_\ell \Psi) + \Omega_0(\Psi) + N \\ \Psi(s=0) = \overline{\Psi}. \end{cases} \tag{3.3.1}$$

Then there exists a unique solution Ψ to (3.3.1) in $C^\infty_s([0,1], H^\infty_x)$. Moreover, the solution satisfies the following properties.

1. *Let $\mathcal{C}_0 > 0$ be a constant such that*

$$\|\Omega^i_1\|_{\mathcal{L}^{\ell_\gamma,\infty}_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} + \|\Omega_0\|_{\mathcal{L}^{\ell_\gamma+1/2,\infty}_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} \leq \mathcal{C}_0.$$

Then for every $-\frac{d}{2} < \gamma' < \frac{d}{2}$, the unique solution Ψ obeys

$$\begin{aligned}&\sum_{k=0}^{1} \Big(\|\nabla_x^{(k)} \Psi\|_{\mathcal{L}^{\ell_{\gamma'},\infty}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} + \|\nabla_x^{(k+1)} \Psi\|_{\mathcal{L}^{\ell_{\gamma'},2}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} \Big) \\ &\qquad \leq C_{d,\gamma,\gamma',\mathcal{C}_0} \Big(\|\overline{\Psi}\|_{\dot{H}^{\gamma'}_x} + \|N\|_{\mathcal{L}^{\ell_{\gamma'}+1,1}_s \dot{\mathcal{H}}^{\gamma'}_x} + \|N\|_{\mathcal{L}^{\ell_{\gamma'}+1,2}_s \dot{\mathcal{H}}^{\gamma'}_x} \Big).\end{aligned} \tag{3.3.2}$$

2. *Let $m \geq 1$ be an integer, and $\mathcal{C}_m > 0$ a constant such that*

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \Omega_1^i\|_{\mathcal{L}^{\ell_\gamma,\infty}_s (\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} + \|\nabla_x^{(k)} \Omega_0\|_{\mathcal{L}^{\ell_\gamma+1/2,\infty}_s (\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} \Big) \leq \mathcal{C}_m.$$

Then for every $-\frac{d}{2} < \gamma' < \frac{d}{2}$, the unique solution Ψ obeys the following smoothing estimates*:*

$$\begin{aligned}&\sum_{k=0}^{m+1} \Big(\|\nabla_x^{(k)} \Psi\|_{\mathcal{L}^{\ell_{\gamma'},\infty}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} + \|\nabla_x^{(k+1)} \Psi\|_{\mathcal{L}^{\ell_{\gamma'},2}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} \Big) \\ &\qquad \leq C_{d,\gamma,\gamma',\mathcal{C}_m} \Big(\|\overline{\Psi}\|_{\dot{H}^{\gamma'}_x} + \|N\|_{\mathcal{L}^{\ell_{\gamma'}+1,1}_s \dot{\mathcal{H}}^{\gamma'}_x} + \sum_{k=0}^{m} \|\nabla_x^{(k)} N\|_{\mathcal{L}^{\ell_{\gamma'}+1,2}_s \dot{\mathcal{H}}^{\gamma'}_x} \Big).\end{aligned} \tag{3.3.3}$$

3. *We have* smooth dependence on parameters*; in particular, the following statement holds:*

   Let $I \subset \mathbb{R}$ be an interval, and suppose that $\Omega_1^i(t)$, $\Omega_0(t)$ and $N(t)$ are parametrized by $t \in I$ so that $\Omega_1^i, \Omega_0, N \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$. For each t, consider the IVP (3.3.1) *with initial data $\overline{\Psi}(t)$ parametrized by $t \in I$ such that $\overline{\Psi} \in C^\infty_t(I, H^\infty_x)$. Then the corresponding solution $\Psi(t)$ satisfies $\Psi \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$.*

Remark 3.3.2. In addition to application to covariant parabolic equations for $F_{\mu\nu}$, we shall also apply this lemma to the parabolic equation for A_0 arising from (dYMHF) in the DeTurck gauge in §3.4.

Proof. The existence of a unique solution $\Psi \in C^\infty_s([0,1], H^\infty_x)$ to (3.3.1) is an easy exercise using the energy estimate for the heat equation, as the coefficients Ω_1^i, Ω_0 and the forcing term N all belong to $C^\infty_s([0,1], H^\infty_x)$. To prove Statements 1 and 2, we shall work with a new variable $\Psi^{\gamma'} := |\partial_x|^{\gamma'} \Psi$, where $-\frac{d}{2} < \gamma' < \frac{d}{2}$. Taking $|\partial_x|^{\gamma'}$ of the equation (3.3.1), we see that $\Psi^{\gamma'}$ obeys the parabolic equation

$$(\partial_s - \triangle) \Psi^{\gamma'} = {}^{(\Psi^{\gamma'})}\mathcal{N},$$

where

$${}^{(\Psi^{\gamma'})}\mathcal{N} := s^{-(1+\gamma')/2} |\nabla_x|^{\gamma'} \Omega_1^\ell (\nabla_\ell \Psi) + s^{-\gamma'/2} |\nabla_x|^{\gamma'} \Omega_0(\Psi) + s^{-\gamma'/2} |\nabla_x|^{\gamma'} N.$$

We shall first prove an *a priori* estimate for (3.3.1), i.e. Statement 1. Take $0 < \epsilon \ll \frac{1}{2}(\gamma' - \frac{d-2}{2}) = \frac{1}{2} - \ell_\gamma$. We claim that for every $(0, \underline{s}] \subset (0, 1]$, we have

$$\begin{aligned} \|^{(\Psi^{\gamma'})}\mathcal{N}\|_{\mathcal{L}_s^{d/4+1,1}\mathcal{L}_x^2(0,\underline{s}]} \leq & C\mathcal{C}_0 \underline{s}^\epsilon \|s^{1/2-\ell_\gamma-\epsilon}\nabla_x \Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \\ & + C\mathcal{C}_0 \underline{s}^\epsilon \|s^{1/2-\ell_\gamma-\epsilon}\Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \\ & + C\|N\|_{\mathcal{L}_s^{\ell_{\gamma'}+1,1}\dot{\mathcal{H}}_x^{\gamma'}(0,\underline{s}]}, \end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned} \|^{(\Psi^{\gamma'})}\mathcal{N}\|_{\mathcal{L}_s^{d/4+1,2}\mathcal{L}_x^2(0,\underline{s}]} \leq & C\mathcal{C}_0 \|s^{1/2-\ell_\gamma}\nabla_x \Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \\ & + C\mathcal{C}_0 \|s^{1/2-\ell_\gamma}\Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \\ & + C\|N\|_{\mathcal{L}_s^{\ell_{\gamma'}+1,2}\dot{\mathcal{H}}_x^{\gamma'}(0,\underline{s}]}. \end{aligned} \tag{3.3.5}$$

Note that for any $1 \leq r \leq \infty$ and $k \geq 0$, we have

$$\|\nabla_x^{(k)}\Psi^{\gamma'}\|_{\mathcal{L}_x^{d/4,r}\mathcal{L}_x^2} = \|\nabla_x^{(k)}\Psi\|_{\mathcal{L}_x^{\ell_{\gamma'},r}\dot{\mathcal{H}}_x^{\gamma'}}$$

and thus, (3.3.2) is a consequence of Theorem 3.1.10 once we establish (3.3.4) and (3.3.5).

We shall prove both estimates simultaneously. Consider first the term $\Omega_1^\ell(\partial_\ell \Psi)$. By Lemma 3.1.4 and the Correspondence Principle,

$$\begin{aligned} s^{-(1+\gamma')/2}\||\nabla_x|^{\gamma'}\Omega_1^\ell(\nabla_\ell\Psi)\|_{\mathcal{L}_x^2} \leq & Cs^{-(1+\gamma')/2}\|\Omega_1^\ell\|_{\dot{\mathcal{H}}_x^{d/2}\cap\mathcal{L}_x^\infty}\||\nabla_x|^{\gamma'}\nabla_\ell\Psi\|_{\mathcal{L}_x^2} \\ \leq & Cs^{-1/2}\|\Omega_1^\ell\|_{\dot{\mathcal{H}}_x^{d/2}\cap\mathcal{L}_x^\infty}\|\nabla_\ell\Psi^{\gamma'}\|_{\mathcal{L}_x^2}. \end{aligned}$$

Multiplying by $s^{d/4+1}$ and integrating over $s \in (0, \underline{s}]$ with respect to $\mathrm{d}s/s$, we get

$$\begin{aligned} \|s^{-(1+\gamma')/2}|\nabla_x|^{\gamma'}\Omega_1^\ell(\nabla_\ell\Psi)\|_{\mathcal{L}_s^{d/4+1,1}\mathcal{L}_x^2(0,\underline{s}]} \leq & C\|\|\Omega_1^\ell\|_{\dot{\mathcal{H}}_x^{d/2}\cap\mathcal{L}_x^\infty}\|\nabla_\ell\Psi^{\gamma'}\|_{\mathcal{L}_x^2}\|_{\mathcal{L}_s^{(d/2+1)/2,1}(0,\underline{s}]} \\ \leq & C\mathcal{C}_0\underline{s}^\epsilon\|s^{1/2-\ell_\gamma-\epsilon}\nabla_x\Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]} \end{aligned}$$

whereas taking the square integral over $s \in (0, \underline{s}]$, we obtain

$$\|s^{-(1+\gamma')/2}|\nabla_x|^{\gamma'}\Omega_1^\ell(\nabla_\ell\Psi)\|_{\mathcal{L}_s^{d/4+1,2}\mathcal{L}_x^2(0,\underline{s}]} \leq C\mathcal{C}_0\|s^{1/2-\ell_\gamma}\nabla_x\Psi^{\gamma'}\|_{\mathcal{L}_s^{d/4,2}\mathcal{L}_x^2(0,\underline{s}]}.$$

Next, consider the term $\Omega_0(\Psi)$. Again by Lemma 3.1.4 and the Correspondence Principle, we have

$$s^{-\gamma'/2}\||\nabla_x|^{\gamma'}\Omega_0(\Psi)\|_{\mathcal{L}^2_x} \leq C s^{-\gamma'/2}\|\Omega_0\|_{\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x}\||\nabla_x|^{\gamma'}\Psi\|_{\mathcal{L}^2_x}$$
$$\leq C\|\Omega_0\|_{\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x}\|\Psi^{\gamma'}\|_{\mathcal{L}^2_x}.$$

Multiplying by $s^{d/2+1}$ and integrating or square-integrating over $(0, \underline{s}]$ with respect to $\mathrm{d}s/s$, we obtain, respectively,

$$\|s^{-\gamma'/2}|\nabla_x|^{\gamma'}\Omega_0(\Psi)\|_{\mathcal{L}^{d/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\|\|\Omega_0\|_{\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x}\|\Psi^{\gamma'}\|_{\mathcal{L}^2_x}\|_{\mathcal{L}^{(d/2+2)/2,1}_s(0,\underline{s}]}$$
$$\leq C\mathcal{C}_0\underline{s}^{\epsilon}\|s^{1/2-\ell_\gamma-\epsilon}\Psi^{\gamma'}\|_{\mathcal{L}^{d/4,2}_s \mathcal{L}^2_x(0,\underline{s}]},$$

and

$$\|s^{-\gamma'/2}|\nabla_x|^{\gamma'}\Omega_0(\Psi)\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\mathcal{C}_0\|s^{1/2-\ell_\gamma}\Psi^{\gamma'}\|_{\mathcal{L}^{d/4,2}_s \mathcal{L}^2_x(0,\underline{s}]}.$$

Finally, for any $1 \leq r \leq \infty$, note that

$$\|s^{-\gamma'/2}|\nabla_x|^{\gamma'}N\|_{\mathcal{L}^{d/4+1,r}_s \mathcal{L}^2_x} \leq C\|N\|_{\mathcal{L}^{\ell_{\gamma'}+1,r}_s \dot{\mathcal{H}}^{\gamma'}_x}.$$

Combining these estimates, (3.3.4) and (3.3.5) follow.

To prove the smoothing estimates in Statement 2, by the second part of Theorem 3.1.10, it suffices to prove the following statement: For every integer $m \geq 1$, we claim

$$\|\nabla^{(m)}_x({}^{(\Psi^{\gamma'})}\mathcal{N})\|_{\mathcal{L}^{d/4+1,2}_s \mathcal{L}^2_x(0,1]} \leq C\mathcal{C}_m\|\Psi^{\gamma'}\|_{\mathcal{L}^{d/4,2}_s \mathcal{H}^{m+1}_x(0,1]} + C\|\nabla^{(m)}_x N\|_{\mathcal{L}^{\ell_{\gamma'}+1,2}_s \dot{\mathcal{H}}^{\gamma'}_x(0,1]}. \tag{3.3.6}$$

This is easily proved by analyzing $\nabla^{(m)}_x({}^{(\Psi^{\gamma'})}\mathcal{N})$ as before, using the Leibniz rule.

Finally, Statement 3 is a easy consequence of the Picard iteration argument used to prove the local well-posedness of (3.3.1); we omit the details. □

3.4 Dynamic Yang-Mills heat flow in the DeTurck gauge

In this section, we shall establish basic properties (e.g. local well-posedness and smoothing) of the *dynamic Yang-Mills heat flow* (dYMHF) on $I \times \mathbb{R}^d \times [0, \infty)$ in the DeTurck gauge $A_s = \partial^\ell A_\ell$, where $I \subset \mathbb{R}$ is an interval and $d \geq 2$. The starting point is the observation that the theory for (cYMHF) developed in §3.2 already takes care of the spatial components of (dYMHF). It only remains to add in the temporal component $F_{s0} = \mathbf{D}^\ell F_{\ell 0}$, which is easy using Lemma 3.3.1 proved in the previous section.

To keep the lengths of the statements reasonable, the main result of this section will be divided into the following two propositions. In the first proposition, we shall establish local existence and uniqueness of (dYMHF) in the DeTurck gauge for initial data in $C_t^\infty(I, H_x^\infty)$. Then, in the second proposition, we shall establish Lipschitz dependence on the initial data.

Proposition 3.4.1 (Local well-posedness of (dYMHF) in the DeTurck gauge, Part I)**.** *Let $d \geq 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ and $I \subset \mathbb{R}$ an interval. Consider the dynamic Yang-Mills heat flow* (dYMHF) *on $I \times \mathbb{R}^d \times [0,1]$ in the DeTurck gauge $A_s = \partial^\ell A_\ell$. Let $\overline{A}_\mu$ be a $\mathfrak{g}$-valued 1-form on $I \times \mathbb{R}^d \times \{0\}$ such that $\overline{A}_\mu \in C_t^\infty(I, H_x^\infty)$ and*

$$\|\overline{A}\|_{\dot{H}_x^\gamma} \leq \delta_P, \tag{3.4.1}$$

where $\delta_P = \delta_P(d,\gamma)$ is the constant in Proposition 3.2.1.

Then there exists a unique solution A_μ to (dYMHF) *under the DeTurck gauge condition $A_s = \partial^\ell A_\ell$ on $I \times \mathbb{R}^d \times [0,1]$ such that $A_\mu \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$. Moreover, the solution satisfies the following estimates.*

1. *The spatial components A_i obey* (3.2.4) *and* (3.2.9)*;*
2. *The s-component $A_s = \partial^\ell A_\ell$ obeys* (3.2.13) *and* (3.2.14)*;*
3. *For each integer $m \geq 0$ and $-\frac{d}{2} < \gamma' < \frac{d}{2}$, the curvature components $F_0 = (F_{0i})_{i=1,\ldots,d}$ obey*

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} F_0\|_{\mathcal{L}_s^{\ell_{\gamma'},\infty} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} + \|\nabla_x^{(k+1)} F_0\|_{\mathcal{L}_s^{\ell_{\gamma'},2} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \Big) \leq C_{d,\gamma,\gamma',m,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{F}_0\|_{\dot{H}_x^{\gamma'}} ; \tag{3.4.2}$$

4. *Finally, for each integer $m \geq 0$ and $-\frac{d}{2} + 1 < \gamma' < \frac{d}{2}$, the temporal component A_0 obeys*

$$\begin{aligned} \sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} A_0\|_{\mathcal{L}_s^{\ell_{\gamma'},\infty} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} + &\|\nabla_x^{(k+1)} A_0\|_{\mathcal{L}_s^{\ell_{\gamma'},2} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \Big) \\ &\leq C_{d,\gamma,\gamma',m,\|\overline{A}\|_{\dot{H}_x^\gamma}} (\|\overline{A}_0\|_{\dot{H}_x^{\gamma'}} + \|\overline{F}_0\|_{\dot{H}_x^{\gamma'-1}}). \end{aligned} \tag{3.4.3}$$

Remark 3.4.2. We remark that the ranges of γ' for (3.4.2) and (3.4.3) are not the largest possible, but will be sufficient for our use.

Proof. Fix $t \in I$; we shall often suppress t for the simplicity of notation. Let $B_i = B_i(t)$ be the solution to the initial value problem

$$\begin{cases} \mathbf{D}_s B_i - \mathbf{D}^\ell \mathbf{D}_\ell B_i = 2[F_i{}^\ell, B_\ell], \\ B_i(s=0) = \overline{F}_{0i}. \end{cases} \tag{3.4.4}$$

on $\{t\} \times \mathbb{R}^d \times [0,1]$, and $A_0 = A_0(t)$ the solution to the problem

$$\begin{cases} \partial_s A_0 - \triangle A_0 = [A^\ell, \partial_\ell A_0] - [A^\ell, B_\ell], \\ A_0(s=0) = \overline{A}_0. \end{cases} \tag{3.4.5}$$

on $\{t\} \times \mathbb{R}^d \times [0,1]$.

Expanding out the covariant derivatives and $F_i{}^\ell$, we see that (3.4.4) is of the form

$$(\partial_s - \triangle) B_i = \mathcal{O}(A, \partial_x B) + \mathcal{O}(\partial_x A, B) + \mathcal{O}(A, A, B).$$

By Proposition 3.2.1 (which is applicable thanks to (3.4.1)), we may appeal to Lemma 3.3.1 to obtain a unique ($\mathfrak{g}$-valued) solution $B_i \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ to (3.4.4). Fix $-\frac{d}{2} < \gamma' < \frac{d}{2}$. Thanks to the smoothing estimates (3.2.9) for A_i, the hypotheses of Statements 1 and 2 of Lemma 3.3.1hold. Therefore, for each $m \geq 0$, B obeys

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} B\|_{\mathcal{L}^{\ell_{\gamma'},\infty}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} + \|\nabla_x^{(k+1)} B\|_{\mathcal{L}^{\ell_{\gamma'},2}_s \dot{\mathcal{H}}^{\gamma'}_x (0,1]} \Big) \leq C_{d,\gamma,\gamma',m,\|\overline{A}\|_{\dot{H}^{\gamma}_x}} \|\overline{F}_0\|_{\dot{H}^{\gamma'}_x}. \tag{3.4.6}$$

Next, using the fact that B_i is a $\mathfrak{g}$-valued (spatial) 1-form in $C^\infty_{t,s}(I \times [0,1], H^\infty_x)$, appealing to Lemma 3.3.1 gives a unique $\mathfrak{g}$-valued solution $A_0 \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ to (3.4.5). Let $-\frac{d}{2} + 1 < \gamma'' < \frac{d}{2}$. By the smoothing estimates (3.2.9) and (3.4.6) (with $\gamma' = \gamma'' - 1$) for A_i and B_i, respectively, the hypotheses of Statement 1 and 2 of Lemma 3.3.1 hold. (Note that for B_i, we need $-\frac{d}{2} < \gamma' = \gamma'' - 1 < \frac{d}{2}$.) Thus, for each integer $m \geq 0$, A_0 satisfies

$$\begin{aligned} \sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} A_0\|_{\mathcal{L}^{\ell_{\gamma''},\infty}_s \dot{\mathcal{H}}^{\gamma''}_x (0,1]} &+ \|\nabla_x^{(k+1)} A_0\|_{\mathcal{L}^{\ell_{\gamma''},2}_s \dot{\mathcal{H}}^{\gamma''}_x (0,1]} \Big) \\ &\leq C_{d,\gamma,\gamma',\gamma'',m,\|\overline{A}\|_{\dot{H}^{\gamma}_x}} (\|\overline{A}_0\|_{\dot{H}^{\gamma''}_x} + \|\overline{F}_0\|_{\dot{H}^{\gamma''-1}_x}). \end{aligned} \tag{3.4.7}$$

In conclusion, we have achieved the following so far: There exist unique $\mathfrak{g}$-valued solutions B_i and A_0 in $C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ to (3.4.4) and (3.4.5), respectively. Moreover, they satisfy the estimates (3.4.6) and (3.4.7) for $-\frac{d}{2} < \gamma' < \frac{d}{2}$ and $-\frac{d}{2} + 1 < \gamma'' < \frac{d}{2}$, respectively.

Note that if (A_μ, A_s) were a smooth solution to (dYMHF), then by uniqueness, the solution to

(3.4.4) would be exactly F_{0i} and A_0 would indeed satisfy (3.4.5) with $B_\ell = F_{0\ell}$. Conversely, if B_i and A_0 defined as above satisfies $B_i = F_{0i}$, where $F_{0i} := \partial_i A_0 - \partial_0 A_i + [A_i, A_0]$, then it is not difficult to verify that (A_0, A_i, A_s) with $A_s = \partial^\ell A_\ell$ would be a solution to (dYMHF). This would prove the first two statements of the proposition. Moreover, from the estimates (3.4.6) and (3.4.7), the remaining two statements would follow as well.

Therefore, our goal in the remainder of this proof is to show that $F_{0i} = B_i$ on $I \times \mathbb{R}^d \times [0,1]$. We shall begin by rewriting the equation (3.4.5) as follows:

$$F_{s0} = \mathbf{D}^\ell F_{\ell 0} + [A^\ell, F_{0\ell} - B_\ell].$$

Using the Bianchi identity (as in §2.1), we see that F_{0i} satisfies the following parabolic equation.

$$\mathbf{D}_s F_{0i} - \mathbf{D}^\ell \mathbf{D}_\ell F_{0i} = 2[F_i{}^\ell, F_{0\ell}] - \mathbf{D}_i [A^\ell, F_{0\ell} - B_\ell].$$

Let $\delta F_{0i} := F_{0i} - B_i$. Subtracting (3.4.4) from the preceding equation, we arrive at

$$\mathbf{D}_s(\delta F_{0i}) - \mathbf{D}^\ell \mathbf{D}_\ell(\delta F_{0i}) = -2[F_i{}^\ell, \delta F_{0\ell}] - \mathbf{D}_i [A^\ell, \delta F_{0\ell}].$$

where $\delta F_{0i}(s = 0) = 0$. Recall, furthermore, that $A_i, B_i, A_0 \in C^\infty_s([0,1], H^\infty_x)$; then it follows that $F_{0i} \in C^\infty_s([0,1], H^\infty_x)$ as well. Appealing to the uniqueness statement of Lemma 3.3.1, we conclude that $F_{0i} = B_i$. □

The following proposition regarding the Lipschitz dependence on the initial data can be proved in a similar manner; we leave the details to the reader.

Proposition 3.4.3 (Local well-posedness of (dYMHF) in the DeTurck gauge, Part II)**.** *Let $d \geq 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ and $I \subset \mathbb{R}$ an interval. Let $A_\mu, A'_\mu \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ be the solutions to* (dYMHF) *under the DeTurck condition $A_s = \partial^\ell A_\ell$ with initial data $\overline{A}_\mu, \overline{A}'_\mu \in C^\infty_t(I \times H^\infty_x)$ satisfying* (3.4.1)*, respectively, given by Proposition 3.4.1. Then the difference of the two solutions obey the following estimates.*

1. *The difference between the spatial components δA_i obeys* (3.2.5) *and* (3.2.10)*;*

2. *The difference between s-components $\delta A_s = \partial^\ell(\delta A_\ell)$ obeys* (3.2.15) *and* (3.2.16)*;*

3. *For each integer $m \geq 0$ and $-\frac{d}{2} < \gamma' < \frac{d}{2}$, the difference between curvature components*

$\delta F_0 = (\delta F_{0i})_{i=1,\dots,d}$ *obey*

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} F_0\|_{\mathcal{L}_s^{\ell_{\gamma'},\infty} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} + \|\nabla_x^{(k+1)} F_0\|_{\mathcal{L}_s^{\ell_{\gamma'},2} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \Big) \le C_{d,\gamma,\gamma',m,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma},\|\overline{F}_0\|_{\dot{H}_x^{\gamma'}},\|\overline{F}'_0\|_{\dot{H}_x^{\gamma'}}} (\|\delta\overline{A}\|_{\dot{H}_x^\gamma} + \|\delta\overline{F}_0\|_{\dot{H}_x^{\gamma'}}); \tag{3.4.8}$$

4. *Finally, for each integer* $m \ge 0$ *and* $-\frac{d}{2}+1 < \gamma' < \frac{d}{2}$, *the difference between the temporal components* δA_0 *obeys*

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} (\delta A_0)\|_{\mathcal{L}_s^{\ell_{\gamma'},\infty} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} + \|\nabla_x^{(k+1)} (\delta A_0)\|_{\mathcal{L}_s^{\ell_{\gamma'},2} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \Big) \tag{3.4.9}$$

$$\le C_{d,\gamma,\gamma',m,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma},\|\overline{A}_0\|_{\dot{H}_x^{\gamma'}},\|\overline{A}'_0\|_{\dot{H}_x^{\gamma'}},\|\overline{F}_0\|_{\dot{H}_x^{\gamma'}},\|\overline{F}'_0\|_{\dot{H}_x^{\gamma'}}} (\|\delta\overline{A}\|_{\dot{H}_x^\gamma} + \|\delta\overline{A}_0\|_{\dot{H}_x^{\gamma'}} + \|\delta\overline{F}_0\|_{\dot{H}_x^{\gamma'-1}}).$$

3.5 Estimates for gauge transform to the caloric gauge

In the previous sections, we analyzed (cYMHF) and (dYMHF) under the DeTurck gauge condition $A_s = \partial^\ell A_\ell$. However, as discussed in the Introduction, for the purpose of analyzing the time-evolution of the system (HPYM), we need to convert to the *caloric gauge* $A_s = 0$. In this section, we shall present estimates for the gauge transform from the DeTurck gauge to the caloric gauge.

As in the previous section, we shall divide the main result of this section into two propositions. The first one will be about a single gauge transform to the caloric gauge, whereas in the second one, we shall be concerned with the difference between gauge transforms corresponding to two nearby solutions. For simplicity, we shall only consider H_x^∞ initial data. The results will be stated only for (cYMHF), but we remark that these apply equally well to (dYMHF) on each fixed t-slice as (cYMHF) is a part of the latter system. We shall defer their proofs to §A.5.

Proposition 3.5.1 (Gauge transformation from the DeTurck to caloric gauge, Part I)**.** *Let* $d \ge 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$. *Fix* $s_1 \in [0,1]$. *Let* $\overline{A}_i$ *be a* $\mathfrak{g}$*-valued 1-form in* $H_x^\infty(\mathbb{R}^d)$ *satisfying* (3.2.3), *and* $A_i \in C_s^\infty([0,1], H_x^\infty)$ *the corresponding unique solution to* (cYMHF) *in the DeTurck gauge* $A_s = \partial^\ell A_\ell$. *Consider the ODE*

$$\begin{cases} \partial_s U = U A_s \\ U(s = s_1) = \mathrm{Id}, \end{cases} \tag{3.5.1}$$

on $\mathbb{R}^d \times [0,1]$.

Then there exists a unique solution U *to* (3.5.1), *which is a* $\mathfrak{G}$*-valued function on* $\mathbb{R}^d \times [0,1]$ *such* *that* $U - \mathrm{Id} \in C_s^\infty([0,1], H_x^\infty)$*. Moreover, the solution satisfies the following properties.*

1. *The unique solution* U *obeys the estimate*

$$\|U - \mathrm{Id}\|_{L_s^\infty \dot{H}_x^{\gamma+1}[0,1]} + \|U - \mathrm{Id}\|_{L_s^\infty(\dot{H}_x^{d/2} \cap L_x^\infty)[0,1]} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.5.2}$$

2. *We have* smooth dependence on parameter*; in particular, the following statement holds:*

   Suppose that $\overline{A}_i(t) \in H_x^\infty$ *is a family of initial data satisfying* (3.2.3)*, which is parametrized by* $t \in I$ *(*$I \subset \mathbb{R}$ *is an interval) and* $\overline{A}_i \in C_t^\infty(I, H_x^\infty)$*. Then the corresponding gauge transform* $U(t)$ *satisfies* $U - \mathrm{Id} \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$*.*

3. *Furthermore, if* $s_1 > 0$*, then for every integer* $k \geq 1$*,* U *obeys the following estimate on* $(0, s_1]$*:*

$$\|s^{k/2} \partial_x^{(k)} (U - \mathrm{Id})\|_{L_s^\infty \dot{H}_x^{\gamma+1}(0,s_1]} \leq C_{d,\gamma,k,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.5.3}$$

   On $(s_1, 1]$*, we have*

$$s_1^{k/2} \|\partial_x^{(k)} (U - \mathrm{Id})\|_{L_s^\infty \dot{H}_x^{\gamma+1}(s_1,1]} \leq C_{d,\gamma,k,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.5.4}$$

4. *Finally, all of the above properties remain true with* U *replaced by* U^{-1}*.*

Next, we state some estimates regarding the difference between two gauge transforms corresponding to two nearby solution of (cYMHF).

Proposition 3.5.2 (Gauge transformation from the DeTurck to caloric gauge, Part II)**.** *Let* $d \geq 2$*,* $\frac{d-2}{2} < \gamma < \frac{d}{2}$ *and fix* $s_1 \in [0,1]$*. Let* $\overline{A}_i$*,* $\overline{A}_i'$ *be* $\mathfrak{g}$*-valued 1-forms in* $H_x^\infty(\mathbb{R}^d)$ *satisfying* (3.2.3)*, and* $A_i, A_i' \in C_s^\infty([0,1], H_x^\infty)$ *the corresponding unique solutions to* (cYMHF) *in the DeTurck gauge* $A_s = \partial^\ell A_\ell$*. Moreover, let* U, U' *be the unique solutions to* (3.5.1) *corresponding to* A_i, A_i'*, respectively, given by Proposition 3.5.1. Then the difference between two gauge transforms satisfies the following properties.*

1. *The difference* δU *obeys the estimate*

$$\|\delta U\|_{L_s^\infty \dot{H}_x^{\gamma+1}[0,1]} + \|\delta U\|_{L_s^\infty(\dot{H}_x^{d/2} \cap L_x^\infty)[0,1]} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta \overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.5.5}$$

2. *Furthermore, if* $s_1 > 0$*, then for every integer* $k \geq 1$*,* δU *obeys the following estimate holds on*

$(0, s_1]$:

$$\|s^{k/2}\partial_x^{(k)}(\delta U)\|_{L^\infty_s \dot{H}^{\gamma+1}_x(0,s_1]} \leq C_{d,\gamma,k,\|\overline{A}\|_{\dot{H}^\gamma_x},\|\overline{A}'\|_{\dot{H}^\gamma_x}} \|\delta\overline{A}\|_{\dot{H}^\gamma_x}. \tag{3.5.6}$$

On $(s_1, 1]$*, we have instead*

$$s_1^{k/2}\|\partial_x^{(k)}(\delta U)\|_{L^\infty_s \dot{H}^{\gamma+1}_x(s_1,1]} \leq C_{d,\gamma,k,\|\overline{A}\|_{\dot{H}^\gamma_x},\|\overline{A}'\|_{\dot{H}^\gamma_x}} \|\delta\overline{A}\|_{\dot{H}^\gamma_x}, \tag{3.5.7}$$

3. *Finally, all of the above properties remain true with* δU *replaced by* δU^{-1}*.*

Remark 3.5.3. We remark that Proposition 3.5.1 (along with the difference estimates provided by Proposition 3.5.2) will be for us an analogue of Uhlenbeck's lemma [37], on which the work [15] crucially rely. Given a connection 1-form A_i, Uhlenbeck's lemma asserts, roughly speaking, the existence of a gauge transform with good regularity properties to the Coulomb gauge, *provided that* either the $L^{d/2}_x$ norm of F_{ij} or L^d norm of A_i is small. Note that the Coulomb gauge condition is synonymous with setting the curl-free part of A_i zero. On the other hand, heuristically, an application of this proposition with $s_1 > 0$, combined with the smoothing estimates for (cYMHF) and (dYMHF) in the DeTurck gauge in §3.2 – §3.4, amounts to transforming a given initial data set to another whose curl-free part is 'smoother'. The advantage of Propositon 3.5.1 is that it only requires smallness of a (scaling-)*sub-critical* quantity $\|\overline{A}\|_{\dot{H}^\gamma_x}$, compared to the *scaling-invariant* norms in the case of Uhlenbeck's lemma.

In the simpler case of an abelian gauge gauge theory, e.g. Maxwell's equations, this heuristic can be demonstrated in a more concrete manner as follows: In this case, the connection component A_s will exist all the way to $s \to \infty$, and will converge to zero in a suitable sense. Note furthermore that $\partial^\ell F_{s\ell} = \partial_s(\partial^\ell A_\ell) - \triangle A_s = 0$. Therefore, this proposition, if applied with '$s_1 = \infty$', transforms the initial data to one such that the curl-free part is zero, i.e. one satisfying the Coulomb gauge condition.

We shall end this section with useful lemmas that relate the estimates for U, U^{-1}, δU, δU^{-1} obtained in the previous proposition to those for the corresponding gauge transformation of A_μ and covariant $\mathfrak{g}$-valued tensors.

Lemma 3.5.4 (Estimates for gauge transformation, Part I)**.** *Let* U *be a* $\mathfrak{G}$*-valued function in* $C^\infty_s([0,1], H^\infty_x)$*,* B *a* $\mathfrak{g}$*-valued function in* $C^\infty_s([0,1], H^\infty_x)$ *and* $-\frac{d}{2} < \gamma < \frac{d}{2}$*. Then the following statements hold*

1. *Suppose that there exists* $\mathcal{U}_0 > 0$ *such that*

$$\begin{aligned}\|U - \mathrm{Id}\|_{\mathcal{L}^{\ell_\gamma - 1/2, \infty}_s \dot{\mathcal{H}}^{\gamma+1}_x(0,1]} + \|U - \mathrm{Id}\|_{\mathcal{L}^\infty_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} \leq& \mathcal{U}_0, \\ \|U^{-1} - \mathrm{Id}\|_{\mathcal{L}^{\ell_\gamma - 1/2, \infty}_s \dot{\mathcal{H}}^{\gamma+1}_x(0,1]} + \|U^{-1} - \mathrm{Id}\|_{\mathcal{L}^\infty_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} \leq& \mathcal{U}_0.\end{aligned} \tag{3.5.8}$$

Then for $\ell \in \mathbb{R}$, $1 \leq r \leq \infty$ *and* $-\frac{d}{2} < \gamma' < \frac{d}{2}$, *we have*

$$\|UBU^{-1}\|_{\mathcal{L}^{\ell, r}_s \dot{\mathcal{H}}^{\gamma'}_x(0,1]} \leq C_{d,\gamma,\gamma',\mathcal{U}_0} \|B\|_{\mathcal{L}^{\ell, r}_s \dot{\mathcal{H}}^{\gamma'}_x(0,1]}. \tag{3.5.9}$$

Furthermore, $\partial_i U U^{-1}$ *obeys*

$$\|\partial_i U U^{-1}\|_{\mathcal{L}^{\ell_\gamma, \infty}_s \dot{\mathcal{H}}^{\gamma}_x(0,1]} \leq C_{d,\gamma,\mathcal{U}_0} \mathcal{U}_0. \tag{3.5.10}$$

2. *Let* $m \geq 0$ *be an integer. Suppose that there exists* $\mathcal{U}_m > 0$ *such that* $\mathcal{U}_0 \leq \mathcal{U}_m$ *and*

$$\sum_{k=0}^{m} \Big(\|\nabla^{(k)}_x (U - \mathrm{Id})\|_{\mathcal{L}^{\ell_\gamma - 1/2, \infty}_s \dot{\mathcal{H}}^{\gamma+1}_x(0,1]} + \|\nabla^{(k)}_x (U^{-1} - \mathrm{Id})\|_{\mathcal{L}^{\ell_\gamma - 1/2, \infty}_s \dot{\mathcal{H}}^{\gamma+1}_x(0,1]} \Big) \leq \mathcal{U}_m. \tag{3.5.11}$$

Then for any $\ell \in \mathbb{R}$, $1 \leq r \leq \infty$ *and* $-\frac{d}{2} < \gamma' < \frac{d}{2}$, *we have*

$$\|\nabla^{(m)}_x (UBU^{-1})\|_{\mathcal{L}^{\ell, r}_s \dot{\mathcal{H}}^{\gamma'}_x(0,1]} \leq C_{d,\gamma,\gamma',m,\mathcal{U}_m} \sum_{k=0}^{m} \|\nabla^{(k)}_x B\|_{\mathcal{L}^{\ell, r}_s \dot{\mathcal{H}}^{\gamma'}_x(0,1]}. \tag{3.5.12}$$

Furthermore, for any integer $0 \leq k \leq m$, $\nabla^{(k)}_x (\partial_i U U^{-1})$ *obeys*

$$\|\nabla^{(k)}_x (\partial_i U U^{-1})\|_{\mathcal{L}^{\ell_\gamma, \infty}_s \dot{\mathcal{H}}^{\gamma}_x(0,1]} \leq C_{d,\gamma,\mathcal{U}_m} \mathcal{U}_m. \tag{3.5.13}$$

Remark 3.5.5. Let $\overline{A}_i$ be a $\mathfrak{g}$-valued 1-form in H^∞_x satisfying the hypotheses of Proposition 3.5.1. Then the hypothesis (3.5.8) is satisfied with $\mathcal{U}_0 = C_{d,\gamma,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}$ by (3.5.2) of Proposition 3.5.1. Moreover, if $s_1 = 1$, then the hypothesis (3.5.11) is satisfied for every integer $m \geq 0$ with $\mathcal{U}_m := C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}$, by (3.5.3) of the same proposition.

Proof. This is an easy consequence of Lemma A.3.1. The following observation may be useful: Suppose that (3.5.8) and (3.5.11) hold. By Gagliardo-Nirenberg and the fact that $\ell_\gamma < \frac{1}{2}$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla^{(k)}_x (U - \mathrm{Id})\|_{\mathcal{L}^\infty_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} + \|\nabla^{(k)}_x (U^{-1} - \mathrm{Id})\|_{\mathcal{L}^\infty_s(\dot{\mathcal{H}}^{d/2}_x \cap \mathcal{L}^\infty_x)(0,1]} \Big) \leq \mathcal{U}_m. \qquad \square$$

The following is a difference analogue of the preceding lemma, whose proof we omit.

Lemma 3.5.6 (Estimates for gauge transformation, Part II). *Let U, U' be $\mathfrak{G}$-valued functions in $C_s^\infty([0,1], H_x^\infty)$, B, B' be $\mathfrak{g}$-valued functions in $C_s^\infty([0,1], H_x^\infty)$ and $-\frac{d}{2} < \gamma < \frac{d}{2}$. Recall the notations $\delta(UBU^{-1})$ and $\delta(\partial_i U U^{-1})$ from Lemma A.3.1.Then the following statements hold*

1. *Suppose that there exist $\mathcal{U}_0, \delta\mathcal{U}_0 > 0$ such that U and U^{-1} obey (3.5.8) and*

$$\begin{aligned} \|\delta U\|_{\mathcal{L}_s^{\ell_\gamma - 1/2, \infty} \dot{\mathcal{H}}_x^{\gamma+1}(0,1]} + \|\delta U\|_{\mathcal{L}_s^\infty (\dot{\mathcal{H}}_x^{d/2} \cap \mathcal{L}_x^\infty)(0,1]} \leq \delta\mathcal{U}_0, \\ \|\delta U^{-1}\|_{\mathcal{L}_s^{\ell_\gamma - 1/2, \infty} \dot{\mathcal{H}}_x^{\gamma+1}(0,1]} + \|\delta U^{-1}\|_{\mathcal{L}_s^\infty (\dot{\mathcal{H}}_x^{d/2} \cap \mathcal{L}_x^\infty)(0,1]} \leq \delta\mathcal{U}_0. \end{aligned} \tag{3.5.14}$$

   Then for $\ell \in \mathbb{R}$, $1 \leq r \leq \infty$ and $-\frac{d}{2} < \gamma' < \frac{d}{2}$, we have

$$\|\delta(UBU^{-1})\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \leq C_{d,\gamma,\gamma',\mathcal{U}_0} \|\delta B\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} + C_{d,\gamma,\gamma',\mathcal{U}_0} \delta\mathcal{U}_0 \|B\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]}. \tag{3.5.15}$$

   Furthermore, $\delta(\partial_i U U^{-1})$ obeys

$$\|\delta(\partial_i U U^{-1})\|_{\mathcal{L}_s^{\ell_\gamma, \infty} \dot{\mathcal{H}}_x^{\gamma}(0,1]} \leq C_{d,\gamma,\mathcal{U}_0} \delta\mathcal{U}_0. \tag{3.5.16}$$

2. *Let $m \geq 0$ be an integer. Suppose that there exist $\mathcal{U}_m, \delta\mathcal{U}_m > 0$ such that*

$$\mathcal{U}_0 \leq \mathcal{U}_m, \quad \delta\mathcal{U}_0 \leq \delta\mathcal{U}_m,$$

   U and U' obey (3.5.11) and

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} (\delta U)\|_{\mathcal{L}_s^{\ell_\gamma - 1/2, \infty} \dot{\mathcal{H}}_x^{\gamma+1}(0,1]} + \|\nabla_x^{(k)} (\delta U^{-1})\|_{\mathcal{L}_s^{\ell_\gamma - 1/2, \infty} \dot{\mathcal{H}}_x^{\gamma+1}(0,1]} \Big) \leq \delta\mathcal{U}_m. \tag{3.5.17}$$

   Then for any $\ell \in \mathbb{R}$, $1 \leq r \leq \infty$ and $-\frac{d}{2} < \gamma' < \frac{d}{2}$, we have

$$\begin{aligned} \|\nabla_x^{(m)} \delta(UBU^{-1})\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \leq & C_{d,\gamma,\gamma',m,\mathcal{U}_m} \sum_{k=0}^{m} \|\nabla_x^{(k)} (\delta B)\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \\ & + C_{d,\gamma,\gamma',m,\mathcal{U}_m} \delta\mathcal{U}_m \sum_{k=0}^{m} \|\nabla_x^{(k)} B\|_{\mathcal{L}_s^{\ell, r} \dot{\mathcal{H}}_x^{\gamma'}(0,1]}. \end{aligned} \tag{3.5.18}$$

Furthermore, for any $0 \le k \le m$, $\nabla_x^{(k)} \delta(\partial_i U U^{-1})$ *obeys*

$$\|\nabla_x^{(k)} \delta(\partial_i U U^{-1})\|_{\mathcal{L}_s^{\ell_\gamma,\infty} \dot{\mathcal{H}}_x^\gamma(0,1]} \le C_{d,\gamma,\mathcal{U}_m} \delta \mathcal{U}_m. \tag{3.5.19}$$

Remark 3.5.7. Let $\overline{A}_i$, $\overline{A}_i'$ be $\mathfrak{g}$-valued 1-form in H_x^∞ satisfying the hypotheses of Proposition 3.5.2. Then as before, the hypothesis (3.5.14) is satisfied with $\delta\mathcal{U}_0 = C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta\overline{A}\|_{\dot{H}_x^\gamma}$ by (3.5.5) of Proposition 3.5.2. Moreover, if $s_1 = 1$, then the hypothesis (3.5.17) is satisfied for every integer $m \ge 0$ with $\delta\mathcal{U}_m := C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta\overline{A}\|_{\dot{H}_x^\gamma}$, by (3.5.6) of the same proposition.

3.6 Yang-Mills heat flows in the caloric gauge

In this section, we shall establish local well-posedness of (cYMHF) and (dYMHF) in the caloric gauge $A_s = 0$. In fact, the former system is exactly the original Yang-Mills heat flow

$$\partial_s A_i = \mathbf{D}^\ell F_{\ell i}, \tag{YMHF}$$

so our approach give an alternative proof of the classical local well-posedness result for (YMHF), established in [27] in dimensions 2 and 3.

Proposition 3.6.1 (Local well-posedness of (YMHF))**.** *Let* $d \ge 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ *and define* $q \ge 2$ *by* $\frac{d}{q} = \frac{d}{2} - \gamma$. *Consider the Yang-Mills heat flow*

$$\partial_s A_i = \mathbf{D}^\ell F_{\ell i}. \tag{eq:YMHF}$$

Let $\overline{A}_i$ *be a* $\mathfrak{g}$*-valued 1-form on* $\mathbb{R}^d \times \{0\}$ *such that* $\overline{A}_i \in H_x^\infty$ *and*

$$\|\overline{A}\|_{\dot{H}_x^\gamma} \le \delta_P, \tag{3.6.1}$$

where $\delta_P = \delta_P(d,\gamma)$ *is the constant in Proposition 3.2.1.*

Then there exists a unique solution A_i *to* (YMHF) *such that* $A_i \in C_s^\infty([0,1], H_x^\infty)$. *Moreover, the solution satisfies the following properties.*

1. *The solution* A_i *obeys the estimate*

$$\|A\|_{L_s^\infty \dot{H}_x^\gamma[0,1]} \le C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.6.2}$$

2. *Consider an additional initial data* $\overline{A}_i' \in H_x^\infty$ *satisfying* (3.6.1)*, and let* $A_i' \in C_s^\infty([0,1], H_x^\infty)$ *be the corresponding unique solution to* (YMHF)*. Then the difference* δA *between the two solutions obeys the estimate*

$$\|\delta A\|_{L_s^\infty \dot{H}_x^\gamma [0,1]} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma}} \|\delta \overline{A}\|_{\dot{H}_x^\gamma}. \tag{3.6.3}$$

3. *We have* smooth dependence on the initial data*; in particular, the following statement holds: Suppose that* $\overline{A}_i(t) \in H_x^\infty$ *is a family of initial data satisfying* (3.6.1)*, which is parametrized by* $t \in I$ *(*$I \subset \mathbb{R}$ *is an interval) and* $\overline{A}_i \in C_t^\infty(I, H_x^\infty)$*. Then the corresponding solution* $A_i(t)$ *to* (YMHF) *satisfies* $A_i \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$*.*

Remark 3.6.2. Note that no smoothing estimate for A_i is claimed. Although we do have such estimates in the DeTurck gauge (see Proposition 3.2.5), the gauge transform U to the caloric gauge with $U(s=0) = \mathrm{Id}$ is not bounded in $\dot{H}_x^{\gamma+k}$ for $k > 0$. This is consistent with the hypothesis $s_1 > 0$ in Statement 3 of Proposition 3.5.1.

Proof. Let $(\widetilde{A}_i, \widetilde{A}_s)$ be the solution in $C_s^\infty([0,1], H_x^\infty)$ to (3.2.2) with initial data $\widetilde{A}_i(s=0) = \overline{A}_i$, given by Proposition 3.2.1. Moreover, let $U \in C_s^\infty([0,1], H_x^\infty)$ be the gauge transform of $(\widetilde{A}_i, \widetilde{A}_s)$ from the DeTurck to caloric gauge with $U(s=0) = \mathrm{Id}$, which is given by Proposition 3.5.1 with $s_1 = 0$. Then, by construction, the gauge transform (A_i, A_s) of $(\widetilde{A}_i, \widetilde{A}_s)$ by U, i.e.

$$A_i := U\widetilde{A}_i U^{-1} - \partial_i U U^{-1}, \quad A_s := U\widetilde{A}_s U^{-1} - \partial_s U U^{-1},$$

is a solution to (cYMHF) in the caloric gauge, or equivalently, to (YMHF). Moreover, A_i belongs to the class $C_s^\infty([0,1], H_x^\infty)$. Then, thanks to the estimates and properties of $\widetilde{A}_i$ and U as in Propositions 3.2.1, 3.5.1 and 3.5.2, along with Lemmas 3.5.4 and 3.5.6, Statements 1 – 3 of the proposition follow. (For an additional initial data $\overline{A}_i'$, we construct the corresponding solution A_i' in the identical manner.) It is therefore only left to establish the *uniqueness* of A_i, which is a consequence of the following lemma. □

Lemma 3.6.3. *Let* $d \geq 2$*. Consider solutions* A_i, A_i' *to* (YMHF) *on a common* s*-interval* $J = [0, s_0]$ *belonging to* $C_s^\infty([0,s_0], H_x^\infty)$*. If their initial data coincide, i.e.* $\overline{A}_i = \overline{A}_i'$ *on* $\mathbb{R}^d \times \{0\}$*, then so do the solutions, i.e.* $A_i = A_i'$ *on* $\mathbb{R}^d \times [0, s_0]$*.*

Proof. By a simple continuous induction argument, it suffices to prove that the solutions coincide on an arbitrarily short interval $[0, s_1]$, where $s_1 > 0$. Let $(\widetilde{A}_i, \widetilde{A}_s) \in C_s^\infty([0,s_1], H_x^\infty)$ be the unique solution to (cYMHF) in the DeTurck gauge $\widetilde{A}_s = \partial^\ell \widetilde{A}_\ell$ with $\overline{A}_i$ as the initial data, which is given by Proposition 3.2.1. Furthermore, let $U \in C_s^\infty([0, s_0], H_s^\infty)$ be the unique solution to the ODE

$$\begin{cases} \partial_s U = & U \widetilde{A}_s \\ U(s=0) = & \mathrm{Id}. \end{cases} \tag{3.6.4}$$

given by Proposition 3.5.1. Without loss of generality, we may assume that one of the solutions, say A_i, is equal to $\widetilde{A}_i$ gauge-transformed by U, i.e.

$$A_i = U\widetilde{A}_i U^{-1} - \partial_i U U^{-1}.$$

The goal is to show that $A'_i = A_i$ using the uniqueness of $\widetilde{A}_i$ and U in the DeTurck gauge. The key is to note that the equation for a (smooth) gauge transform W' from the caloric to DeTurck gauge, namely

$$-\partial_s W W^{-1} = \partial^\ell (W A'_\ell W^{-1} - \partial_\ell W W^{-1}), \tag{3.6.5}$$

is a parabolic equation for a $\mathfrak{G}$-valued function W on $\mathbb{R}^d \times [0, \infty)$ (thanks to the fact that $A'_i \in C_s^\infty([0, s_0], H_x^\infty)$ is $\mathfrak{g}$-valued). By the standard theory of semi-linear parabolic equation, for sufficiently small $s_1 > 0$, there exists a unique $\mathfrak{G}$-valued function W which solves (3.6.5) with initial data $W(s=0) = \mathrm{Id}$ and satisfies $W - \mathrm{Id} \in C_s^\infty([0, s_1], H_x^\infty)$.

Consider now the $\mathfrak{g}$-valued 1-form $\widetilde{A}'_i$, which is the gauge transform of A'_i by W defined by

$$\widetilde{A}'_i := W A'_i W^{-1} - \partial_i W W^{-1}.$$

Obviously, $\widetilde{A}'_i$ is a solution to (cYMHF). Moreover, we see also that $\widetilde{A}_i$ belongs to $C_s^\infty([0, s_1], H_x^\infty)$ and satisfies the DeTurck gauge condition by (3.6.5). Taking $s_1 > 0$ smaller if necessary, we may apply the uniqueness statement of (rescaled) Proposition 3.2.1 and conclude that $\widetilde{A}'_i = \widetilde{A}_i$ on $\mathbb{R}^d \times [0, s_1]$. Finally, note that W^{-1} belongs to $C_s^\infty([0, s_1], H_x^\infty)$ and solves the ODE (3.6.4), thanks to $A'_s = 0$. We therefore conclude $W^{-1} = U$ on $\mathbb{R}^d \times [0, s_1]$, by uniqueness for ODEs. Thus $A'_i = A_i$ on $[0, s_1]$, as desired. □

Next, we shall formulate and prove a local well-posedness statement for (dYMHF) in the caloric gauge.

Proposition 3.6.4 (Local well-posedness of (dYMHF) in the caloric gauge)**.** *Let $d \geq 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ and $I \subset \mathbb{R}$ an interval. Consider the dynamic Yang-Mills heat flow* (dYMHF) *on $I \times \mathbb{R}^d \times [0, 1]$ in* *the caloric gauge $A_s = 0$. Let $\overline{A}_\mu$ be a $\mathfrak{g}$-valued 1-form on $I \times \mathbb{R}^d \times \{0\}$ such that $\overline{A}_\mu \in C_t^\infty(I, H_x^\infty)$ and*

$$\|\overline{A}\|_{\dot{H}_x^\gamma} \leq \delta_P, \tag{3.6.6}$$

where $\delta_P = \delta_P(d,\gamma)$ is the constant in Proposition 3.2.1.

Then there exists a unique solution A_μ to (dYMHF) *in the caloric gauge $A_s = 0$ on $I \times \mathbb{R}^d \times [0,1]$ such that $A_\mu \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$. Moreover, the solution satisfies the following estimates.*

1. *The spatial components A_i obey* (3.6.2)*, whereas the curvature components F_{0i} obey*

$$\|F_0\|_{L_s^\infty \dot{H}_x^{\gamma'}[0,1]} \leq C_{d,\gamma,\gamma',\|\overline{A}\|_{\dot{H}_x^\gamma}} \|\overline{F}_0\|_{\dot{H}_x^{\gamma'}}. \tag{3.6.7}$$

   for each $-\frac{d}{2} < \gamma' < \frac{d}{2}$.

2. *Consider an additional initial data $\overline{A}'_\mu \in C_t^\infty(I, H_x^\infty)$ satisfying* (3.6.6)*, and let $A'_i \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$ be the corresponding unique solution to* (dYMHF) *in the caloric gauge. Then the difference δA between the spatial components of the two solutions obeys* (3.6.3)*, whereas the difference δF_{0i} between the curvature components F_{0i}, F'_{0i} obeys*

$$\|\delta F_0\|_{L_s^\infty \dot{H}_x^{\gamma'}[0,1]} \leq C_{d,\gamma,\gamma',\|\overline{A}\|_{\dot{H}_x^\gamma},\|\overline{A}'\|_{\dot{H}_x^\gamma},\|\overline{F}_0\|_{\dot{H}_x^{\gamma'}},\|\overline{F}'_0\|_{\dot{H}_x^{\gamma'}}} (\|\delta \overline{A}\|_{\dot{H}_x^\gamma} + \|\delta \overline{F}_0\|_{\dot{H}_x^{\gamma'}}). \tag{3.6.8}$$

   for each $-\frac{d}{2} < \gamma' < \frac{d}{2}$.

Remark 3.6.5. In the above proposition, we have omitted the statements of estimates for A_0 and δA_0, for they will not be of use later.

Proof. Let $(\widetilde{A}_\mu, \widetilde{A}_s)$ be the solution in $C_{t,s}^\infty(I \times [0,1], H_x^\infty)$ to (dYMHF) in the DeTurck gauge with initial data $\widetilde{A}_\mu(s=0) = \overline{A}_\mu$, given by Proposition 3.4.1. Let $U \in C_{t,s}^\infty(I \times [0,1], H_x^\infty)$ be the gauge transform of $(\widetilde{A}_\mu, \widetilde{A}_s)$ to the caloric gauge with $U(s=0) = \mathrm{Id}$, given by Proposition 3.5.1 with $s_1 = 0$. Then, by construction, the gauge transform (A_μ, A_s) of $(\widetilde{A}_\mu, \widetilde{A}_s)$ by U, i.e.

$$A_\mu := U\widetilde{A}_\mu U^{-1} - \partial_\mu U U^{-1}, \quad A_s := U\widetilde{A}_s U^{-1} - \partial_s U U^{-1},$$

is a solution to (dYMHF) in the caloric gauge, which furthermore satisfies $A_\mu \in C_{t,x}^\infty(I \times [0,1], H_x^\infty)$. By Proposition 3.6.1, we see that A_i and δA_i obey (3.6.2) and (3.6.3), respectively. (Here, A'_μ is constructed from $\overline{A}'_\mu$ in the identical manner.) On the other hand, since

$$F_{0i} = U\widetilde{F}_{0i}U^{-1},$$

the estimates (3.6.7) and (3.6.8) follow from Propositions 3.4.1, 3.4.3, 3.5.1 and 3.5.2, as well as Lemmas 3.5.4 and 3.5.6. Thus, as before, we are only left to establish *uniqueness* of the solution A_μ, which is achieved by the following lemma. $\square$

Lemma 3.6.6. *Let $d \geq 2$ and $I \subset \mathbb{R}$ be an interval. Consider solutions A_μ, A'_μ to* (dYMHF) *in the caloric gauge on $I \times \mathbb{R}^d \times [0, s_0]$ for some $s_0 > 0$ belonging to $C^\infty_{t,s}(I \times [0, s_0], H^\infty_x)$. If their initial data coincide, i.e. $\overline{A}_\mu = \overline{A}'_\mu$ on $I \times \mathbb{R}^d \times \{0\}$, then so do the solutions, i.e. $A_\mu = A'_\mu$ on $I \times \mathbb{R}^d \times [0, s_0]$.*

Proof. For each fixed t, note that the spatial components $A_i(t)$ satisfy (YMHF). Therefore, by Lemma 3.6.3, it follows that $A_i = A'_i$. Thus we are only left to show $A_0 = A'_0$.

Observe that $\delta F_{0i} = F_{0i} - F'_{0i}$ now obeys the linear parabolic equation

$$(\partial_s - \mathbf{D}^\ell \mathbf{D}_\ell)(\delta F_{0i}) = -2[\delta {F_0}^\ell, F_{i\ell}].$$

Note furthermore that $\delta F_{0i}(s = 0) = 0$. Applying the uniqueness statement of Lemma 3.3.1 (scaling $[0, 1]$ to $[0, s_0]$), we see that $F_{0i} = F'_{0i}$ on $I \times \mathbb{R}^d \times [0, s_0]$. Then, by the dynamic Yang-Mills heat flow, $\partial_s A_0 = \mathbf{D}^\ell F_{\ell 0} = \partial_s A'_0$ everywhere. Since $\overline{A}_0 = \overline{A}'_0$, it follows that $A_0 = A'_0$ on $I \times \mathbb{R}^d \times [0, s_0]$, which concludes the proof. $\square$

3.7 Transformation to the caloric-temporal gauge

As discussed in the Introduction, after solving (dYMHF) on $[0, s_0]$ from a solution $A^\dagger_\mu$ to (YM) at $\{s = 0\}$, we need to impose the *caloric-temporal gauge* condition

$$\begin{cases} A_s = 0, & \text{everywhere,} \\ \underline{A}_0 = 0, & \text{along } s = s_0. \end{cases}$$

in order to proceed to the analysis of the time evolution. The purpose of this section is to formulate and prove theorems to achieve this gauge transformation, along with appropriate estimates. The general idea is very similar to that of §3.6, but the key difference is that we shall choose $s_1 = 1 \neq 0$

in the application of Proposition 3.5.1. This will allow us to keep the smoothing estimates from the analysis in §3.2 – §3.4, which is central to the our entire approach.

We shall make the statements of the following theorems slightly general by not requiring $A^{\dagger}_{\mu}$ to be a solution to (YM). We remind the reader the convention that $\mathbf{a}, \mathbf{b}, \dots$ run over all indices $(x^0, x^1, \dots, x^d, s)$ on $I \times \mathbb{R}^d \times [0,1]$.

Theorem 3.7.1 (Transformation to the caloric-temporal gauge, Part I)**.** *Let $d \geq 2$ and $\frac{d-2}{2} < \gamma < \frac{d}{2}$. Let $A^{\dagger}_{\mu}$ be a $\mathfrak{g}$-valued 1-form on $I \times \mathbb{R}^d \times \{s = 0\}$ ($I \subset \mathbb{R}$ is an interval) such that $A^{\dagger}_{\mu} \in C^{\infty}_t(I, H^{\infty}_x)$ and satisfies*

$$\sup_{t \in I} \|A^{\dagger}(t)\|_{\dot{H}^{\gamma}_x} < \delta_P \tag{3.7.1}$$

where $\delta_P = \delta_P(d, \gamma) > 0$ is the constant in Proposition 3.2.1.

Then there exists a gauge transform $V \in C^{\infty}_t(I, H^{\infty}_x)$ and a solution $A_{\mathbf{a}} \in C^{\infty}_{t,s}(I \times [0,1], H^{\infty}_x)$ to (dYMHF) *such that*

$$\overline{A}_{\mu} := A_{\mu}(s=0) = V A^{\dagger}_{\mu} V^{-1} - \partial_{\mu} V V^{-1}, \tag{3.7.2}$$

and satisfies the caloric-temporal gauge conditions, *i.e. $A_s = 0$ everywhere and $\underline{A}_0 := A_0(s = 1) = 0$ along $s = 1$.*

Assume $0 \in I$ and define $\mathring{A}_i := A^{\dagger}_i(t = 0)$, $\mathring{E}_i := F^{\dagger}_{0i}(t = 0)$. Define furthermore $\underline{A}_{\mu} := A_{\mu}(s = 1)$. Then the gauge transform V and solution $A_{\mathbf{a}}$ satisfy the following statements:

1. *For every integer $m \geq 0$, $\underline{A}_i$ and F_{si} obey the following estimates at $t = 0$.*

$$\sum_{k=0}^{m} \|\partial^{(k)}_x \underline{A}_i\|_{\dot{H}^{\gamma}_x} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}^{\gamma}_x}} \|\mathring{A}\|_{\dot{H}^{\gamma}_x}, \tag{3.7.3}$$

$$\sum_{k=0}^{m} \Big(\|\nabla^{(k)}_x F_{si}\|_{\mathcal{L}^{\ell_\gamma+1,\infty}_s \dot{\mathcal{H}}^{\gamma}_x} + \|\nabla^{(k)}_x F_{si}\|_{\mathcal{L}^{\ell_\gamma+1,2}_s \dot{\mathcal{H}}^{\gamma}_x} \Big) \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}^{\gamma}_x}} \|\mathring{A}\|_{\dot{H}^{\gamma}_x}. \tag{3.7.4}$$

2. *Define $\mathring{V} := V(t = 0)$. Then $\mathring{V} - \mathrm{Id}, \mathring{V}^{-1} - \mathrm{Id} \in H^{\infty}_x$ and obey*

$$\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{\gamma+1}_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{d/2}_x \cap L^{\infty}_x} \leq C_{d,\gamma,\|\mathring{A}\|_{\dot{H}^{\gamma}_x}} \|\mathring{A}\|_{\dot{H}^{\gamma}_x}, \tag{3.7.5}$$

$$\|\mathring{V}^{-1} - \mathrm{Id}\|_{\dot{H}^{\gamma+1}_x} + \|\mathring{V}^{-1} - \mathrm{Id}\|_{\dot{H}^{d/2}_x \cap L^{\infty}_x} \leq C_{d,\gamma,\|\mathring{A}\|_{\dot{H}^{\gamma}_x}} \|\mathring{A}\|_{\dot{H}^{\gamma}_x}. \tag{3.7.6}$$

3. *Finally, for every integer $m \geq 0$, $\partial_t \underline{A}_i$ and $\partial_t F_{si}$ obey the following estimates at $t = 0$.*

$$\sum_{k=0}^{m} \|\partial_x^{(k)} \partial_t \underline{A}_i\|_{\dot{H}_x^{\gamma-1}} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}, \tag{3.7.7}$$

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \nabla_t F_{si}\|_{\mathcal{L}_s^{\ell\gamma+1,\infty} \dot{\mathcal{H}}_x^{\gamma-1}} + \|\nabla_x^{(k)} \nabla_t F_{si}\|_{\mathcal{L}_s^{\ell\gamma+1,2} \dot{\mathcal{H}}_x^{\gamma-1}} \Big) \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}. \tag{3.7.8}$$

The following is an analogue of Theorem 3.7.1 for differences.

Theorem 3.7.2 (Transformation to the caloric-temporal gauge, Part II)**.** *Let $d \geq 2$, $\frac{d-2}{2} < \gamma < \frac{d}{2}$ and $A^\dagger_\mu$, $(A')^\dagger_\mu$ $\mathfrak{g}$-valued 1-forms in $C_t^\infty(I \times H_x^\infty)$ ($I \subset \mathbb{R}$ is an interval) satisfying* (3.7.1) *and $0 \in I$. Denote by $(A_{\mathbf{a}}, V)$, $(A'_{\mathbf{a}}, V')$ the solution to* (dYMHF) *in the caloric-temporal gauge and the gauge transform, respectively, obtained from $A^\dagger_\mu$, $(A')^\dagger_\mu$ as in Theorem 3.7.1, in that order.*

1. *For every integer $m \geq 0$, $\delta \underline{A}_i$ and δF_{si} obey the following estimates at $t = 0$.*

$$\sum_{k=0}^{m} \|\partial_x^{(k)} (\delta \underline{A}_i)\|_{\dot{H}_x^\gamma} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma}} \|\delta \mathring{A}\|_{\dot{H}_x^\gamma}, \tag{3.7.9}$$

$$\begin{aligned} \sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} (\delta F_{si})\|_{\mathcal{L}_s^{\ell\gamma+1,\infty} \dot{\mathcal{H}}_x^{\gamma}} + \|\nabla_x^{(k)} (\delta F_{si})\|_{\mathcal{L}_s^{\ell\gamma+1,2} \dot{\mathcal{H}}_x^{\gamma}} \Big) \\ \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma}} \|\delta \mathring{A}\|_{\dot{H}_x^\gamma}. \end{aligned} \tag{3.7.10}$$

2. *Define $\delta \mathring{V} := \mathring{V} - \mathring{V}'$ and $\delta \mathring{V}^{-1} := \mathring{V}^{-1} - (\mathring{V}')^{-1}$. Then the following estimates hold.*

$$\|\delta \mathring{V}\|_{\dot{H}_x^{\gamma+1}} + \|\delta \mathring{V}\|_{\dot{H}_x^{d/2} \cap L_x^\infty} \leq C_{d,\gamma,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma}} \|\delta \mathring{A}\|_{\dot{H}_x^\gamma}, \tag{3.7.11}$$

$$\|\delta \mathring{V}^{-1}\|_{\dot{H}_x^{\gamma+1}} + \|\delta \mathring{V}^{-1}\|_{\dot{H}_x^{d/2} \cap L_x^\infty} \leq C_{d,\gamma,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma}} \|\delta \mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.12}$$

3. *Finally, for every integer $m \geq 0$, $\partial_t \underline{A}_i$ and $\partial_t F_{si}$ obey the following estimates at $t = 0$.*

$$\begin{aligned} \sum_{k=0}^{m} \|\partial_x^{(k)} \partial_t (\delta \underline{A}_i)\|_{\dot{H}_x^{\gamma-1}} \\ \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma},\|\mathring{E}\|_{\dot{H}_x^{\gamma-1}},\|\mathring{E}'\|_{\dot{H}_x^{\gamma-1}}} (\|\delta \mathring{A}\|_{\dot{H}_x^\gamma} + \|\delta \mathring{E}\|_{\dot{H}_x^{\gamma-1}}), \end{aligned} \tag{3.7.13}$$

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \nabla_t (\delta F_{si})\|_{\mathcal{L}_s^{\ell_\gamma+1,\infty} \dot{\mathcal{H}}_x^{\gamma-1}} + \|\nabla_x^{(k)} \nabla_t (\delta F_{si})\|_{\mathcal{L}_s^{\ell_\gamma+1,2} \dot{\mathcal{H}}_x^{\gamma-1}} \Big) \tag{3.7.14}$$
$$\leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma},\|\mathring{E}\|_{\dot{H}_x^{\gamma-1}},\|\mathring{E}'\|_{\dot{H}_x^{\gamma-1}}} (\|\delta \mathring{A}\|_{\dot{H}_x^\gamma} + \|\delta \mathring{E}\|_{\dot{H}_x^{\gamma-1}}).$$

Remark 3.7.3. By scaling, Theorems 3.7.1 and 3.7.2 may be applied to $A^\dagger_\mu$, $(A')^\dagger_\mu$ with large $L^\infty_t \dot{H}^\gamma_x$-norm. Indeed, suppose that $A^\dagger_\mu$ satisfies all of the hypotheses of Theorem 3.7.1 except (3.7.1). Then the conclusion of Theorem 3.7.1 would hold with $[0,1]$ replaced by $[0, s_0]$, where s_0 is a positive constant depending on $\|A^\dagger\|_{L^\infty_t \dot{H}^\gamma_x}$ in a non-increasing manner. Accordingly, we shall define $\underline{A}_\mu := A_\mu(s = s_0)$. Then the estimates stated in Theorems 3.7.1 and 3.7.2 for F_{si}, $\mathring{V}$ and their differences would continue to hold, with only $[0,1]$ replaced by $[0, s_0]$. For $\underline{A}_\mu$, however, we must put an appropriate weight of s_0. More precisely, instead of (3.7.3), (3.7.7), (3.7.9) and (3.7.13), we would have, respectively, the following estimates for every integer $m \geq 0$:

$$\sum_{k=0}^{m} s_0^{k/2} \|\partial_x^{(k)} \underline{A}_i\|_{\dot{H}_x^\gamma} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}, \tag{3.7.3$'$}$$

$$\sum_{k=0}^{m} s_0^{k/2} \|\partial_x^{(k)} \partial_t \underline{A}_i\|_{\dot{H}_x^{\gamma-1}} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}, \tag{3.7.7$'$}$$

$$\sum_{k=0}^{m} s_0^{k/2} \|\partial_x^{(k)} (\delta \underline{A}_i)\|_{\dot{H}_x^\gamma} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma}} \|\delta \mathring{A}\|_{\dot{H}_x^\gamma}, \tag{3.7.9$'$}$$

$$\sum_{k=0}^{m} s_0^{k/2} \|\partial_x^{(k)} \partial_t (\delta \underline{A}_i)\|_{\dot{H}_x^{\gamma-1}} \tag{3.7.13$'$}$$
$$\leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma},\|\mathring{A}'\|_{\dot{H}_x^\gamma},\|\mathring{E}\|_{\dot{H}_x^{\gamma-1}},\|\mathring{E}'\|_{\dot{H}_x^{\gamma-1}}} (\|\delta \mathring{A}\|_{\dot{H}_x^\gamma} + \|\delta \mathring{E}\|_{\dot{H}_x^{\gamma-1}}).$$

In the remainder of this section, we shall prove Theorems 3.7.1 and 3.7.2.

Proof of Theorem 3.7.1. In this proof, we shall adopt the following convention: A solution to (dYMHF) in the *DeTurck gauge* will be denoted $A_{\mathbf{a}}$, whereas that in the *caloric-temporal gauge* will be denoted by $\widetilde{A}_{\mathbf{a}}$. Although this is contrary to the notations in the statement of the above theorems (in which $\mathbf{A}_{\mathbf{a}}$ had been used for the latter), it will be far more efficient as most of the work will be done in the DeTurck gauge.

Step 1. Construction of gauge transform to caloric-temporal gauge. Let us begin by applying Proposition 3.4.1 to the initial data $A^\dagger_\mu$, from which we obtain a solution $A_{\mathbf{a}} \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ of (dYMHF) in the DeTurck gauge $A_s = \partial^\ell A_\ell$ such that $A_\mu(s = 0) = A^\dagger_\mu$. Our goal is to exhibit a gauge transform U which would transform $A_{\mathbf{a}}$ to a solution

$$\widetilde{A}_{\mathbf{a}} = U A_{\mathbf{a}} U^{-1} - \partial_{\mathbf{a}} U U^{-1} \tag{3.7.15}$$

to (dYMHF) in the caloric-temporal gauge $\widetilde{A}_s = 0, \underline{\widetilde{A}}_0 = 0$.

Observe that the 'temporal' gauge condition, namely $\underline{\widetilde{A}}_0 = 0$, is equivalent to the following ODE for $\underline{U} := U(s=1)$ along $\{s=1\}$:

$$\partial_t \underline{U} = \underline{U}\, \underline{A}_0. \tag{3.7.16}$$

Similarly, the caloric gauge condition $A_s = 0$ is equivalent to the following ODE for every $t \in I$:

$$\partial_s U = U A_s. \tag{3.7.17}$$

Let us solve (3.7.16) with

$$\underline{U}(t=0) = \mathrm{Id}, \tag{3.7.18}$$

and in turn (3.7.17) for every $t \in I$ with

$$U(s=1) = \underline{U}(t). \tag{3.7.19}$$

As $\underline{A}_0 \in C^\infty_t(I, H^\infty_x)$ and $A_s \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$, we then obtain a unique solution $U - \mathrm{Id} \in C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ satisfying (3.7.16) – (3.7.19). By construction, the gauge transformed connection 1-form $\widetilde{A}_{\mathbf{a}}$ defined by (3.7.15) belongs to $C^\infty_{t,s}(I \times [0,1], H^\infty_x)$ and satisfies the caloric-temporal gauge conditions $\widetilde{A}_s = 0$ and $\underline{\widetilde{A}}_0 = 0$. Defining $V(t,x) := U(t,x,0)$, we have $V - \mathrm{Id} \in C^\infty_t(I, H^\infty_x)$ (3.7.2) as desired.

Step 2. Proof of (3.7.3) – (3.7.6) *at* $t = 0$. Next, assuming $0 \in I$, we shall focus on obtaining *quantitative estimates* (3.7.3) – (3.7.8) at $t = 0$. Recall the notations $\mathring{A}_i := A^\dagger_i(t=0)$, $\mathring{E}_i := F^\dagger_{0i}(t=0)$ and $\mathring{V} := V(t=0) = U(t=0, s=0)$. In this step, we work exclusively on $t = 0$, i.e. $\{0\} \times \mathbb{R}^d \times [0,1]$, and derive the estimates (3.7.3) – (3.7.6) concerning only spatial derivatives.

To begin with, by the smoothing estimates (3.2.9), note that the following statement holds: For every integer $m \geq 0$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla^{(k)}_x A\|_{\mathcal{L}^{\ell_\gamma,\infty}_s \dot{\mathcal{H}}^\gamma_x(0,1]} + \|\nabla^{(k)}_x A\|_{\mathcal{L}^{\ell_\gamma,2}_s \dot{\mathcal{H}}^{\gamma+1}_x(0,1]} \Big) \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}^\gamma_x}} \|\mathring{A}\|_{\dot{H}^\gamma_x}. \tag{3.7.20}$$

This immediately proves (3.7.3), as $\underline{\widetilde{A}}_i = \underline{A}_i$. To proceed, recall the formula

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

By (3.7.20), Lemma 3.1.14, Lemma 3.1.4 and Lemma 3.1.8 (along with the fact that $\frac{1}{2} - \ell_\gamma > 0$), we have

$$\|\partial_x A\|_{\mathcal{L}_s^{\ell_\gamma+1/2,r}\dot{\mathcal{H}}_x^\gamma(0,1]} \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}, \quad \|\mathcal{O}(A,A)\|_{\mathcal{L}_s^{\ell_\gamma+1/2,r}\dot{\mathcal{H}}_x^\gamma(0,1]} \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}.$$

for $r = 2, \infty$. As a consequence, the following statement for $F := (F_{ij})_{1 \leq i < j \leq d}$ holds: For every integer $m \geq 0$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} F\|_{\mathcal{L}_s^{\ell_\gamma+1/2,\infty}\dot{\mathcal{H}}_x^\gamma(0,1]} + \|\nabla_x^{(k)} F\|_{\mathcal{L}_s^{\ell_\gamma+1/2,2}\dot{\mathcal{H}}_x^\gamma(0,1]} \Big) \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.21}$$

(Note that $\mathcal{L}_s^2$-type estimate holds for $\dot{\mathcal{H}}_x^\gamma$ in this case, unlike (3.7.20).)

By $F_{si} = \mathbf{D}^\ell F_{\ell i}$, using (3.7.20), (3.7.21), Lemma 3.1.14, Lemma 3.1.4 and Lemma 3.1.8, we furthermore derive the following statement: For every integer $m \geq 0$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} F_s\|_{\mathcal{L}_s^{\ell_\gamma+1,\infty}\dot{\mathcal{H}}_x^\gamma(0,1]} + \|\nabla_x^{(k)} F_s\|_{\mathcal{L}_s^{\ell_\gamma+1,2}\dot{\mathcal{H}}_x^\gamma(0,1]} \Big) \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.22}$$

Appealing to (3.5.2) and (3.5.3) of Proposition 3.5.1 with $s_1 = 1$, we have

$$\|U - \mathrm{Id}\|_{L_s^\infty \dot{H}_x^{\gamma+1}[0,1]} + \|U - \mathrm{Id}\|_{L_s^\infty(\dot{H}_x^{d/2} \cap L_x^\infty)[0,1]} \leq C_{d,\gamma,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.23}$$

and for each integer $m \geq 1$,

$$\sum_{k=1}^{m} \|s^{k/2} \partial_x^{(k)} (U - \mathrm{Id})\|_{L_s^\infty \dot{H}_x^{\gamma+1}(0,1]} \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.24}$$

and identical estimates hold also for U^{-1}. Recalling that $V = U(s=0)$, (3.7.5) and (3.7.6) immediately follow. Moreover, by (3.7.22), (3.7.23), (3.7.24) and Lemma 3.5.4, for every integer $m \geq 0$, the following estimates for $\widetilde{F}_{si} = U F_{si} U^{-1}$ holds:

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \widetilde{F}_s\|_{\mathcal{L}_s^{\ell_\gamma+1,\infty}\dot{\mathcal{H}}_x^\gamma(0,1]} + \|\nabla_x^{(k)} \widetilde{F}_s\|_{\mathcal{L}_s^{\ell_\gamma+1,2}\dot{\mathcal{H}}_x^\gamma(0,1]} \Big) \leq C_{d,\delta,m,\|\mathring{A}\|_{\dot{H}_x^\gamma}} \|\mathring{A}\|_{\dot{H}_x^\gamma}. \tag{3.7.25}$$

This proves (3.7.4).

Step 3. Proof of (3.7.7) – (3.7.8) *at* $t = 0$. Finally, let us prove the estimates (3.7.7), (3.7.8) involving a time derivative. The key additional ingredient is the estimate (3.4.2) for F_{0i} with $\gamma' = \gamma - 1$. In the present setting, it may be restated as follows: For each integer $m \geq 0$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} F_0\|_{\mathcal{L}_s^{\ell\gamma+1/2,\infty}\dot{\mathcal{H}}_x^{\gamma-1}(0,1]} + \|\nabla_x^{(k)} F_0\|_{\mathcal{L}_s^{\ell\gamma+1/2,2}\dot{\mathcal{H}}_x^{\gamma}(0,1]} \Big) \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}}} \|\mathring{F}_0\|_{\dot{H}_x^{\gamma-1}}. \tag{3.7.26}$$

Since $U(t=0, s=1) = \mathrm{Id}$ and $\underline{\widetilde{A}}_0 = 0$, we have $\underline{F}_{0i} = \underline{\widetilde{F}}_{0i} = \partial_t \underline{A}_i$. Therefore, (3.7.7) immediately follows.

To proceed to the proof of (3.7.8), we shall begin with the following statement: For every integer $m \geq 0$, we have

$$\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \widetilde{F}_{s0}\|_{\mathcal{L}_s^{\ell\gamma+1,\infty}\dot{\mathcal{H}}_x^{\gamma-1}(0,1]} + \|\nabla_x^{(k)} \widetilde{F}_{s0}\|_{\mathcal{L}_s^{\ell\gamma+1,2}\dot{\mathcal{H}}_x^{\gamma}(0,1]} \Big) \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}}} \|\mathring{F}_0\|_{\dot{H}_x^{\gamma-1}}. \tag{3.7.27}$$

This is an obvious consequence of applying Lemma 3.5.4 to the corresponding estimate for $F_{s0} = \mathbf{D}^\ell F_{\ell 0}$, which in turn follows from (3.7.20), (3.7.26).

Next, we claim that for every integer $m \geq 0$,

$$\begin{aligned} \sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} \widetilde{\mathcal{D}}_t \widetilde{F}_s\|_{\mathcal{L}_s^{\ell\gamma+1,\infty}\dot{\mathcal{H}}_x^{\gamma-1}(0,1]} + \|\nabla_x^{(k)} \widetilde{\mathcal{D}}_t \widetilde{F}_s\|_{\mathcal{L}_s^{\ell\gamma+1,2}\dot{\mathcal{H}}_x^{\gamma-1}(0,1]} \Big) & \\ \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}}} \|\mathring{F}_0\|_{\dot{H}_x^{\gamma-1}}. & \end{aligned} \tag{3.7.28}$$

(Note that $\mathcal{L}_s^2$-type estimate holds for $\dot{\mathcal{H}}_x^{\gamma-1}$.)

Again this follows from Lemma 3.5.4 applied to the corresponding estimate for $\mathbf{D}_0 F_{si}$. To derive the latter, we begin with the identity

$$\mathbf{D}_t F_{si} = \mathbf{D}^\ell \mathbf{D}_\ell F_{0i} - \mathbf{D}_i \mathbf{D}^\ell F_{0\ell} - 2[F_0{}^\ell, F_{i\ell}],$$

which is a consequence of the Bianchi identity

$$\mathbf{D}_0 F_{si} = \mathbf{D}_s F_{0i} - \mathbf{D}_i F_{0s},$$

the covariant parabolic equation for F_{0i}, and the equation $F_{s0} = \mathbf{D}^\ell F_{\ell 0}$. The desired estimate then follows from the estimates (3.7.20), (3.7.21) and (3.7.26).

In view of the formula $\partial_t \widetilde{F}_{si} = \widetilde{\mathbf{D}}_t \widetilde{F}_{si} - [\widetilde{A}_0, \widetilde{F}_{si}]$, we are only left to estimate $[\widetilde{A}_0, \widetilde{F}_{si}]$. We claim that the following statement for $\widetilde{A}_0$ holds: For every integer $m \geq 0$ and any $\gamma < \gamma' < \frac{d}{2}$, we have

$$\sum_{k=0}^{m} \|\nabla_x^{(k)} \widetilde{A}_0\|_{\mathcal{L}_s^{\ell_\gamma,\infty} \dot{\mathcal{H}}_x^{\gamma'}(0,1]} \leq C_{d,\gamma,\gamma',m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}},\|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}} \|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}.$$

Indeed, taking the $\dot{H}_x^{\gamma'}$-norm of the formula

$$\widetilde{A}_0(s) = -\int_s^1 \widetilde{F}_{s0}(s') \, \mathrm{d}s'$$

(which holds thanks to the caloric-temporal gauge condition) and using the estimate (3.7.9), the desired estimate follows. By Lemma 3.1.14, for each integer $m \geq 0$, we then have

$$\sum_{k=0}^{m} \|\nabla_x^{(k)} \widetilde{A}_0\|_{\mathcal{L}_s^{\ell_\gamma,\infty} (\dot{\mathcal{H}}_x^{d/2} \cap \mathcal{L}_x^{\infty})(0,1]} \leq C_{d,\gamma,\gamma',m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}},\|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}} \|\mathring{E}\|_{\dot{H}_x^{\gamma-1}}.$$

Combining the preceding estimate and (3.7.25), we finally arrive at

$$\begin{aligned}\sum_{k=0}^{m} \Big(\|\nabla_x^{(k)} [\widetilde{A}_0, \widetilde{F}_s]\|_{\mathcal{L}_s^{\ell_\gamma+1/2,\infty} \dot{\mathcal{H}}_x^{\gamma-1}(0,1]} + & \|\nabla_x^{(k)} [\widetilde{A}_0, \widetilde{F}_s]\|_{\mathcal{L}_s^{\ell_\gamma+1/2,2} \dot{\mathcal{H}}_x^{\gamma-1}(0,1]} \Big) \\ & \leq C_{d,\gamma,m,\|\mathring{A}\|_{\dot{H}_x^{\gamma}}} \|\mathring{F}_0\|_{\dot{H}_x^{\gamma-1}}.\end{aligned}$$

for every integer $m \geq 0$. Then from (3.7.28), the desired estimate (3.7.8) follows. □

The proof of Theorem 3.7.2 is entirely analogous to Steps 2–3 of the previous proof, and thus will be omitted.

Chapter 4

Proof of the Main LWP Theorem

The goal of this chapter is to establish the Main LWP Theorem for (YM). Accordingly, we shall restrict to the case $d = 3$.

To help the reader quickly grasp the main ideas, we shall begin with an outline of our proof of the Main LWP Theorem in §4.1. Then after some preliminary materials in §4.2, we shall reduce the Main LWP Theorem to Theorems A and B in §4.3.

Theorem A, essentially concerning the gauge transformation procedure from the temporal to caloric-temporal gauge, will follow from more general results already proved in §3.7.

In the remainder of this chapter, we shall prove Theorem B, concerning the time dynamics of the Yang-Mills equations in the caloric-temporal gauge. We shall begin in §4.4 by reducing Theorem B to smaller statements, namely Propositions 4.4.1 - 4.4.4 and Theorems C (Hyperbolic estimates for $\underline{A}_i$) and D (Hyperbolic estimates for F_{si}). In §4.5, we shall analyze the parabolic equations satisfied by F_{si}, $F_{s0} = -w_0$ and w_i; this part will depend heavily on the abstract parabolic theory developed in §3.1.2 – §3.1.4. As a consequence, we shall quickly prove Propositions 4.4.1 - 4.4.4 in §4.6. Finally, in §4.7, we shall prove Theorems C and D by analyzing the wave equations for $\underline{A}_i$ and F_{si}, respectively, in the caloric-temporal gauge.

The materials in this chapter had been previously published in [25, §7 – §10].

4.1 Outline of the argument

Due to the fact that we deal simultaneously with two nonlinear PDEs, namely (YM) and (YMHF), the argument of this chapter is rather lengthy. To help the reader grasp the main ideas, we shall to present an overview of the arguments of this chapter, with the ambition to indicate each of the major difficulties, as well as their resolutions, without being overly technical. We shall respect the numbering of steps in §1.5.

As in §1.5, instead of the full local well-posedness statement, we shall focus on the simpler problem of deriving a local-in-time *a priori* bound of a solution to (YM) in the temporal gauge. In other words, under the assumption that a (suitably smooth and decaying) solution $A^{\dagger}_{\mu}$ to (YM) in the temporal gauge exists on $I \times \mathbb{R}^3$, where $I := (-T_0, T_0) \subset \mathbb{R}$, we aim to prove

$$\| \partial_{t,x} A^{\dagger}_{\mu} \|_{C_t(I, L^2_x)} \leq C_0 \mathring{\mathcal{I}}, \tag{4.1.1}$$

where $\mathring{\mathcal{I}} := \sum_{i=1,2,3} \| (\mathring{A}_i, \mathring{E}_i) \|_{\dot{H}^1_x \times L^2_x}$ measures the size of the initial data, for T_0 sufficiently small compared to $\mathring{\mathcal{I}}$.

Step 0: Scaling and set-up of the bootstrap

Observe that, thanks to the scaling property of (YM) and the sub-criticality of $\mathring{\mathcal{I}}$, it suffices to prove (4.1.1) for $T_0 = 1$, assuming $\mathring{\mathcal{I}}$ is small. We shall use a bootstrap argument to establish (4.1.1). More precisely, under the *bootstrap assumption* that

$$\| \partial_{t,x} A^{\dagger}_{\mu} \|_{C_t((-T,T), L^2_x)} \leq 2 C_0 \mathring{\mathcal{I}} \tag{4.1.2}$$

holds for $0 < T \leq 1$, we shall retrieve (4.1.1) for $I = (-T, T)$ provided that $\mathring{\mathcal{I}}$ is sufficiently small (independent of T). Then, by a standard continuity argument, (4.1.1) will follow for $I = (-1, 1)$.

Steps 1 & 2: Transformation to the caloric-temporal gauge

As discussed earlier, the starting point of our analysis is to smooth out $A^{\dagger}_{\mu}$ by solving the dynamic Yang-Mills heat flow (dYMHF) along $s \in [0, \infty)$ (a newly added variable), and impose the caloric-temporal gauge condition $A_s = 0$ and $\underline{A}_0 = 0$ on the resulting solution to (HPYM)[1]. Taking $\mathring{\mathcal{I}}$ sufficiently small, the bootstrap assumption (4.1.2) allows us to apply Theorem 3.7.1 (Transformation to the caloric-temporal gauge). As a consequence, we shall obtain a gauge transform V on $(-T, T) \times \mathbb{R}^d$ and solution $A_{\mathbf{a}}$ on $(-T, T) \times \mathbb{R}^d \times [0, 1]$ to (HPYM) (as we begin with a solution to (YM)) such that

$$\overline{A}_{\mu} := A_{\mu}(s = 0) = V(A^{\dagger}_{\mu}) V^{-1} - \partial_{\mu} V V^{-1},$$

[1] We remind the reader that (HPYM) is nothing but (dYMHF) with (YM) at $s = 0$.

and $A_{\mathbf{a}}$ is in the caloric-temporal gauge $A_s = 0$ and $\underline{A}_0 := A_0(s=1) = 0$. On $t = 0$, for $F_{si} = \partial_s A_i$ and $\underline{A}_i := A_0(s=1)$, we shall have

$$\|\nabla_x^{(m-1)} \nabla_{t,x} F_s(t=0)\|_{\mathcal{L}_s^{5/4,\infty} \mathcal{L}_x^2} + \|\nabla_x^{(m-1)} \nabla_{t,x} F_s(t=0)\|_{\mathcal{L}_s^{5/4,2} \mathcal{L}_x^2} \leq C_m \mathring{\mathcal{I}}, \tag{4.1.3}$$

$$\|\partial_x^{(k-1)} \partial_{t,x} \underline{A}(t=0)\|_{L_x^2} \leq C_k \mathring{\mathcal{I}}, \tag{4.1.4}$$

up to some integers $m_0, k_0 > 1$, i.e. $1 \leq m \leq m_0$, $1 \leq k \leq k_0$. Moreover, we shall have estimates for the initial gauge transform $\mathring{V} := V(t=0)$ as well. These estimates shall be referred to as *initial data estimates*.

The result described in this step will be made precise in Theorem A (Transformation to the caloric-temporal gauge), stated in §4.3.

Step 3: Analysis of the time evolution

The next step is to propagate the bounds (4.1.3) and (4.1.4) to all $t \in (-T, T)$ by analyzing a system of coupled hyperbolic and parabolic equations derived from (HPYM) in the caloric-temporal gauge.

We shall begin our explanation with a brief overview of the equations of motion for (HPYM). From §2.1, recall the definition of the Yang-Mills tension field $w_\nu(s)$ at $s \in [0,1]$ by

$$w_\nu(s) := \mathbf{D}^\mu F_{\nu\mu}(s).$$

and the *equations of motion* of (HPYM), which are central to the analysis of the t-evolution of $A_{\mathbf{a}}$:

$$\underline{\mathbf{D}}^\mu \underline{F}_{\nu\mu} = \underline{w}_\nu, \tag{4.1.5}$$

$$\mathbf{D}^\mu \mathbf{D}_\mu F_{s\nu} = 2[F_s{}^\mu, F_{\nu\mu}] - 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}] - \mathbf{D}^\ell \mathbf{D}_\ell w_\nu + \mathbf{D}_\nu \mathbf{D}^\ell w_\ell - 2[F_\nu{}^\ell, w_\ell], \tag{4.1.6}$$

$$\mathbf{D}_s F_{\mathbf{ab}} = \mathbf{D}^\ell \mathbf{D}_\ell F_{\mathbf{ab}} - 2[F_{\mathbf{a}}{}^\ell, F_{\mathbf{b}\ell}], \tag{4.1.7}$$

$$\mathbf{D}_s w_\nu = \mathbf{D}^\ell \mathbf{D}_\ell w_\nu + 2[F_\nu{}^\ell, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{\nu\ell} + \mathbf{D}_\ell F_{\nu\mu}]. \tag{4.1.8}$$

Furthermore, $\overline{w}_\nu = 0$.

The equations (4.1.5) and (4.1.6) are the main hyperbolic equations of the system, used to estimate F_{si} and $\underline{A}_i$, respectively. These equations, however, involve additional variables (e.g. F_{s0}, w_μ) which do not satisfy wave equations. Instead, they may be rewritten in terms of F_{si} and $\underline{A}_i$

by inverting the parabolic equations (4.1.7) and (4.1.8). Below, we shall first present the analysis of the parabolic equations (4.1.7) – (4.1.8), and then proceed to the analysis of hyperbolic equations (4.1.5) – (4.1.6).

Analysis of the parabolic equations

The equation (4.1.7) says that each curvature component satisfies a covariant parabolic equation. In view of proving the Main LWP Theorem, of particular interest are the equations

$$\mathbf{D}_s F_{si} - \mathbf{D}^\ell \mathbf{D}_\ell F_{si} = -\, 2[F_s{}^\ell, F_{i\ell}], \tag{4.1.7$'$}$$

$$\mathbf{D}_s F_{s0} - \mathbf{D}^\ell \mathbf{D}_\ell F_{s0} = -\, 2[F_s{}^\ell, F_{0\ell}]. \tag{4.1.7$''$}$$

These equations are estimated by first expanding $\mathbf{D}$, $\mathbf{F}$ in terms of $\mathbf{A}$, and then using $F_{s\mu} = \partial_s A_\mu$, $\underline{A}_0 = 0$ (which hold by the caloric-temporal gauge condition) to reduce all variables to F_{si}, $\underline{A}_i$ and F_{s0}.

Thanks to the smoothing property of (4.1.7$'$), we may (at least heuristically) always exchange derivatives of F_{si} for an appropriate power of s; see §4.5.2 and §4.5.3. These will be useful in the analysis of the wave equation for F_{si}. The second equation (4.1.7$''$) will be used to derive estimates for F_{s0}, which, combined with the caloric-temporal gauge condition, leads to the corresponding estimates for A_0. As $F_{s0} = -w_0$, note that the data for (4.1.7$''$) at $s = 0$ is zero, i.e. $\overline{w}_0 = 0$[2]. This has the implication that A_0 is, in general, is nonlinear in F_{si} and $\underline{A}_i$; see §4.5.4. As a consequence, in the present perturbative setting, it obeys more favorable estimates than A_i.

Next, the Yang-Mills tension field w_i will be estimated using the equation (4.1.8). As in the case of F_{s0}, the data for w_i at $s = 0$ is *zero*, thanks to $\overline{w}_i = 0$. Proceeding as before, w_i will be seen to be nonlinear in F_{si} and $\underline{A}_i$ as well. We refer the reader to §4.5.5.

Analysis of the hyperbolic equations

The key point regarding (4.1.5), which is nothing but the Yang-Mills equations in the temporal gauge with the source $\underline{w}_\mu$, is that its data at $t = 0$ is smooth. Therefore, we shall basically imitate the classical analysis of (YM) in the temporal gauge for smooth initial data, and using the estimate for $\underline{w}_\mu$ (in terms of F_s, $\underline{A}$) proved by the above parabolic analysis. See Theorem C (Hyperbolic estimates for $\underline{A}_i$) in §4.4 for the precise statement and §4.7.1 for more details.

On the other hand, in order to treat (4.1.6), we need to make use of the null structure present at

[2] It is an exercise for the reader to show that (4.1.7$''$) is equivalent to (4.1.8) for $\mu = 0$.

the most dangerous quadratic nonlinearity. It turns out that, for the problem under consideration, all quadratic nonlinearities can be treated just by Strichartz and Sobolev inequalities, except for the single term

$$2[A^{\ell} - \underline{A}^{\ell}, \partial_{\ell} F_{si}].$$

Applying (2.2.2) from §2.2, we see that this possesses a null structure, modulo a term involving $A^{\mathrm{cf}} - \underline{A}^{\mathrm{cf}}$ which is essentially cubic. This will allow us to close the estimates for F_{si}; see Theorem D (Hyperbolic estimates for F_{si}) in §4.4 for the precise statement and §4.7.2 for details.

Provided that $\mathring{\mathcal{I}}$ is sufficiently small, the analysis sketched above will lead (in particular) to the following estimates for $F_{si}(s)$ and $\underline{A}_i$:

$$\|\nabla_x^{(m-1)} F_s\|_{\mathcal{L}_s^{5/4,\infty} \dot{S}^1} + \|\nabla_x^{(m-1)} \nabla_{t,x} F_s\|_{\mathcal{L}_s^{5/4,2} \dot{S}_x^1} \leq C_m \mathring{\mathcal{I}}, \tag{4.1.9}$$

$$\|\partial_x^{(k-1)} \underline{A}\|_{\dot{S}_x^1} \leq C_k \mathring{\mathcal{I}}, \tag{4.1.10}$$

for $1 \leq m \leq m_0$, $1 \leq k \leq k_0$, where all norms are taken over $I \times \mathbb{R}^d \times [0,1]$. The function space $\dot{S}^1$ for the wave equation, defined in §4.2, in particular satisfies $\dot{S}^1 \subset C_t \dot{H}_x^1$.

Step 4: Returning to $A_\mu^\dagger$

The last step is to translate estimates for $\partial_s A_i$ and $\underline{A}_i$, such as (4.1.9), (4.1.10), to those for $A_\mu^\dagger$ so that (4.1.1) is retrieved. One immediate issue is that the naive approach of integrating the estimates (4.1.1) in s fails to bound $\|\partial_{t,x} \overline{A}_\mu\|_{C_t(I, L_x^2)}$ by a logarithm. In order to remedy this issue, let us take the (weakly-parabolic) equation

$$\partial_s A_i = \triangle A_i - \partial^{\ell} \partial_i A_{\ell} + \text{(lower order terms)}.$$

differentiate by $\partial_{t,x}$, multiply by $\partial_{t,x} A_i$ and then integrate the highest order terms by parts over $\mathbb{R}^3 \times [0,1]$. This trick, combined with the $\mathcal{L}_s^2$-type estimates of (4.1.9), overcomes the logarithmic divergence[3]; see Proposition 4.4.2 and its proof in §4.6.

Another issue is that the estimates derived so far, being in the caloric-temporal gauge, are not in the temporal gauge along $s = 0$. Therefore, we are required to control the gauge transform back to the temporal gauge along $s = 0$. For this purpose, we need appropriate estimates for $\overline{A}_0$ in the

[3]It turns out that such a trick is already needed at the stage of deriving estimates such as (4.1.9).

caloric-temporal gauge are needed; see Lemma 4.3.6. These are obtained ultimately as a consequence of the analysis of the hyperbolic equations of (HPYM); see Proposition 4.4.1 and its proof in §4.6.

The precise statement of the end result of Steps 3 and 4 is Theorem B (Time dynamics of (HPYM) in the caloric-temporal gauge), stated in §4.3.

4.2 Preliminaries

In the first subsection, we shall briefly recap the estimates for the linear wave equation which will be needed in this chapter. These estimates will be encapsulated by a function space called $\dot{S}^1$. Then, in §4.2.2, we shall put the function space $\dot{S}^1$ in the framework of abstract parabolic theory, as developed in §3.1.2 – §3.1.4. In the end, a short discussion will be given on the notion of the *associated s-weights*, which is a useful heuristic for figuring out the appropriate weight of s in various instances in this chapter and the next.

4.2.1 Estimates for the linear wave equation and the space $\dot{S}^k$

We summarize the estimates for solutions to an inhomogeneous wave equation that will be used in the following proposition.

Proposition 4.2.1 (Wave estimates)**.** *Let ψ, φ be smooth solutions with a suitable decay towards the spatial infinity (say $\psi, \varphi \in C_t^\infty \mathcal{S}_x$) to the inhomogeneous wave equations*

$$\Box\psi = \mathcal{N}, \qquad \Box\varphi = \mathcal{M},$$

on $(-T,T)\times\mathbb{R}^3$. The following estimates hold.

- **($L_t^\infty L_x^2$ estimate)**

$$\|\partial_{t,x}\psi\|_{L_t^\infty L_x^2((-T,T)\times\mathbb{R}^3)} \leq C\Big(\|(\psi,\partial_0\psi)(t=0)\|_{\dot{H}_x^1\times L_x^2(\mathbb{R}^3)} + \|\mathcal{N}\|_{L_t^1 L_x^2((-T,T)\times\mathbb{R}^3)}\Big) \tag{4.2.1}$$

- **($L_{t,x}^4$-Strichartz estimate)**

$$\begin{aligned}&\|\partial_{t,x}\psi\|_{L_{t,x}^4((-T,T)\times\mathbb{R}^3)}\\ &\quad\leq C\Big(\|(\psi,\partial_0\psi)(t=0)\|_{\dot{H}_x^{3/2}\times\dot{H}_x^{1/2}(\mathbb{R}^3)} + \|\mathcal{N}\|_{L_t^1\dot{H}_x^{1/2}((-T,T)\times\mathbb{R}^3)}\Big).\end{aligned} \tag{4.2.2}$$

- **(Null form estimate)** *For $Q_{ij}(\psi,\phi) := \partial_i\psi\partial_j\phi - \partial_j\psi\partial_i\phi$, we have*

$$\begin{aligned}
&\|Q_{ij}(\psi,\phi)\|_{L^2_{t,x}((-T',T')\times\mathbb{R}^3} \\
&\quad \le C\Big(\|(\psi,\partial_0\psi)(t=0)\|_{\dot H^2_x\times\dot H^1_x(\mathbb{R}^3)} + \|\mathcal{N}\|_{L^1_t\dot H^1_x((-T',T')\times\mathbb{R}^3)}\Big) \\
&\quad\quad \times C\Big(\|(\phi,\partial_0\phi)(t=0)\|_{\dot H^1_x\times L^2_x(\mathbb{R}^3)} + \|\mathcal{M}\|_{L^1_tL^2_x((-T',T')\times\mathbb{R}^3)}\Big).
\end{aligned} \tag{4.2.3}$$

Proof. This is a standard material. For the $L^\infty_t L^2_x$ and the Strichartz estimates, we refer the reader to [31, Chapter III]. For the null form estimate, see the original article [13]. □

Motivated by Proposition 4.2.1, let us define the norms[4] $\dot S^k$ which will be used as a convenient device for controlling the wave-like behavior of certain dynamic variables. Let ψ be a smooth function on $I\times\mathbb{R}^3$ ($I\subset\mathbb{R}$) which decays sufficiently towards the spatial infinity. We start with the norm $\dot S^1$, which we define by

$$\|\psi\|_{\dot S^1(I)} := \|\partial_{t,x}\psi\|_{L^\infty_t L^2_x} + |I|^{1/2}\|\Box\psi\|_{L^2_{t,x}}. \tag{4.2.4}$$

The norms $\dot S^k$ for $k=2,3,4$ are then defined by taking spatial derivatives, i.e.

$$\|\psi\|_{\dot S^k(I)} := \|\partial_x^{(k-1)}\psi\|_{\dot S^1(I)}, \tag{4.2.5}$$

and we furthermore define $\dot S^k$ for $k\ge 1$ a real number by using fractional derivatives. Note the interpolation property

$$\|\psi\|_{\dot S^{k+\theta}(I)} \le C_\theta\|\psi\|^{1-\theta}_{\dot S^k(I)}\|\psi\|^{\theta}_{\dot S^{k+1}(I)}, \quad 0<\theta<1. \tag{4.2.6}$$

The following estimates concerning the $\dot S^k$-norms are an immediate consequence of Proposition 4.2.1 and the fact that $C^\infty_t H^\infty_x$ functions can be approximated by functions in $C^\infty_t\mathcal{S}_x$ with respect to each of the norms involved.

Proposition 4.2.2. *Let $k\ge 1$ be an integer and $\psi,\phi\in C^\infty_t((-T,T),H^\infty_x)$. Then the following estimates hold.*

- **($L^\infty_t L^2_x$ estimate)**

$$\|\partial_x^{(k-1)}\partial_{t,x}\psi\|_{L^\infty_t L^2_x((-T,T)\times\mathbb{R}^3)} \le \|\psi\|_{\dot S^k(-T,T)}. \tag{4.2.7}$$

[4] We remark that $\|\cdot\|_{\dot S^k}$ is a norm after restricted to H^∞_x functions, by Sobolev.

- **($L^4_{t,x}$-Strichartz estimate)**

$$\|\partial_x^{(k-1)}\partial_{t,x}\psi\|_{L^4_{t,x}((-T,T)\times\mathbb{R}^3)} \leq C\|\psi\|_{\dot{S}^{k+1/2}(-T,T)}. \tag{4.2.8}$$

- **(Null form estimate)**

$$\|Q_{ij}(\psi,\phi)\|_{L^2_{t,x}((-T,T)\times\mathbb{R}^3)} \leq C\|\psi\|_{\dot{S}^2(-T,T)}\|\phi\|_{\dot{S}^1(-T,T)}. \tag{4.2.9}$$

On the other hand, in order to control the $\dot{S}^k$ norm of ψ, all one has to do is to estimate the d'Alembertian of ψ along with the initial data. This is the content of the following proposition, which is sometimes referred to as the *energy estimate* in the literature.

Proposition 4.2.3 (Energy estimate)**.** *Let $k \geq 1$ be an integer and $\psi \in C^\infty_t((-T,T), H^\infty_x)$. Then the following estimate holds.*

$$\|\psi\|_{\dot{S}^k(-T,T)} \leq C\Big(\|(\psi,\partial_0\psi)(t=0)\|_{\dot{H}^k_x\times\dot{H}^{k-1}_x(\mathbb{R}^3)} + T^{1/2}\|\Box\psi\|_{L^2_{t,x}((-T,T)\times\mathbb{R}^3)}\Big).$$

Proof. After a standard approximation procedure, this is an immediate consequence of (4.2.1). ☐

4.2.2 Abstract parabolic theory for $\dot{S}^1$

The purpose of this subsection is to put the $\dot{S}^1$-norm in the framework of abstract parabolic theory, as developed in §3.1.2 – §3.1.4.

To begin with, consider the p-normalization of the norm $\dot{S}^1$. Note that the $\dot{S}^1$-norm is homogeneous of degree $2\ell = 1/2$, which is the same as $L^\infty_t\dot{H}^1_x$ (i.e. the energy). For p-normalized version of $\dot{S}^1$, we shall use a set of notations slightly deviating from the rest in order to keep consistency with the intuition that $\|\phi\|_{\dot{S}^1}$ is at the level of $L^\infty_x\dot{H}^1_x$. Indeed, for $m, k \geq 1$ and m an integer, we shall write

$$\|\phi\|_{\dot{\mathcal{S}}^k} := s^{(k-1)/2-1/4}\|\partial_x^{(k-1)}\phi\|_{\dot{S}^1_x}, \quad \|\phi\|_{\widehat{\mathcal{S}}^m} := \sum_{k=1}^m \|\phi\|_{\dot{\mathcal{S}}^k}.$$

Next, we shall prove the following analogue of Proposition 3.1.11 for the $\dot{S}^1$-norm.

Proposition 4.2.4. *Let $d \geq 1$. Then following statements hold.*

1. *Let ψ a function in $C^\infty_{t,s}(I\times J, H^\infty)_x(\mathbb{R}^d))$, where $I, J \subset \mathbb{R}$ are finite intervals. Then for $k \geq 1$, we have*

$$\|\psi\|_{\mathcal{L}^{\ell,p}_s\dot{\mathcal{S}}^k(J)} < \infty$$

if either $1 \leq p \leq \infty$ *and* $\ell - 3/4 + k/2 > 0$*, or* $p = \infty$ *and* $\ell - 3/4 + k/2 = 0$*.*

2. *Furthermore, the norm* $\dot{S}^1_x$ *satisfy the parabolic energy and smoothing estimates* (3.1.8), (3.1.9).

Proof. The proof of the first statement is identical to that for Proposition 3.1.11. For the second one, we begin by observing that

$$\begin{aligned}\|\psi\|_{\mathcal{L}^{\ell,p}_s \dot{\mathcal{S}}^1} &:= \|\nabla_{t,x}\psi\|_{\mathcal{L}^{\ell,p}_s \mathcal{L}^\infty_t \mathcal{L}^2_x} + |I|^{1/2}\|s^{3/4}\Box\psi\|_{\mathcal{L}^{\ell,p}_s \mathcal{L}^2_{t,x}} \\ &\sim \|s^{1/2}\partial_{t,x}\psi(t=0)\|_{\mathcal{L}^{\ell,p}_s \mathcal{L}^2_x} + |I|^{1/2}\|s^{3/4}\Box\psi\|_{\mathcal{L}^{\ell,p}_s \mathcal{L}^2_{t,x}}\end{aligned}$$

for every $\ell \geq 0$ and $1 \leq p \leq \infty$, where $A \sim B$ means that A, B are comparable, i.e. there exist $C > 0$ such that $A \leq CB, B \leq CA$. One direction is trivial, whereas the other follows from the energy estimate. Using furthermore the fact that $\partial_{t,x}, \Box$ commute with $(\partial_s - \triangle)$, this statement follows from Statement 2 of Proposition 3.1.11. □

4.2.3 Associated s-weights for variables of (HPYM)

Let us consider the system (HPYM), introduced in §1.5. Associated to each variable of (HPYM) is a power of s, which represents the expected size of the variable in a dimensionless norm (say $L^\infty_{t,s,x}$); we call this the *associated s-weight* of the variable. The notion of associated s-weights provides a useful heuristic which will make keeping track of these weights easier in the rest of the thesis.

The associated s-weights for the 'spatial variables' $A = A_i, F = F_{ij}, F_s = F_{si}$ are derived directly from scaling considerations, and as such easy to determine. Indeed, as we expect that $\|\partial_x A_i\|_{L^2_x}$ should stay bounded for every t, s, using the scaling heuristics $\partial_x \sim s^{-1/2}$ and $L^2_x \sim s^{3/4}$, it follows that $A_i \sim s^{-1/4}$. The worst term in F_{ij} is at the level of $\partial_x A$, so $F_{ij} \sim s^{-3/4}$, and similarly $F_{si} \sim s^{-5/4}$.

The associated s-weights for w_ν is s^{-1}, which is actually better than that which comes from scaling considerations (which is $s^{-5/4}$). To see why, observe that w_ν satisfies a parabolic equation $(\partial_s - \triangle)w_\nu = {}^{(w_\nu)}\mathcal{N}$ with zero data at $s = 0$.[5] Duhamel's principle then tells us that $w_\nu \sim s^{(w_\nu)}\mathcal{N}$. Looking at the equation (4.1.8), we see that ${}^{(w_\nu)}\mathcal{N} \sim s^{-2}$, from which we conclude $w_\nu \sim 1$. Note that as $w_0 = -F_{s0}$, this shows that the 'temporal variables' A_0, F_{s0} behave better than their 'spatial' counterparts.

We summarize the associated s-weights for important variables as follows.

[5] We remind the reader, that this is a consequence of the original Yang-Mills equations $\mathbf{D}^\mu F_{\nu\mu} = 0$ at $s = 0$.

$$A_i \sim s^{-1/4} \qquad A_0 \sim s^0 \qquad F_{\mu\nu} \sim s^{-3/4}$$
$$F_{si} \sim s^{-5/4} \quad F_{s0} \sim s^{-1} \qquad w_\mu \sim s^{-1}.$$

Accordingly, when we control the sizes of these variables, they will be weighted by the inverse of their respective associated weights.

As we always work on a finite s-interval J such that $J \subset [0, 1]$, extra powers of s compared to the inverse of the associated s-weight should be considered favorable when estimating. For example, it is easier to estimate $\|A_i\|_{\mathcal{L}_s^{1/4+\ell,\infty}\dot{\mathcal{H}}_x^1}$ when $\ell > 0$ than $\ell = 0$. (Compare Lemma 4.5.2 with Proposition 4.4.2.) Informally, when it suffices to control a variable with more power of s, say s^ℓ, compared to the associated s-weight, we shall say that there is an *extra s-weight of s^ℓ*. Thanks to the sub-critical nature of the problem, such extra weights will be abundant, and this will simplify the analysis in many places.

It is also useful to keep in mind the following heuristics.

$$\partial_{t,x}, \mathbf{D}_{t,x} \sim s^{-1/2}, \quad \partial_s, \mathbf{D}_s \sim s^{-1}, \quad L_t^q L_x^r \sim s^{1/(2q)+3/(2r)}$$

4.3 Reduction of the Main LWP Theorem to Theorems A and B

In the first subsection, we shall state and prove some preliminary results that we shall need in this section. These will include a H^2 local well-posedness statement for the Yang-Mills equations in the temporal gauge (Theorem 4.3.4), an approximation lemma for the initial data (Lemma 4.3.5) and a gauge transform lemma (Lemma 4.3.6). Next, we shall state Theorems A (Transformation to caloric-temporal gauge) and B (Analysis of time dynamics in the caloric-temporal gauge), and show that the proof of the Main LWP Theorem is reduced to that of Theorems A and B by a simple bootstrap argument involving a gauge transformation. Theorem A will be an immediate consequence of Theorems 3.7.1 and 3.7.2 proved in the previous chapter. The remainder of this chapter will therefore be devoted to the proof of Theorem B.

4.3.1 Preliminary results

We shall begin this subsection by making a number of important definitions. Let us define the notion of *regular* solutions, which are smooth solutions with appropriate decay towards the spatial infinity.

Definition 4.3.1 (Regular solutions)**.** We say that a representative $A_\mu : I \times \mathbb{R}^d \to \mathfrak{g}$ of a classical solution to (YM) is *regular* if $A_\mu \in C^\infty_t(I, H^\infty_x)$. Furthermore, we say that a smooth solution $A_{\mathbf{a}}$ on $I \times \mathbb{R}^d \times J$ to (HPYM) is *regular* if $A_{\mathbf{a}} \in C^\infty_{t,s}(I \times J, H^\infty_x)$.

In relation to regular solutions, we also define the notion of a *regular gauge transform*, which is basically that which keeps the 'regularity' of the connection 1-form.

Definition 4.3.2 (Regular gauge transform)**.** We say that a gauge transform U on $I \times \mathbb{R}^3 \times J$ is a *regular gauge transform* if $U - \mathrm{Id}, U^{-1} - \mathrm{Id} \in C_{t,s}(I \times J, H^\infty_x)$. The notion of a *regular gauge transform* on $I \times \mathbb{R}^3$ is defined similarly.

We remark that a regular solution (whether to (YM) or (HPYM)) remains regular under a regular gauge transform.

Let us also give the definition of *regular initial data sets* for (YM).

Definition 4.3.3 (Regular initial data sets)**.** We say that an initial data set $(\mathring{A}_i, \mathring{E}_i)$ to (YM) is *regular* if, in addition to satisfying the constraint equation (1.1.1), $\mathring{A}_i, \mathring{E}_i \in H^\infty_x$.

We shall now present some results which are needed to prove the Main LWP Theorem. The first result we present is a local well-posedness result for initial data with higher regularity. For this purpose, we have an H^2 local well-posedness theorem, which is essentially due to Eardley-Moncrief [9]. However, as we do not assume anything on the L^2_x norm of the initial data $\mathring{A}_i$ (in particular, it does not need to belong to L^2_x), we need a minor variant of the theorem proved in [9].

In order to state the theorem, let us define the space $\widehat{H}^2_x$ to be the closure of $\mathcal{S}_x(\mathbb{R}^3)$ with respect to the partially homogeneous Sobolev norm $\|\phi\|_{\widehat{H}^2_x} := \|\partial_x \phi\|_{H^1_x}$. The point, of course, is that this norm[6] does not contain the L^2_x norm.

Theorem 4.3.4 (H^2 local well-posedness of Yang-Mills)**.** *Let $(\mathring{A}_i, \mathring{E}_i)$ be an initial data set satisfying* (1.1.1) *such that $\partial_x \mathring{A}_i, \mathring{E}_i \in H^1_x$.*

1. *There exists $T = T(\|(\mathring{A}, \mathring{E})\|_{\widehat{H}^2_x \times H^1_x}) > 0$, which is non-increasing in $\|(\mathring{A}, \mathring{E})\|_{\widehat{H}^2_x \times H^1_x}$, such that a unique solution A_μ to* (YM) *in the temporal gauge with the prescribed initial data satisfying*

$$A_i \in C_t((-T,T), \widehat{H}^2_x), \quad \partial_t A_i \in C_t((-T,T), H^1_x) \tag{4.3.1}$$

exists on $(-T,T) \times \mathbb{R}^3$.

[6] That $\|\cdot\|_{\widehat{H}^2_x}$ is indeed a norm when restricted to $\widehat{H}^2_x$ follows from Sobolev.

2. *Furthermore,* persistence of higher regularity *holds, in the following sense: If* $\partial_x \mathring{A}, \mathring{E} \in H^m_x$ *(for an integer* $m \geq 1$*), then the solution* A_i *obtained in Statement 1 satisfies* $\partial_{t,x} A_i \in C^{k_1}_t((-T,T), H^{k_2}_x)$ *for non-negative integers* k_1, k_2 *such that* $k_1 + k_2 \leq m$.

   In particular, if $(\mathring{A}_i, \mathring{E}_i)$ *is a regular initial data set, then the corresponding solution* A_μ *is a regular solution to* (YM) *in the temporal gauge.*

3. *Finally, we have the following* continuation criterion: *If* $\sup_{t\in(-T',T')} \|\partial_{t,x} A\|_{H^1_x} < \infty$*, then the solution given by Statement 1 can be extended past* $(-T', T')$*, while retaining the properties stated in Statements 1 and 2.*

Proof. It is not difficult to see that the iteration scheme introduced in Klainerman-Machedon [15, Proposition 3.1] goes through with the above norm, from which Statements $1-3$ follow. A cheaper way of proving Theorem 4.3.4 is to note that $\|\mathring{A}_i\|_{H^2_x(B)} \leq C\|\mathring{A}_i\|_{\widehat{H}^2_x(\mathbb{R}^3)}$, $\|\mathring{E}_i\|_{H^1_x(B)} \leq \|\mathring{E}_i\|_{H^1_x(\mathbb{R}^3)}$ uniformly for all unit balls in $\mathbb{R}^3$. This allows us to apply the localized local well-posedness statement Proposition 3.1 of [15] to each ball, and glue these local solutions to form a global solution via a domain of dependence argument. □

Next, we shall prove a technical lemma, which shows that an arbitrary admissible H^1_x initial data set can be approximated by a sequence of regular initial data sets.

Lemma 4.3.5 (Approximation lemma)**.** *Any admissible* H^1_x *initial data set* $(\mathring{A}_i, \mathring{E}_i) \in (\dot{H}^1_x \cap L^3_x) \times L^2_x$ *can be approximated by a sequence of regular initial data sets* $(\mathring{A}_{(n)i}, \mathring{E}_{(n)i})$ *satisfying the constraint equation* (1.1.1)*. More precisely, the initial data sets* $(\mathring{A}_{(n)i}, \mathring{E}_{(n)i})$ *may be taken to satisfy the following properties.*

1. $\mathring{A}_{(n)}$ *is smooth, compactly supported, and*

2. $\mathring{E}_{(n)} \in H^\infty_x$.

Proof. This proof can essentially be read off from [15, Proposition 1.2]. We reproduce it below for the convenience of the reader.

Choose compactly supported, smooth sequences $\mathring{A}_{(n)i}, \mathring{F}_{(n)i}$ such that $\mathring{A}_{(n)i} \to \mathring{A}_i$ in $\dot{H}^1_x \cap L^3_x$ and $\mathring{F}_{(n)i} \to \mathring{E}_i$ in L^2_x. Let us denote the covariant derivative associated to $\mathring{A}_{(n)}$ by $\mathbf{D}_{(n)}$. Using the fact that $(\mathring{A}_i, \mathring{E}_i)$ satisfies the constraint equation (1.1.1) in the distributional sense and the $\dot{H}^1_x \subset L^6_x$

Sobolev, we see that for any test function φ,

$$
\begin{aligned}
&|\int (\mathbf{D}^{\ell}_{(n)}\, \mathring{F}_{(n)\ell}, \varphi)\, \mathrm{d}x| = |\int (\mathbf{D}^{\ell}_{(n)}\, \mathring{F}_{(n)\ell} - \mathbf{D}^{\ell}\, \mathring{E}_{\ell}, \varphi)\, \mathrm{d}x| \\
&=| \int -(\mathring{F}_{(n)\ell} - \mathring{E}_{\ell}, \partial^{\ell}\varphi) + ([\mathring{A}^{\ell}_{(n)} - \mathring{A}^{\ell}, \mathring{F}_{(n)\ell}] + [\mathring{A}^{\ell}, \mathring{F}_{(n)\ell} - \mathring{E}_{\ell}], \varphi)\, \mathrm{d}x| \\
&\leq \Big(\|\mathring{F}_{(n)} - \mathring{E}\|_{L^2_x} + \|\mathring{A}_{(n)} - \mathring{A}\|_{L^3_x} \|\mathring{F}_{(n)}\|_{L^2_x} + \|\mathring{A}\|_{L^3_x} \|\mathring{F}_{(n)} - \mathring{E}\|_{L^2_x} \Big) \|\varphi\|_{\dot{H}^1_x}.
\end{aligned}
$$

In view of the L^3_x, L^2_x convergence of $\mathring{A}_{(n)}, \mathring{F}_{(n)}$ to $\mathring{A}, \mathring{E}$, respectively, it follows that

$$
\mathbf{D}^{\ell}_{(n)} \mathring{F}_{(n)\ell} \in \dot{H}^{-1}_x \text{ for each } n, \quad \|\mathbf{D}^{\ell}_{(n)}\, \mathring{F}_{(n)\ell}\|_{\dot{H}^{-1}_x} \to 0 \quad \text{ as } n \to \infty,
$$

where $\dot{H}^{-1}_x$ is the dual space of $\dot{H}^1_x$ (defined to be the closure of Schwartz functions on $\mathbb{R}^3$ under the $\dot{H}^1_x$-norm).

Let us now define $\mathring{E}_{(n)i} := \mathring{F}_{(n)i} + \mathbf{D}_{(n)i}\phi_{(n)}$, where the $\mathfrak{g}$-valued function $\phi_{(n)}$ is constructed by solving the elliptic equation

$$
\mathbf{D}^{\ell}_{(n)}\, \mathbf{D}_{(n)\ell}\phi_{(n)} = -\mathbf{D}^{\ell}_{(n)}\, \mathring{F}_{(n)\ell}, \tag{4.3.2}
$$

imposing a suitable decay condition at infinity; we want, in particular, to have $\phi_{(n)} \in \dot{H}^1_x \cap L^6_x$. This ensures that $(\mathring{A}_{(n)i}, \mathring{E}_{(n)i})$ satisfies the constraint equation. Furthermore, in view of the fact that $\mathring{A}_{(n)}, \mathring{F}_{(n)}$ are smooth and compactly supported, it is clear that $\mathbf{D}_{(n)}\phi_{(n)}$ belongs to any H^k_x for $k \geq 0$, and hence so does $\mathring{E}_{(n)}$. Therefore, in order to prove the lemma, it is only left to prove $\mathbf{D}_{(n)}\phi_{(n)} \to 0$ in L^2_x.

Multiplying (4.3.2) by $\phi_{(n)}$ and integrating by parts, we obtain

$$
\int |\mathbf{D}_{(n)}\phi_{(n)}|^2\, \mathrm{d}x \leq \|\mathbf{D}^{\ell}_{(n)} \mathring{F}_{(n)\ell}\|_{\dot{H}^{-1}_x} \|\phi_{(n)}\|_{\dot{H}^1_x}. \tag{4.3.3}
$$

On the other hand, expanding out $\mathbf{D}_{(n)}$, we have

$$
\|\phi_{(n)}\|_{\dot{H}^1_x} \leq \|\mathbf{D}_{(n)}\phi_{(n)}\|_{L^2_x} + \|\mathring{A}_{(n)}\|_{L^3_x} \|\phi_{(n)}\|_{L^6_x}. \tag{4.3.4}
$$

Recall Kato's inequality (for a proof, see Lemma 5.2.1), which shows that $|\partial_i |\phi_{(n)}|| \leq |\mathbf{D}_{(n)i}\phi_{(n)}|$ in the distributional sense. Combining this with the $\dot{H}^1_x \subset L^6_x$ Sobolev inequality for $|\phi_{(n)}|$, we get

$$
\|\phi_{(n)}\|_{L^6_x} \leq C \|\mathbf{D}_{(n)}\phi_{(n)}\|_{L^2_x}. \tag{4.3.5}
$$

Combining (4.3.3) - (4.3.5) and canceling a factor of $\|\mathbf{D}_{(n)}\phi_{(n)}\|_{L^2_x}$, we arrive at

$$\|\mathbf{D}_{(n)}\phi_{(n)}\|_{L^2_x} \leq \|\mathbf{D}^{\ell}_{(n)}(\mathring{F}_{(n)\ell})\|_{\dot{H}^{-1}_x}(1 + C\|\mathring{A}_{(n)}\|_{L^3_x}) \to 0,$$

as desired. □

Given a time interval $I \subset \mathbb{R}$, we claim the existence of norms $\overline{\mathcal{A}}_0(I)$ and $\delta\overline{\mathcal{A}}_0(I)$ for A_0 and δA_0 on I, respectively, for which the following lemma holds. The significance of these norms will be that they can be used to estimate the gauge transform back to the original temporal gauge.

Lemma 4.3.6 (Estimates for gauge transform to temporal gauge)**.** *For a $\mathfrak{g}$-valued function $\overline{A}_0 \in C^\infty_t((-T,T), H^\infty_x)$, consider the following ODE on $(-T,T) \times \mathbb{R}^3$:*

$$\begin{cases} \partial_t V = V\overline{A}_0 \\ V(t=0) = \mathring{V}. \end{cases} \tag{4.3.6}$$

1. *Suppose that $\mathring{V}$ is a $\mathfrak{G}$-valued function such that $\mathring{V} - \mathrm{Id} \in H^\infty_x$. Then there exists a unique solution V to the ODE (4.3.6) such that $V - \mathrm{Id} \in C^\infty_t((-T,T), H^\infty_x)$. Moreover, the solution V obeys the following estimates:*

$$\begin{aligned} &\|V - \mathrm{Id}\|_{L^\infty_t \dot{H}^2_x(-T,T)} + \|V - \mathrm{Id}\|_{L^\infty_t(\dot{H}^{d/2}_x \cap L^\infty_x)(-T,T)} \\ &\quad \leq C_{\overline{\mathcal{A}}_0(-T,T)}\Big(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^2_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + \overline{\mathcal{A}}_0(-T,T)\Big), \end{aligned} \tag{4.3.7}$$

$$\begin{aligned} &\|\partial_t(V - \mathrm{Id})\|_{L^\infty_t \dot{H}^1_x(-T,T)} + \|\partial_t(V - \mathrm{Id})\|_{L^\infty_t \dot{H}^{(d-2)/2}_x(-T,T)} \\ &\quad \leq C_{\overline{\mathcal{A}}_0(-T,T)} \cdot \overline{\mathcal{A}}_0(-T,T)\Big(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + 1\Big). \end{aligned} \tag{4.3.8}$$

2. *Let $A'_0 \in C^\infty_t((-T,T), H^\infty_x)$ be a $\mathfrak{g}$-valued function and $\mathring{V}'$ a $\mathfrak{G}$-valued smooth function such that $\mathring{V}' - \mathrm{Id} \in H^\infty_x$. Let V' be the solution to the ODE (4.3.6) with A_0 and $\mathring{V}$ replaced by A'_0, $\mathring{V}'$, respectively. Without loss of generality, assume that $\overline{\mathcal{A}}'_0(-T,T) \leq \overline{\mathcal{A}}_0(-T,T)$. Then the difference $\delta V := V - V'$ obeys the following estimates:*

$$\begin{aligned} &\|\delta V\|_{L^\infty_t \dot{H}^2_x(-T,T)} + \|\delta V\|_{L^\infty_t(\dot{H}^{d/2}_x \cap L^\infty_x)(-T,T)} \\ &\quad \leq C_{\overline{\mathcal{A}}_0(-T,T)}(\|\delta\mathring{V}\|_{\dot{H}^2_x} + \|\delta\mathring{V}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \\ &\qquad + C_{\overline{\mathcal{A}}_0(-T,T)} \cdot \delta\overline{\mathcal{A}}_0(-T,T)\Big(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^2_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + 1\Big), \end{aligned} \tag{4.3.9}$$

$$\begin{aligned}&\|\partial_t(\delta V)\|_{L^\infty_t \dot{H}^1_x(-T,T)} + \|\partial_t(\delta V)\|_{L^\infty_t \dot{H}^{(d-2)/2}_x(-T,T)} \\ &\quad \leq C_{\overline{\mathcal{A}}_0(-T,T)} \cdot \overline{\mathcal{A}}_0(-T,T)\|\delta\mathring{V}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + C_{\overline{\mathcal{A}}_0(-T,T)} \cdot \delta\overline{\mathcal{A}}_0(-T,T)\Big(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + 1\Big).\end{aligned} \tag{4.3.10}$$

3. *Finally, all of the above statement remain true with* V, δV, $\mathring{V}$, $\delta\mathring{V}$ *replaced by* V^{-1}, δV^{-1}, $\mathring{V}^{-1}$ *and* $\delta\mathring{V}^{-1}$, *respectively.*

The precise definition of $\overline{\mathcal{A}}_0, \delta\overline{\mathcal{A}}_0$ will be given in §4.4.1, whereas we defer the proof of Lemma 4.3.6 until Appendix A.

Next, we shall prove a simple lemma which will be used to estimate the L^3_x norm of our solution.

Lemma 4.3.7. *Let* $\psi = \psi(t,x)$ *be a function defined on* $(-T,T) \times \mathbb{R}^3$ *such that* $\psi(0) \in L^3_x$ *and* $\partial_{t,x}\psi \in C_t L^2_x$. *Then* $\psi \in C_t L^3_x$ *and the following estimate holds.*

$$\sup_{t\in(-T,T)} \|\psi(t)\|_{L^3_x} \leq \|\psi(0)\|_{L^3_x} + CT^{1/2}\|\partial_{t,x}\psi\|_{L^\infty_t L^2_x}. \tag{4.3.11}$$

Proof. By a standard approximation procedure, it suffices to consider $\psi = \psi(t,x)$ defined on $(-T,T) \times \mathbb{R}^3$ which is smooth in time and Schwartz in space. For $t \in (-T,T)$, we estimate via Hölder, Sobolev and the fundamental theorem of calculus as follows:

$$\begin{aligned}\|\psi(t) - \psi(0)\|_{L^3_x} \leq & \|\psi(t) - \psi(0)\|^{1/2}_{L^2_x}\|\psi(t) - \psi(0)\|^{1/2}_{L^6_x} \\ \leq & C(\int_0^t \|\partial_t\psi(t')\|_{L^2_x}\,\mathrm{d}t')^{1/2}(\|\partial_x\psi(t)\|_{L^2_x} + \|\partial_x\psi(0)\|_{L^2_x})^{1/2} \\ \leq & CT^{1/2}\|\partial_t\psi\|^{1/2}_{L^\infty_t L^2_x}\|\partial_x\psi\|^{1/2}_{L^\infty_t L^2_x} \leq CT^{1/2}\|\partial_{t,x}\psi\|_{L^\infty_t L^2_x}.\end{aligned}$$

By the triangle inequality, (4.3.11) follows. □

4.3.2 Reduction of the Main LWP Theorem

Let $A_{\mathbf{a}}, A'_{\mathbf{a}}$ be regular solutions to (HPYM) (defined in §1.5) on $I \times \mathbb{R}^3 \times [0,1]$. For $t \in I$, define the norms $\mathcal{I}(t)$ and $\delta\mathcal{I}(t)$ which measure the sizes of $A_{\mathbf{a}}$ and $\delta A_{\mathbf{a}}$, respectively, at t as follows:

$$\mathcal{I}(t) := \sum_{k=1}^{10}\Big[\|\nabla_{t,x}F_s(t)\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{H}}^{k-1}_x} + \|\nabla_{t,x}F_s(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k-1}_x}\Big] + \sum_{k=1}^{31}\|\partial_{t,x}\underline{A}(t)\|_{\dot{H}^{k-1}_x}.$$

$$\delta\mathcal{I}(t) := \sum_{k=1}^{10}\Big[\|\nabla_{t,x}(\delta F_s)(t)\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{H}}^{k-1}_x} + \|\nabla_{t,x}(\delta F_s)(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k-1}_x}\Big] + \sum_{k=1}^{31}\|\partial_{t,x}(\delta\underline{A})(t)\|_{\dot{H}^{k-1}_x}.$$

For $t = 0$, which will be the most frequently used case in this chapter, we shall often omit writing t. That is, $\mathcal{I} := \mathcal{I}(0)$ and $\delta\mathcal{I} := \delta\mathcal{I}(0)$.

Now the following theorem is an immediate consequence of Theorems 3.7.1 and 3.7.2 proved in Chapter 3.

Theorem A (Transformation to caloric-temporal gauge)**.** *Let $0 < T \leq 1$, and $A^{\dagger}_{\mu}$ a regular solution to the Yang-Mills equation in the temporal gauge $A^{\dagger}_{0} = 0$ on $(-T, T) \times \mathbb{R}^3$ with the initial data $(\mathring{A}_i, \mathring{E}_i)$ at $t = 0$. Define $\mathring{\mathcal{I}} := \|\mathring{A}\|_{\dot{H}^1_x} + \|\mathring{E}\|_{L^2_x}$. Suppose that*

$$\sup_{t \in (-T,T)} \sup_{i} \|A^{\dagger}_{i}(t)\|_{\dot{H}^1_x} < \delta_P, \tag{4.3.12}$$

where δ_P is the small constant in Proposition 3.2.1. Then the following statements hold.

1. *There exists a regular gauge transform $V = V(t, x)$ on $(-T, T) \times \mathbb{R}^3$ and a regular solution $A_{\mathbf{a}}$ to* (HPYM) *on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ such that*

$$\overline{A}_{\mu} = V(A^{\dagger}_{\mu})V^{-1} - \partial_{\mu} V V^{-1}, \tag{4.3.13}$$

   where $\overline{A}_{\mu} := A_{\mu}(s = 0)$.

2. *Furthermore, the solution $A_{\mathbf{a}}$ satisfies the caloric-temporal gauge condition, i.e. $A_s = 0$ everywhere and $\underline{A}_0 = 0$.*

3. *Let $(A')^{\dagger}_{\mu}$ be another regular solution to the Yang-Mills equation in the temporal gauge with the initial data $(\mathring{A}'_i, \mathring{E}'_i)$ satisfying $\|(\mathring{A}, \mathring{E})\|_{\dot{H}^1_x \times L^2_x} \leq \mathring{\mathcal{I}}$ and (4.3.12). Let $A'_{\mathbf{a}}$ be the solution to* (HPYM) *in the caloric-temporal gauge obtained from $(A')^{\dagger}_{\mu}$ as in Statements 1 and 2. Then the following initial data estimates hold:*

$$\mathcal{I} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}, \quad \delta\mathcal{I} \leq C_{\mathring{\mathcal{I}}} \cdot \delta\mathring{\mathcal{I}}, \tag{4.3.14}$$

   where $\delta\mathring{\mathcal{I}} := \|\delta\mathring{A}\|_{\dot{H}^1_x} + \|\delta\mathring{E}\|_{L^2_x}$.

4. *Let V' be the gauge transform obtained from $(A')^{\dagger}_{i}$ as in Statement 1, and let us write $\mathring{V} := V(t = 0), \mathring{V}' := V'(t = 0)$. For the latter two gauge transforms, the following estimates hold:*

$$\|\partial^{(2)}_x(\mathring{V} - \mathrm{Id})\|_{L^2_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}, \tag{4.3.15}$$

$$\|\partial_x^{(2)}(\delta\mathring{V})\|_{L^2_x} + \|(\delta\mathring{V})\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \leq C_{\mathring{\mathcal{I}}} \cdot \delta\mathring{\mathcal{I}}. \tag{4.3.16}$$

The same estimates with $\mathring{V}$ and $\delta\mathring{V}$ replaced by $\mathring{V}^{-1}$ and $\delta\mathring{V}^{-1}$, respectively, also hold.

Our next theorem will concern the time dynamics of (HPYM).

Theorem B (Time dynamics of (HPYM) in the caloric-temporal gauge)**.** *Let $0 < T \leq 1$, and $A_{\mathbf{a}}$ a regular solution to the hyperbolic-parabolic Yang-Mills system* (HPYM) *on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ in the caloric-temporal gauge. Recall the notation $\overline{A}_\mu := A_\mu(s = 0)$. Then there exists $\delta_H > 0$ such that if*

$$\mathcal{I} < \delta_H, \tag{4.3.17}$$

then the following estimate holds.

$$\|\partial_{t,x}\overline{A}_i\|_{C_t((-T,T),L^2_x)} + \overline{\mathcal{A}}_0(-T, T) \leq C\mathcal{I}. \tag{4.3.18}$$

Also, if $A'_{\mathbf{a}}$ is an additional solution to (HPYM) *on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$ in the caloric-temporal gauge which also satisfies* (4.3.17)*, then the following estimate for the difference holds as well:*

$$\|\partial_{t,x}\overline{A}_i - \partial_{t,x}\overline{A}'_i\|_{C_t((-T,T),L^2_x)} + \delta\overline{\mathcal{A}}_0(-T, T) \leq C_{\mathcal{I}} \cdot \delta\mathcal{I}. \tag{4.3.19}$$

The goal of the remainder of this section is to prove the Main LWP Theorem, assuming Theorems A and B.

Proof of the Main LWP Theorem. In view of Lemma 4.3.5 (approximation lemma) and the fact that we are aiming to prove the difference estimates (1.7.2) and (1.7.3), we shall first consider initial data sets $(\mathring{A}_i, \mathring{E}_i)$ which are regular in the sense of Definition 4.3.3. Also, for the purpose of stating the estimates for differences, we shall consider an additional regular initial data set $(\mathring{A}'_i, \mathring{E}'_i)$. The corresponding solution will be also marked by a prime. The statements in this proof concerning a solution A should be understood as being applicable to both A and A'.

Observe that $\mathring{\mathcal{I}}$ does not contain the L^3_x norm of $\mathring{A}$, and has the scaling property.

$$\mathring{\mathcal{I}} \to \lambda^{-1/2}\mathring{\mathcal{I}}$$

under the scaling of the Yang-Mills equations. This allows us to treat the 'local-in-time, large-data' case on an equal footing as the 'unit-time, small-data' case. More precisely, we shall assume by scaling that $\mathring{\mathcal{I}}$ is sufficiently small, and prove that the solution to the Yang-Mills equation exists on the time interval $(-1, 1)$. Unravelling the scaling at the end, the Main LWP Theorem will follow. We remark that the length of the time interval of existence obtained by this method will be of size $\sim \|(\mathring{A}, \mathring{E})\|^{-2}_{\dot{H}^1_x \times L^2_x}$.

Using Theorem 4.3.4, we obtain a unique solution $A^\dagger_\mu$ to the hyperbolic Yang-Mills equation (YM) under the temporal gauge condition $A^\dagger_0 = 0$. We remark that this solution is *regular* by persistence of regularity. Denote by $T_\star$ the largest number $T > 0$ such that the solution $A^\dagger_\mu$ exists smoothly on $(-T, T) \times \mathbb{R}^3$, and furthermore satisfies the following estimates for some $B > 0$ and $C_{\mathring{\mathcal{I}},B} > 0$:

$$\begin{cases} \|\partial_{t,x} A^\dagger_i\|_{C_t((-T,T),L^2_x)} \leq B\mathring{\mathcal{I}}, \\ \|\partial_{t,x} A^\dagger_i - \partial_{t,x}(A')^\dagger_i\|_{C_t((-T,T),L^2_x)} \leq C_{\mathring{\mathcal{I}},B} \cdot \delta\mathring{\mathcal{I}}, \\ \|A^\dagger_i - (A')^\dagger_i\|_{C_t((-T,T),L^3_x)} \leq C_{\mathring{\mathcal{I}},B} \cdot \delta\mathring{\mathcal{I}} + C_{\mathring{\mathcal{I}},B}\|\mathring{A} - \mathring{A}'\|_{L^3_x}. \end{cases} \tag{4.3.20}$$

The goal is to show that $T_\star \geq 1$, provided that $\mathring{\mathcal{I}} > 0$ is small enough.

We shall proceed by a bootstrap argument. In view of the continuity of the norms involved, the inequalities (4.3.20) are satisfied for $T > 0$ sufficiently small if $B \geq 2$ and $C_{\mathring{\mathcal{I}},B} \geq 2$, say. Next, we claim that if we assume

$$\|\partial_{t,x} A^\dagger_i\|_{C_t((-T,T),L^2_x)} \leq 2B\mathring{\mathcal{I}}. \tag{4.3.21}$$

then we can recover (4.3.20) by assuming $\mathring{\mathcal{I}}$ to be small enough and $T \leq 1$.

Assuming the claim holds, let us first complete the proof of the Main LWP Theorem. Indeed, suppose that (4.3.20) holds for some $0 \leq T < 1$. Applying the difference estimate in (4.3.20) to infinitesimal translations of $\mathring{A}, \mathring{E}$ and using the translation invariance of the Yang-Mills equation, we obtain

$$\|\partial_x \partial_{t,x} A^\dagger_i\|_{C_t((-T,T),L^2_x)} < \infty.$$

This, in turn, allows us to apply Theorem 4.3.4 (H^2 local well-posedness) to ensure that the solution $A^\dagger_i$ extends uniquely as a regular solution to a larger time interval $(-T-\epsilon, T+\epsilon)$ for some $\epsilon > 0$. Taking $\epsilon > 0$ smaller if necessary, we can also ensure that the bootstrap assumption (4.3.21) holds and $T + \epsilon \leq 1$. This, along with the claim, allows us to set up a continuity argument to show that a regular solution $A^\dagger_i$ exists uniquely on the time interval $(-1, 1)$ and furthermore satisfies (4.3.20) with $T = 1$. From (4.3.20), the estimates (1.7.1) - (1.7.3) follow immediately for regular initial data sets. Then by Lemma 4.3.5 and the difference estimates (1.7.2) and (1.7.3), these results are extended to admissible initial data sets and solutions, which completes the proof of the Main

LWP Theorem[7].

Let us now prove the claim. Assuming $2B\mathring{\mathcal{I}} < \delta_P$, we can apply Theorem A. This provides us with a regular gauge transform V and a regular solution $A_{\mathbf{a}}$ to (HPYM) satisfying the caloric-temporal gauge condition, along with the following estimates at $t = 0$:

$$\|\partial_x^{(2)}(\mathring{V} - \mathrm{Id})\|_{L^2_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}},$$
$$\mathcal{I} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}, \quad \|\underline{A}_i(t=0)\|_{L^3_x} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}} + \|\mathring{A}\|_{L^3_x}.$$

The same estimate as the first holds with $\mathring{V}$ replaced by $\mathring{V}^{-1}$. We remark that all the constants stated above are independent of $B > 0$. Applying Theorem B with $\mathring{\mathcal{I}}$ small enough (so that $\mathcal{I} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}$ is also small), we have

$$\|\partial_{t,x}\overline{A}_i\|_{C_t((-T,T),L^2_x)} + \overline{\mathcal{A}}_0(-T,T) \leq C\mathcal{I} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}.$$

Note that V is a solution to the ODE (4.3.6), which is unique by the standard ODE theory. Furthermore, in view of the estimates we have for $\overline{\mathcal{A}}_0(-T,T)$ and $\mathring{V}$ in terms of $\mathring{\mathcal{I}}$, we may invoke Lemma 4.3.6 to estimate the gauge transform V and V^{-1} in terms of $\mathring{\mathcal{I}}$. Then using the previous estimate and the gauge transform formula

$$A_i^\dagger = V^{-1}\overline{A}_i V - \partial_i(V^{-1})V,$$

and Lemma A.3.1, we obtain

$$\|\partial_{t,x}A_i^\dagger\|_{C_t((-T,T),L^2_x)} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}.$$

Applying Lemma 4.3.7 and the initial data estimate for the L^3_x norm of $\underline{A}_i$, we also get

$$\|A_i^\dagger\|_{C_t((-T,T),L^3_x)} < C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}} + C_{\mathring{\mathcal{I}}}\|\mathring{A}\|_{L^3_x}.$$

Furthermore, applying a similar procedure to the difference, we arrive at

$$\|\partial_{t,x}A_i^\dagger - \partial_{t,x}(A')_i^\dagger\|_{C_t((-T,T),L^2_x)} \leq C_{\mathring{\mathcal{I}}} \cdot \delta\mathring{\mathcal{I}},$$

$$\|A_i^\dagger - (A')_i^\dagger\|_{C_t((-T,T),L^3_x)} \leq C_{\mathring{\mathcal{I}}} \cdot \delta\mathring{\mathcal{I}} + \|\mathring{A} - \mathring{A}'\|_{L^3_x}.$$

[7] We remark that Statements 1 and 2 of the Main LWP Theorem follows from the persistence of regularity statement in Theorem 4.3.4.

Therefore, taking $B > 0$ sufficiently large (while keeping $2B\mathring{\mathcal{I}} < \delta_P$), we recover (4.3.20). $\square$

The rest of this chapter will be devoted to a proof of Theorem B.

4.4 Definition of norms and reduction of Theorem B

In this section, we shall first introduce the various norms which will be used in the sequel. Then we shall reduce Theorem B to six smaller statements: Propositions 4.4.1, 4.4.2, 4.4.3 and 4.4.4, and Theorems C (Hyperbolic estimates for $\underline{A}_i$) and D (Hyperbolic estimates for F_{si}).

4.4.1 Definition of norms

In this subsection, we shall define the norms $\overline{\mathcal{A}}_0$, $\underline{\mathcal{A}}$, $\mathcal{F}$ and $\mathcal{E}$, along with their difference analogous.

Let $I \subset \mathbb{R}$ be a time interval. The norms $\overline{\mathcal{A}}_0(I)$ and $\delta\overline{\mathcal{A}}_0(I)$, which are used to estimate the gauge transform back to the temporal gauge $A_0 = 0$ at $s = 0$, are defined by

$$\overline{\mathcal{A}}_0(I) := \|\overline{A}_0\|_{L^\infty_t \dot{H}^{1/2}_x} + \|\overline{A}_0\|_{L^\infty_t \dot{H}^1_x} + \|\overline{A}_0\|_{L^1_t(\dot{H}^{3/2}_x \cap L^\infty_x)} + \|\overline{A}_0\|_{L^1_t \dot{H}^2_x}.$$

$$\delta\overline{\mathcal{A}}_0(I) := \|\delta\overline{A}_0\|_{L^\infty_t \dot{H}^{1/2}_x} + \|\delta\overline{A}_0\|_{L^\infty_t \dot{H}^1_x} + \|\delta\overline{A}_0\|_{L^1_t(\dot{H}^{3/2}_x \cap L^\infty_x)} + \|\delta\overline{A}_0\|_{L^1_t \dot{H}^2_x}.$$

where $\overline{A}_0 := A_0(s=0), \delta\overline{A}_0 := A_0(s=0)$.

The norms $\underline{\mathcal{A}}(I)$ and $\delta\underline{\mathcal{A}}(I)$, which control the sizes of $\underline{A}_i$ and $\delta\underline{A}_i$, respectively, are defined by

$$\underline{\mathcal{A}}(I) := \|\underline{A}\|_{L^\infty_t \dot{H}^{31}_x} + \|\partial_0(\partial \times \underline{A})\|_{L^\infty_t \dot{H}^{29}_x} + \sum_{k=1}^{30} \|\underline{A}\|_{\dot{S}^k},$$

$$\delta\underline{\mathcal{A}}(I) := \|\delta\underline{A}\|_{L^\infty_t \dot{H}^{31}_x} + \|\partial_0(\partial \times (\delta\underline{A}))\|_{L^\infty_t \dot{H}^{29}_x} + \sum_{k=1}^{30} \|\delta\underline{A}\|_{\dot{S}^k}.$$

Here, $(\partial_x \times B)_i := \epsilon_{ijk}\partial^j B^k$, where ϵ_{ijk} is the Levi-Civita symbol, i.e. the completely anti-symmetric 3-tensor on $\mathbb{R}^3$ with $\epsilon_{123} = 1$.

Next, let us define the norms $\mathcal{F}(I)$ and $\delta\mathcal{F}(I)$, which control the sizes of F_{si} and δF_{si}, respectively.

$$\mathcal{F}(I) := \sum_{k=1}^{10} \Big(\|F_s\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{S}}^k(0,1]} + \|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^k(0,1]}\Big),$$

$$\delta\mathcal{F}(I) := \sum_{k=1}^{10} \Big(\|\delta F_s\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{S}}^k(0,1]} + \|\delta F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^k(0,1]}\Big).$$

We remark that $\mathcal{F}(I)$ (also $\delta\mathcal{F}(I)$) controls far less derivatives compared to $\underline{\mathcal{A}}(I)$. Nevertheless, it is still possible to close a bootstrap argument on $\mathcal{F} + \underline{\mathcal{A}}$, thanks to the fact that F_{si} satisfies a parabolic equation, which gives smoothing effects. The difference between the numbers of controlled derivatives, in turn, allows us to be lenient about the number of derivatives of $\underline{A}_i$ we use when studying the wave equation for F_{si}. We refer the reader to Remark 4.5.9 for a more detailed discussion.

For $t \in I$, we define $\mathcal{E}(t)$ and $\delta\mathcal{E}(t)$, which control the sizes of low derivatives of $F_{s0}(t)$ and $\delta F_{s0}(t)$, respectively, by

$$\mathcal{E}(t) := \sum_{m=1}^{3} \Big(\|F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \Big),$$

$$\delta\mathcal{E}(t) := \sum_{m=1}^{3} \Big(\|\delta F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \Big).$$

We furthermore define $\mathcal{E}(I) := \sup_{t\in I} \mathcal{E}(t)$ and $\delta\mathcal{E}(I) := \sup_{t\in I} \delta\mathcal{E}(t)$.

4.4.2 Statement of Propositions 4.4.1 - 4.4.4 and Theorems C, D

For the economy of notation, we shall omit the dependence of the quantities and norms on the time interval $(-T, T)$; in other words, all quantities and space-time norms below should be understood as being defined over the time interval $(-T, T)$ with $0 < T \leq 1$.

Proposition 4.4.1 (Improved estimates for A_0)**.** *Let $A_{\mathbf{a}}$, $A'_{\mathbf{a}}$ be regular solutions to* (HPYM) *in the caloric-temporal gauge and $0 < T \leq 1$. Then the following estimates hold.*

$$\overline{\mathcal{A}}_0 \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.4.1}$$

$$\delta\overline{\mathcal{A}}_0 \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \delta\mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.4.2}$$

Proposition 4.4.2 (Improved estimates for A_i)**.** *Let $A_{\mathbf{a}}$, $A'_{\mathbf{a}}$ be regular solutions to* (HPYM) *in the caloric-temporal gauge and $0 < T \leq 1$. Then the following estimates hold.*

$$\sup_i \sup_{0\leq s\leq 1} \|A_i(s)\|_{\dot{S}^1} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}}),$$

$$\sup_i \sup_{0\leq s\leq 1} \|\delta A_i(s)\|_{\dot{S}^1} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\delta\mathcal{F} + \delta\underline{\mathcal{A}}).$$

Proposition 4.4.3 (Estimates for $\mathcal{E}$)**.** *Let $A_{\mathbf{a}}, A'_{\mathbf{a}}$ be regular solutions to* (HPYM) *in the caloric-*

temporal gauge and $0 < T \leq 1$*. Suppose furthermore that the smallness assumption*

$$\mathcal{F} + \underline{\mathcal{A}} \leq \delta_E,$$

holds for sufficiently small $\delta_E > 0$*. Then the following estimates hold.*

$$\mathcal{E} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.4.3}$$

$$\delta\underline{\mathcal{E}} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.4.4}$$

Proposition 4.4.4 (Continuity properties of $\mathcal{F}, \underline{\mathcal{A}}$)**.** *Let* $A_{\mathbf{a}}$*,* $A'_{\mathbf{a}}$ *be regular solutions to* (HPYM) *in the caloric-temporal gauge on some interval* $I_0 := (-T_0, T_0)$*. For* $\mathcal{F} = \mathcal{F}(I), \underline{\mathcal{A}} = \underline{\mathcal{A}}(I)$ $(I \subset I_0)$ *and their difference analogues, the following continuity properties hold.*

- *The norms* $\mathcal{F}(-T,T)$ *and* $\underline{\mathcal{A}}(-T,T)$ *are continuous as a function of* T *(where* $0 < T < T_0$*).*
- *Similarly, the norms* $\delta\mathcal{F}(-T,T)$ *and* $\delta\underline{\mathcal{A}}(-T,T)$ *are continuous as a function of* T*.*
- *We furthermore have*

$$\limsup_{T\to 0+} \Big(\mathcal{F}(-T,T) + \underline{\mathcal{A}}(-T,T)\Big) \leq C\,\mathcal{I},$$

$$\limsup_{T\to 0+} \Big(\delta\mathcal{F}(-T,T) + \delta\underline{\mathcal{A}}(-T,T)\Big) \leq C\,\delta\mathcal{I}.$$

Theorem C (Hyperbolic estimates for $\underline{A}_i$)**.** *Let* $A_{\mathbf{a}}$*,* $A'_{\mathbf{a}}$ *be regular solutions to* (HPYM) *in the caloric-temporal gauge and* $0 < T \leq 1$*. Then the following estimates hold.*

$$\underline{\mathcal{A}} \leq C\mathcal{I} + T\Big(C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2\Big), \tag{4.4.5}$$

$$\delta\underline{\mathcal{A}} \leq C\delta\mathcal{I} + T\Big(C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \delta\mathcal{E} + C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}})\Big). \tag{4.4.6}$$

Theorem D (Hyperbolic estimates for F_{si})**.** *Let* $A_{\mathbf{a}}$*,* $A'_{\mathbf{a}}$ *be regular solutions to* (HPYM) *in the caloric-temporal gauge and* $0 < T \leq 1$*. Then the following estimates hold.*

$$\mathcal{F} \leq C\mathcal{I} + T^{1/2} C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.4.7}$$

$$\delta\mathcal{F} \leq C\delta\mathcal{I} + T^{1/2} C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.4.8}$$

A few remarks are in order concerning the above statements.

The significance of Propositions 4.4.1 and 4.4.2 is that they allow us to pass from the quantities $\mathcal{F}$ and $\underline{\mathcal{A}}$ to the norms of A_i and A_0 on the left-hand side of (4.3.18). Unfortunately, a naive approach to any of these will fail, leading to a logarithmic divergence. The structure of (HPYM), therefore, has to be used in a crucial way in order to overcome this.

Proposition 4.4.3, which will be proved in §4.5.4, deserves some special remarks. This is a perturbative result for the parabolic equation for F_{s0}, meaning that we need some smallness to estimate the nonlinearity. However, the latter fact has the implication that the required smallness cannot come from the size of the time interval, but rather only from the size of the data $(\mathcal{F} + \underline{\mathcal{A}})$ or the size of the s-interval. It turns out that this feature causes a little complication in the proof of global well-posedness. Therefore, in Chapter 5, we shall prove a modified version of Proposition 4.4.3, using more covariant techniques to analyze the (covariant) parabolic equation for F_{0i}, which allows one to get around this issue.

In this work, to opt for simplicity, we have chosen to fix the s-interval to be $[0, 1]$ and make $\mathcal{I}$ (therefore $\mathcal{F} + \underline{\mathcal{A}}$) small by scaling, exploiting the fact that $\mathcal{I}$ is *sub-critical* with respect to the scaling of the equation. We remark, however, that it would have been just as fine to keep $\mathcal{I}$ large and obtain smallness by shrinking the size of the s-interval.

The proof of Theorem B will be via a bootstrap argument for $\mathcal{F} + \underline{\mathcal{A}}$, and Proposition 4.4.4 provides the necessary continuity properties. In fact, Proposition 4.4.4 is a triviality in view of the simplicity of our function spaces and the fact that $A_{\mathbf{a}}, A'_{\mathbf{a}}$ are regular solutions. On the other hand, Theorems C and D, obtained by analyzing the hyperbolic equations for $\underline{A}_i$ and F_{si}, respectively, give the main driving force of the bootstrap argument. Observe that these estimates themselves do not require any smallness. This will prove to be quite useful in the proof of global well-posedness in Chapter 5.

As we need to use some results derived from the parabolic equations of (HPYM), we shall defer the proofs of Propositions 4.4.1 - 4.4.4, along with further discussion, until §4.6. The proofs of Theorems C and D will be the subject of §4.7.

4.4.3 Proof of Theorem B

Assuming the above statements, we are ready to prove Theorem B.

Proof of Theorem B. Let $A_{\mathbf{a}}$, $A'_{\mathbf{a}}$ be regular solutions to (HPYM) in the caloric-temporal gauge, defined on $(-T, T) \times \mathbb{R}^3 \times [0, 1]$. As usual, $\mathcal{I}$ will control the sizes of both $A_{\mathbf{a}}$ and $A'_{\mathbf{a}}$ at $t = 0$, in the manner described in Theorem A.

Let us prove (4.3.18). We claim that

$$\mathcal{F}(-T,T) + \underline{\mathcal{A}}(-T,T) \leq B\mathcal{I} \tag{4.4.9}$$

for a large constant B to be determined later, and $\mathcal{I} < \delta_H$ with $\delta_H > 0$ sufficiently small. By taking B large enough, we obviously have $\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq B\mathcal{I}$ for $T' > 0$ sufficiently small by Proposition 4.4.4. This provides the starting point of the bootstrap argument.

Next, for $0 < T' \leq T$, let us assume the following *bootstrap assumption*:

$$\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq 2B\mathcal{I}. \tag{4.4.10}$$

The goal is to improve this to $\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq B\mathcal{I}$.

Taking $2B\mathcal{I}$ to be sufficiently small, we can apply Proposition 4.4.3 and estimate $\mathcal{E} \leq C_{\mathcal{F},\underline{\mathcal{A}}}(\mathcal{F} + \underline{\mathcal{A}})^2$. (We remark that in order to close the bootstrap, it is important that $\mathcal{E}$ is at least quadratic in $(\mathcal{F} + \underline{\mathcal{A}})$.) Combining this with Theorems C and D, and removing the powers of T' by using the fact that $T' \leq T \leq 1$, we obtain

$$\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq C\mathcal{I} + C_{\mathcal{F}(-T',T'),\underline{\mathcal{A}}(-T',T')}(\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T'))^2.$$

Using the bootstrap assumption (4.4.10) and taking $2B\mathcal{I}$ to be sufficiently small, we can absorb the last term into the left-hand side and obtain

$$\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq C\mathcal{I}.$$

Therefore, taking B sufficiently large, we beat the bootstrap assumption, i.e. $\mathcal{F}(-T',T') + \underline{\mathcal{A}}(-T',T') \leq B\mathcal{I}$. Using this, a standard continuity argument gives (4.4.9) as desired.

From (4.4.9), estimate (4.3.18) follows immediately by Propositions 4.4.1, 4.4.2 and 4.4.3.

Next, let us turn to (4.3.19). By essentially repeating the above proof for $\delta\mathcal{F} + \delta\underline{\mathcal{A}}$, and using the estimate (4.4.10) as well, we obtain the following difference analogue of (4.4.9):

$$\delta\mathcal{F}(-T,T) + \delta\underline{\mathcal{A}}(-T,T) \leq C_{\mathcal{I}} \cdot \delta\mathcal{I}. \tag{4.4.11}$$

From (4.4.9) and (4.4.11), estimate (4.3.19) follows by Propositions 4.4.1, 4.4.2 and 4.4.3. $\square$

4.5 Parabolic equations of (HPYM)

In this section, we shall analyze the parabolic equations of (HPYM) for the variables F_{si}, F_{s0} and w_i. The results of this analysis will provide one of the 'analytic pillars' of the proof of Theorem B that had been outlined in §4.4, the other 'pillar' being the hyperbolic estimates in §4.7. Moreover, the hyperbolic estimates in §4.7 will depend heavily on the results of this section as well.

As this section is a bit long, let us start with a brief outline. Beginning in §4.5.1 with some preliminaries, we shall prove in §4.5.2 smoothing estimates for F_{si} (Proposition 4.5.8), which will allow us to control higher derivatives of $\partial_{t,x} F_{si}$ in terms of $\mathcal{F}$, provided that we control high enough derivatives of $\partial_{t,x} \underline{A}_i$. In §4.5.3, we shall also prove that F_{si} itself (i.e. without any derivative) can be controlled in $L^\infty_t L^2_x$ and $L^4_{t,x}$ by $\mathcal{F} + \underline{\mathcal{A}}$ as well (Proposition 4.5.11). Next, in §4.5.4, we shall study the parabolic equation for F_{s0}. Two main results of this subsection are Propositions 4.5.13 and 4.5.14. The former states that low derivatives of F_{s0} (i.e. $\mathcal{E}$) can be controlled *under the assumption that* $\mathcal{F} + \underline{\mathcal{A}}$ *is small*, whereas the latter says that once $\mathcal{E}$ is under control, higher derivatives of F_{s0} can be controlled (with out any smallness assumption) as long as high enough derivatives of $\underline{A}_i$ are under control. Finally, in §4.5.5, we shall derive parabolic estimates for w_i (Proposition 4.5.17). Although these are similar to those proved for F_{s0}, it will be important (especially in view of the proof of finite energy global well-posedness in Chapter 5) to note that no smallness of $\mathcal{F} + \underline{\mathcal{A}}$ is required in this part.

Throughout the section, we shall always work with regular solutions $A_{\mathbf{a}}, A'_{\mathbf{a}}$ to (HPYM) on $I \times \mathbb{R}^3 \times [0, 1]$, where $I = (-T, T)$.

4.5.1 Preliminary estimates

Let us begin with a simple integral inequality.

Lemma 4.5.1. *For* $\delta > 0$ *and* $1 \leq q \leq p \leq \infty$*, the following estimate holds.*

$$\| \int_s^1 (s/s')^\delta f(s') \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^p_s(0,1]} \leq C_{\delta,p,q} \|f\|_{\mathcal{L}^q_s(0,1]}.$$

Proof. This is rather a standard fact about integral operators. By interpolation, it suffices to consider the three cases $(p, q) = (1, 1), (\infty, 1)$ and (∞, ∞). The first case follows by Fubini, using the fact that $\sup_{0<s'\leq 1} \int_0^1 1_{[0,\infty)}(s'-s)(s/s')^\delta \, \mathrm{d}s/s \leq C_\delta$, as $\delta > 0$. On the other hand, the second and the third cases (i.e. $p = \infty$ and $q = 1, \infty$) follow by Hölder, using furthermore the fact that $\sup_{0<s,s'\leq 1} 1_{[0,\infty)}(s'-s)(s/s')^\delta \leq 1$ and $\sup_{0<s\leq 1} \int_0^1 1_{[0,\infty)}(s'-s)(s/s')^\delta \, \mathrm{d}s'/s' \leq C_\delta$, respectively.

□

By the caloric-temporal gauge condition, we have $\partial_s A_\mu = F_{s\mu}$. Therefore, we can control A_μ with estimates for $F_{s\mu}$ and $\underline{A}_\mu$. The following two lemmas make this idea precise.

Lemma 4.5.2. *Let X be a homogeneous norm of degree $2\ell_0$. Suppose furthermore that the caloric gauge condition $A_s = 0$ holds. Then for $k, \ell \geq 0$ and $1 \leq q \leq p \leq \infty$ such that $1/4 + k/2 + \ell - \ell_0 > 0$, the following estimate holds.*

$$\|A_i\|_{\mathcal{L}_s^{1/4+\ell,p} \dot{\mathcal{X}}^k(0,1]} \leq C(\|F_{si}\|_{\mathcal{L}_s^{5/4,q} \dot{\mathcal{X}}^k(0,1]} + \|\underline{A}_i\|_{\dot{X}^k}).$$

where C depends on p, q and $r(\ell, k, \ell_0) := 1/4 + k/2 + \ell - \ell_0$.

Proof. By the caloric gauge condition $A_s = 0$, it follows that $\partial_s A_i = F_{si}$. By the fundamental theorem of calculus, we have

$$A_i(s) = -\int_s^1 s' F_{si}(s') \frac{\mathrm{d}s'}{s'} + \underline{A}_i.$$

Let us take the $\mathcal{L}_s^{1/4+\ell,p} \dot{\mathcal{X}}^k(0,1]$-norm of both sides. Defining $r(\ell, k, \ell_0) = 1/4 + k/2 + \ell - \ell_0$, we easily compute

$$\begin{aligned} \| \int_s^1 s' F_{si}(s') \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}_s^{1/4+\ell,p} \dot{\mathcal{X}}^k(0,1]} =& \| \int_s^1 (s/s')^{r(\ell,k,\ell_0)} (s')^\ell (s')^{5/4} \|F_{si}(s')\|_{\dot{\mathcal{X}}^k(s')} \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}_s^p(0,1]} \\ \leq& \| \int_s^1 (s/s')^{r(\ell,k,\ell_0)} (s')^{5/4} \|F_{si}(s')\|_{\dot{\mathcal{X}}^k(s')} \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}_s^p(0,1]}. \end{aligned}$$

where on the second line we used $\ell \geq 0$. Since $r > 0$, we can use Lemma 4.5.1 to estimate the last line by $C_{p,q,r} \|F_{si}\|_{\mathcal{L}_s^{5/4,q} \dot{\mathcal{X}}^k(0,1]}$.

On the other hand, $\underline{A}_i$ is independent of s, and therefore

$$\|\underline{A}_i\|_{\mathcal{L}_s^{1/4+\ell,p} \dot{\mathcal{X}}^k(0,1]} = \|s^{r(\ell,k,\ell_0)}\|_{\mathcal{L}_s^p(0,1]} \|\underline{A}_i\|_{\dot{X}^k} \leq C_{p,q,r} \|\underline{A}_i\|_{\dot{X}^k},$$

where the last inequality holds as $r > 0$. □

The following analogous lemma for A_0, whose proof we omit, can be proved by a similar argument.

Lemma 4.5.3. *Let X be a homogeneous norm of degree $2\ell_0$. Suppose furthermore that the caloric-temporal gauge condition $A_s = 0$, $\underline{A}_0 = 0$ holds. Then for $k, \ell \geq 0$ and $1 \leq q \leq p \leq \infty$ such that*

$k/2 + \ell - \ell_0 > 0$*, the following estimate holds.*

$$\|A_0\|_{\mathcal{L}^{\ell,p}_s \dot{\mathcal{X}}^k(0,1]} \leq C \|F_{s0}\|_{\mathcal{L}^{1,q}_s \dot{\mathcal{X}}^k(0,1]},$$

where C *depends on* p, q *and* $r'(\ell, k, \ell_0) := k/2 + \ell - \ell_0$.

Some of the most frequently used choices of X are $X = \dot{S}^k$ for Lemma 4.5.2, $X = L^2_t \dot{H}^k_x$ for Lemma 4.5.3, and $X = \dot{H}^k_x, \dot{W}^{k,\infty}_x$ for both. Moreover, these lemmas will frequently applied to norms which can be written as a sum of such norms, e.g. $\mathcal{L}^{\ell,p}_s \mathcal{H}^m_x$, which is the sum of $\mathcal{L}^{\ell,p}_s \dot{\mathcal{H}}^k_x$ norms for $k = 0, \dots, m$.

As an application of the previous lemmas, we end this subsection with estimates for some components of the curvature 2-form and its covariant derivative.

Lemma 4.5.4 (Bounds for F_{0i})**.** *Suppose that the caloric-temporal gauge condition* $A_s = 0$, $\underline{A}_0 = 0$ *holds. Then:*

1. *The following estimate holds for* $2 \leq p \leq \infty$*:*

$$\begin{aligned} \|F_{0i}(t)\|_{\mathcal{L}^{3/4,p}_s \dot{\mathcal{H}}^{1/2}_x} \leq & C_p \big(\|\nabla_0 F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{1/2}_x} + \|\partial_0 \underline{A}_i(t)\|_{\dot{H}^{1/2}_x} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{3/2}_x} \\ & + \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} (\|\nabla_x F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x} + \|\partial_x \underline{A}_i(t)\|_{L^2_x}) \big). \end{aligned} \tag{4.5.1}$$

2. *For any* $2 \leq p \leq \infty$ *and* $k \geq 1$ *an integer, we have*

$$\begin{aligned} \|F_{0i}(t)\|_{\mathcal{L}^{3/4,p}_s \dot{\mathcal{H}}^k_x} \leq & C_{p,k} \big(\|\nabla_0 F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^k_x} + \|\partial_0 \underline{A}_i(t)\|_{\dot{H}^k_x} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+1}_x} \\ & + \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x} (\|\nabla_x F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^k_x} + \|\partial_x \underline{A}_i(t)\|_{H^k_x}) \big). \end{aligned} \tag{4.5.2}$$

3. *For any* $2 \leq p \leq \infty$ *and* $k \geq 0$ *an integer, we have*

$$\begin{aligned} \|F_{0i}\|_{\mathcal{L}^{3/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \leq & C_{p,k} \big(\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+3/2}} + \|\underline{A}_i\|_{\dot{S}^{k+3/2}} + T^{1/4} \sup_{t \in I} \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+7/4}_x} \\ & + T^{1/4} \sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x} (\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}} + \|\underline{A}_i\|_{\widehat{S}^{k+1}}) \big). \end{aligned} \tag{4.5.3}$$

Proof. Let us begin with the identity

$$F_{0i} = \partial_0 A_i - \partial_i A_0 + [A_0, A_i] = s^{-1/2} \nabla_0 A_i + s^{-1/2} \nabla_i A_0 + [A_0, A_i],$$

Applying Lemma 4.5.2 to $s^{-1/2} \nabla_0 A_i$ and Lemma 4.5.3 to $s^{-1/2} \nabla_x A_0$, the estimates (4.5.1) and

(4.5.2) are reduced to the product estimates

$$\|[A_0, A_i](t)\|_{\mathcal{L}_s^{3/4,p}\dot{\mathcal{H}}_x^{1/2}} \leq C_p \|\nabla_x F_{s0}(t)\|_{\mathcal{L}_s^{1,2}\mathcal{L}_x^2}(\|\nabla_x F_{si}(t)\|_{\mathcal{L}_s^{5/4,2}\mathcal{L}_x^2} + \|\partial_x \underline{A}_i(t)\|_{L_x^2}), \tag{4.5.4}$$

$$\|[A_0, A_i](t)\|_{\mathcal{L}_s^{3/4,p}\dot{\mathcal{H}}_x^{k}} \leq C_p \|\nabla_x F_{s0}(t)\|_{\mathcal{L}_s^{1,2}\mathcal{H}_x^k}(\|\nabla_x F_{si}(t)\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k} + \|\partial_x \underline{A}_i(t)\|_{H_x^k}), \tag{4.5.5}$$

respectively.

Let us start with the product estimate

$$\|\phi_1\phi_2\|_{\dot{H}_x^{1/2}} \leq C\|\phi_1\|_{\dot{H}_x^1}\|\phi_2\|_{\dot{H}_x^1}, \tag{4.5.6}$$

which follows from the product rule for homogeneous Sobolev norms (Lemma 3.1.3). Applying the Correspondence Principle and Lemma 3.1.8, we obtain

$$\|[A_0, A_i](t)\|_{\mathcal{L}_s^{3/4,p}\dot{\mathcal{H}}_x^{1/2}} \leq C_p \|A_0(t)\|_{\mathcal{L}_s^{0+3/8,\infty}\dot{\mathcal{H}}_x^1}\|A_i(t)\|_{\mathcal{L}_s^{1/4+1/8,p}\dot{\mathcal{H}}_x^1}$$

Note the extra weights of $s^{3/8}$ and $s^{1/8}$ for A_0 and A_i, respectively. Applying Lemma 4.5.3 to A_0 and Lemma 4.5.2 to A_i, the desired estimate (4.5.4) follows.

The other product estimate (4.5.5) can be proved by a similar argument, this time starting with $\|\phi_1\phi_2\|_{\dot{H}_x^1} \leq C\|\phi_1\|_{\dot{H}_x^{5/4}}\|\phi_2\|_{\dot{H}_x^{5/4}}$, (which follows again from Lemma 3.1.3) instead of (4.5.6), and using Leibniz's rule to deal with the cases $k \geq 2$.

Finally, let us turn to (4.5.3). We use Lemma 4.5.2 and Strichartz to control $s^{-1/2}\nabla_0 A_i$, and Lemma 4.5.3, Hölder in time and Sobolev for $s^{-1/2}\nabla_x A_0$. Then we are left to establish

$$\|[A_0, A_i]\|_{\mathcal{L}_s^{3/4,p}\mathcal{L}_t^4\dot{\mathcal{W}}_x^{k,4}} \leq C_p T^{1/4} \sup_{t\in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}_s^{1,2}\mathcal{H}_x^{k+1}}(\|F_{si}\|_{\mathcal{L}_s^{5/4,2}\widehat{\mathcal{S}}^{k+1}} + \|\underline{A}_i\|_{\widehat{\mathcal{S}}^{k+1}}). \tag{4.5.7}$$

To prove (4.5.7), one starts with $\|\phi_1\phi_2\|_{L_t^4 L_x^4} \leq C|I|^{1/4}\|\phi_1\|_{L_t^\infty \dot{H}_x^{5/4}}\|\phi_2\|_{L_t^\infty \dot{H}_x^1}$, (which follows via Hölder and Sobolev) instead of (4.5.6). Using Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain for $k \geq 0$

$$\|[A_0, A_i]\|_{\mathcal{L}_s^{3/4,p}\mathcal{L}_t^4\dot{\mathcal{W}}_x^{k,4}} \leq CT^{1/4}\sum_{j=0}^{k} \|A_0\|_{\mathcal{L}_s^{0+5/16,\infty}\mathcal{L}_t^\infty\dot{\mathcal{H}}_x^{j+5/4}}\|A_i\|_{\mathcal{L}_s^{1/4+1/16,p}\mathcal{L}_t^\infty\dot{\mathcal{H}}_x^{k+1-j}}$$

Now we are in position to apply Lemmas 4.5.2 and 4.5.3 to A_i and A_0, respectively. Using furthermore $\|\nabla_x F_{si}\|_{\mathcal{L}_s^{5/4,2}\mathcal{L}_t^\infty\mathcal{H}_x^k} \leq \|F_{si}\|_{\mathcal{L}_s^{5/4,2}\widehat{\mathcal{S}}^{k+1}}$, $\|\partial_x\underline{A}_i\|_{L_t^\infty H_x^k} \leq \|\underline{A}_i\|_{\widehat{\mathcal{S}}^{k+1}}$, (4.5.7) follows. □

By the same proof applied to δF_{0i}, we obtain the following difference analogue of Lemma 4.5.4.

Lemma 4.5.5 (Bounds for δF_{0i})**.** *Suppose that the caloric-temporal gauge condition $A_s = 0$, $\underline{A}_0 = 0$ holds (for both A and A'). Then:*

1. *The following estimate holds for $2 \leq p \leq \infty$:*

$$\begin{aligned}
\|\delta F_{0i}(t)\|_{\mathcal{L}^{3/4,p}_s \dot{\mathcal{H}}^{1/2}_x} \leq & C_p(\|\nabla_0(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{1/2}_x} + \|\partial_0(\delta \underline{A}_i)(t)\|_{\dot{H}^{1/2}_x} + \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{3/2}_x}) \\
& + C_{\|\nabla_x F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x}, \|\partial_x \underline{A}_i(t)\|_{L^2_x}} \cdot \|\nabla_x(\delta F_{s0})(t)\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} \\
& + C_{\|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x}} \cdot (\|\nabla_x(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x} + \|\partial_x(\delta \underline{A}_i)(t)\|_{L^2_x})
\end{aligned} \tag{4.5.8}$$

2. *For any $2 \leq p \leq \infty$ and $k \geq 1$, we have*

$$\begin{aligned}
\|\delta F_{0i}(t)\|_{\mathcal{L}^{3/4,p}_s \dot{\mathcal{H}}^{k}_x} \leq & C_{p,k}(\|\nabla_0(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k}_x} + \|\partial_0(\delta \underline{A}_i)(t)\|_{\dot{H}^{k}_x}) + \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+1}_x} \\
& + C_{\|\nabla_x F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^k_x}, \|\partial_x \underline{A}_i(t)\|_{H^k_x}} \cdot \|\nabla_x(\delta F_{s0})(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x} \\
& + C_{\|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x}} \cdot (\|\nabla_x(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^k_x} + \|\partial_x(\delta \underline{A}_i)(t)\|_{H^k_x}).
\end{aligned} \tag{4.5.9}$$

3. *For any $2 \leq p \leq \infty$ and $k \geq 0$, we have*

$$\begin{aligned}
\|\delta F_{0i}\|_{\mathcal{L}^{3/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \leq & C_{p,k}(\|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+3/2}} + \|\delta \underline{A}_i\|_{\dot{S}^{k+3/2}} + T^{1/4} \sup_{t \in I} \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+7/4}_x}) \\
& + T^{1/4} C_{\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}}, \|\underline{A}_i\|_{\widehat{S}^{k+1}}} \cdot \sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x} \\
& + T^{1/4} C_{\sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x}} \cdot (\|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}} + \|\delta \underline{A}_i\|_{\widehat{S}^{k+1}}).
\end{aligned} \tag{4.5.10}$$

Next, we derive estimates for $\mathbf{D}_0 F_{ij} + \mathbf{D}_i F_{0j}$.

Lemma 4.5.6 (Bounds for $\mathbf{D}_0 F_{ij}$ and $\mathbf{D}_i F_{0j}$)**.** *Suppose that the caloric-temporal gauge condition $A_s = 0$, $\underline{A}_0 = 0$ holds.*

1. *For any $2 \leq p \leq \infty$ and $k \geq 0$, we have*

$$\begin{aligned}
& \|\mathbf{D}_0 F_{ij}(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} + \|\mathbf{D}_i F_{0j}(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} \\
& \quad \leq C_{p,k}\Big(\|\nabla_0 F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k+1}_x} + \|\partial_0 \underline{A}_i(t)\|_{\dot{H}^{k+1}_x} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+2}_x} \\
& \qquad + (\|\nabla_{t,x} F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k+1}_x} + \|\partial_{t,x} \underline{A}_i(t)\|_{H^{k+1}_x} + \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})^2 \\
& \qquad + (\|\nabla_{t,x} F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k+1}_x} + \|\partial_{t,x} \underline{A}_i(t)\|_{H^{k+1}_x} + \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})^3\Big).
\end{aligned} \tag{4.5.11}$$

2. For any $2 \leq p \leq \infty$ and $k \geq 0$, we have

$$\begin{aligned}
&\|\mathbf{D}_0 F_{ij}\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} + \|\mathbf{D}_i F_{0j}\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \\
&\quad \leq C_{p,k} \Big(\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+5/2}} + \|\underline{A}_i\|_{\dot{S}^{k+5/2}_x} + T^{1/4} \sup_{t \in I} \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+11/4}_x} \\
&\qquad + (\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+2}} + \|\underline{A}_i\|_{\widehat{S}^{k+2}} + \sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})^2 \\
&\qquad + (\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+2}} + \|\underline{A}_i\|_{\widehat{S}^{k+2}} + \sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})^3 \Big).
\end{aligned} \tag{4.5.12}$$

Proof. The proof proceeds in a similar manner as Lemma 4.5.4. We shall give a treatment of the contribution of the term $\|\mathbf{D}_0 F_{ij}\|$, and leave the similar case of $\|\mathbf{D}_i F_{0j}\|$ to the reader.

Our starting point is the schematic identity

$$\mathbf{D}_0 F_{ij} = s^{-1} \mathcal{O}(\nabla_0 \nabla_x A) + s^{-1/2} \mathcal{O}(A_0, \nabla_x A) + s^{-1/2} \mathcal{O}(A, \nabla_0 A) + \mathcal{O}(A, A, A_0), \tag{4.5.13}$$

which can be checked easily by expanding $\mathbf{D}_0 F_{ij}$ in terms of A_μ.

The first term on the right-hand side of (4.5.13) is acceptable for both (4.5.11) and (4.5.12), thanks to Lemma 4.5.2. Therefore, it remains to treat only the bilinear and trilinear terms in (4.5.13).

Let us begin with the proof of (4.5.11). For the bilinear terms (i.e. the second and the third terms), we start with the inequality $\|\phi_1 \phi_2\|_{L^2_x} \leq C \|\phi_1\|_{\dot{H}^1_x} \|\phi_2\|_{\dot{H}^{1/2}_x}$, which follows from Lemma 3.1.3. Applying Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain for $k \geq 0$

$$\begin{aligned}
&\|s^{-1/2} \mathcal{O}(A_0, \nabla_x A)(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} + \|s^{-1/2} \mathcal{O}(A, \nabla_0 A)(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} \\
&\quad \leq C \|\nabla_x A_0\|_{\mathcal{L}^{0+3/8,\infty}_s \mathcal{H}^k_x} \|\nabla_x A\|_{\mathcal{L}^{1/4+1/8,p}_s \mathcal{H}^{k+1}_x} + \|\nabla_x A\|_{\mathcal{L}^{1/4+1/8,\infty}_s \mathcal{H}^k_x} \|\nabla_0 A\|_{\mathcal{L}^{1/4+1/8,p}_s \mathcal{H}^{k+1}_x}.
\end{aligned}$$

Applying Lemma 4.5.2 to A and Lemma 4.5.3 to A_0, we see that the bilinear terms on the right-hand side of (4.5.13) are also okay.

Finally, for the trilinear term, we start with the inequality $\|\phi_1 \phi_2 \phi_3\|_{L^2_x} \leq C \|\phi_1\|_{\dot{H}^1_x} \|\phi_2\|_{\dot{H}^1_x} \|\phi_3\|_{\dot{H}^1_x}$. By Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain for $k \geq 0$

$$\|\mathcal{O}(A, A, A_0)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} \leq \|\nabla_x A\|_{\mathcal{L}^{1/4+1/6,\infty}_s \mathcal{H}^k_x} \|\nabla_x A\|_{\mathcal{L}^{1/4+1/6,p}_s \mathcal{H}^k_x} \|\nabla_x A_0\|_{\mathcal{L}^{0+5/12,\infty}_s \mathcal{H}^k_x}.$$

Applying Lemma 4.5.2 to A and Lemma 4.5.3 to A_0, we see that the last term on the right-hand side of (4.5.13) is acceptable. This proves (4.5.11).

Next, let us prove (4.5.12), which proceeds in an analogous way. For the bilinear terms, we begin with the obvious inequality $\|\phi_1\phi_2\|_{L^4_{t,x}} \leq C\|\phi_1\|_{L^\infty_{t,x}}\|\phi_2\|_{L^4_{t,x}}$. Applying Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14, we obtain for $k \geq 0$

$$\begin{aligned}
\|s^{-1/2}\mathcal{O}(A_0, \nabla_x A)\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} &+ \|s^{-1/2}\mathcal{O}(A, \nabla_0 A)\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \\
&\leq C \sup_{t\in I} \|\nabla_x A_0(t)\|_{\mathcal{L}^{0+3/8,\infty}_s \mathcal{H}^{k+1}_x} \|\nabla_x A\|_{\mathcal{L}^{1/4+1/8,p}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x} \\
&\quad + C\|\nabla_x A\|_{\mathcal{L}^{1/4+1/8,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{k+1}_x} \|\nabla_0 A\|_{\mathcal{L}^{1/4+1/8,p}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x}.
\end{aligned}$$

Using Strichartz and the Correspondence Principle, we can estimate $\|\nabla_{t,x} A\|_{\mathcal{L}^{1/4+1/8,p}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x} \leq C\|A\|_{\mathcal{L}^{1/4+1/8,p}_s \widehat{\mathcal{S}}^{k+2}}$. Then applying Lemma 4.5.2 to A and Lemma 4.5.3 to A_0, it easily follows that the bilinear terms on the right-hand side of (4.5.13) are acceptable.

For the trilinear term, we start with the inequality

$$\|\phi_1\phi_2\phi_3\|_{L^4_{t,x}} \leq C\|\phi_1\|_{L^4_t L^{12}_x}\|\phi_2\|_{L^\infty_t L^{12}_x}\|\phi_3\|_{L^\infty_t L^{12}_x}.$$

By Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain

$$\begin{aligned}
\|\mathcal{O}(A, A, A_0)\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \leq & C\|A\|_{\mathcal{L}^{1/4+1/6,p}_s \mathcal{L}^4_t \mathcal{W}^{k,12}_x} \|A\|_{\mathcal{L}^{1/4+1/6,\infty}_s \mathcal{L}^\infty_t \mathcal{W}^{k,12}_x} \\
& \cdot \sup_{t\in I} \|A_0(t)\|_{\mathcal{L}^{0+5/12,\infty}_s \mathcal{W}^{k,12}_x}.
\end{aligned}$$

Using Strichartz and the Correspondence Principle, the first factor

$$\|A\|_{\mathcal{L}^{1/4+1/6,\infty}_s \mathcal{L}^4_t \mathcal{W}^{k,12}_x}$$

is estimated by $C\|A\|_{\mathcal{L}^{1/4+1/6,p}_s \widehat{\mathcal{S}}^{k+2}}$. Next, using interpolation and the Correspondence Principle, we estimate the second factor $\|A\|_{\mathcal{L}^{1/4+1/6,\infty}_s \mathcal{L}^\infty_t \mathcal{W}^{k,12}_x}$ by $\|\nabla_x A\|_{\mathcal{L}^{1/4+1/6,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{k+1}_x}$. Finally, for the last factor, let us estimate $\|A_0(t)\|_{\mathcal{L}^{0+5/12,\infty}_s \dot{\mathcal{W}}^{k,12}_x} \leq C\|\nabla_x A_0(t)\|_{\mathcal{L}^{0+5/12,\infty}_s \mathcal{H}^{k+1}_x}$. At this point, we can simply apply Lemma 4.5.2 to A and Lemma 4.5.3 to A_0, and conclude that the trilinear term is acceptable as well. This proves (4.5.12). □

Finally, by essentially the same proof, we can prove an analogue of Lemma 4.5.6 for $\delta\mathbf{D}_0 F_{ij} := \mathbf{D}_0 F_{ij} - \mathbf{D}'_0 F'_{ij}$ and $\delta\mathbf{D}_i F_{0j} := \mathbf{D}_i F_{0j} - \mathbf{D}'_i F'_{0j}$, whose statement we give below.

Lemma 4.5.7 (Bounds for $\delta\mathbf{D}_0 F_{ij}$ and $\delta\mathbf{D}_i F_{0j}$)**.** *Suppose that the caloric-temporal gauge condition $A_s = 0$, $\underline{A}_0 = 0$ holds (for both A and A').*

1. For any $2 \le p \le \infty$ and $k \ge 0$, we have

$$\begin{aligned}
&\|\delta \mathbf{D}_0 F_{ij}(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} + \|\delta \mathbf{D}_i F_{0j}(t)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^k_x} \\
&\qquad \le C(\|\nabla_{t,x}(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k+1}_x} + \|\partial_{t,x}(\delta \underline{A}_i)(t)\|_{H^{k+1}_x} + \|\nabla_x(\delta F_{s0})(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x}),
\end{aligned} \tag{4.5.14}$$

where $C = C_{p,k}(\|\nabla_{t,x} F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k+1}_x}, \|\partial_{t,x}\underline{A}_i(t)\|_{H^{k+1}_x}, \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})$ is positive and non-decreasing in its arguments.

2. For any $2 \le p \le \infty$ and $k \ge 0$, we have

$$\begin{aligned}
&\|\delta \mathbf{D}_0 F_{ij}\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} + \|\delta \mathbf{D}_i F_{0j}\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{k,4}_x} \\
&\qquad \le C_{p,k}(\|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+5/2}} + \|\delta \underline{A}_i\|_{\dot{S}^{k+5/2}} + T^{1/4} \sup_{t \in I} \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{k+11/4}_x}) \\
&\qquad\quad + C(\|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+2}} + \|\delta \underline{A}_i\|_{\widehat{S}^{k+2}} + \sup_{t \in I} \|\nabla_x(\delta F_{s0})(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x}),
\end{aligned} \tag{4.5.15}$$

where $C = C_{p,k}(\|F_{si}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+2}}, \|\underline{A}_i\|_{\widehat{S}^{k+2}}, \sup_{t \in I} \|\nabla_x F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{k+1}_x})$ on the last line is positive and non-decreasing in its arguments.

4.5.2 Parabolic estimates for F_{si}

Recall that F_{si} satisfies the covariant parabolic equation

$$\mathbf{D}_s F_{si} - \mathbf{D}^\ell \mathbf{D}_\ell F_{si} = -2[F_s{}^\ell, F_{i\ell}].$$

Under the caloric gauge condition $A_s = 0$, expanding covariant derivatives and $F_{i\ell}$, we obtain a semi-linear heat equation for F_{si}, which looks schematically as follows:

$$^{(F_{si})}\mathcal{N} := (\partial_s - \triangle) F_{si} = s^{-1/2} \mathcal{O}(A, \nabla_x F_s) + s^{-1/2} \mathcal{O}(\nabla_x A, F_s) + \mathcal{O}(A, A, F_s).$$

Note that $\mathcal{F}$ already controls some derivatives of F_{si}. Starting from this, the goal is to prove estimates for higher derivative of F_{si}.

Proposition 4.5.8. *Suppose $0 < T \le 1$, and that the caloric-temporal gauge condition holds.*

1. For any $k \ge 0$, we have

$$\|\nabla_{t,x} F_{si}(t)\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{H}}^k_x(0,1]} + \|\nabla_{t,x} F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k+1}_x(0,1]} \le C_{k, \mathcal{F}, \|\partial_{t,x} \underline{A}(t)\|_{H^k_x}} \cdot \mathcal{F}. \tag{4.5.16}$$

2. *For $1 \le k \le 25$, we have*

$$\|F_{si}\|_{\mathcal{L}_s^{5/4,\infty}\dot{\mathcal{S}}^k(0,1]} + \|F_{si}\|_{\mathcal{L}_s^{5/4,2}\dot{\mathcal{S}}^k(0,1]} \le C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{F}. \tag{4.5.17}$$

Statement 1 of the proposition states, heuristically, that in order to control $k+2$ derivatives of F_{si} in the $\mathcal{L}_s^2$ sense, we need $\mathcal{F}$ and a control of $k+1$ derivatives of $\underline{A}_i$. This numerology is important for closing the bootstrap for the quantity $\underline{\mathcal{A}}$. On the other hand, in Statement 2, we obtain a uniform estimate in terms only of $\mathcal{F}$ and $\underline{\mathcal{A}}$, thanks to the restriction of the range of k. We refer the reader to Remark 4.5.9 for more discussion.

Proof. Step 1: Proof of (1). Fix $t \in (-T, T)$. Let us start with the obvious inequalities

$$\begin{cases} \|\partial_{t,x}(\phi_1 \partial_x \phi_2)\|_{L^2_x} \le C\|\partial_{t,x}\phi_1\|_{\dot{H}^{1/2}_x}\|\phi_2\|_{\dot{H}^1_x} + C\|\phi_1\|_{L^\infty_x}\|\partial_{t,x}\phi_2\|_{\dot{H}^1_x}, \\ \|\partial_{t,x}(\phi_1\phi_2\phi_3)\|_{L^2_x} \le C\sum_\sigma \|\phi_{\sigma(1)}\|_{\dot{H}^{4/3}_x}\|\phi_{\sigma(2)}\|_{\dot{H}^{4/3}_x}\|\partial_{t,x}\phi_{\sigma(3)}\|_{\dot{H}^{1/3}_x}, \end{cases} \tag{4.5.18}$$

where the sum $\sum_\sigma$ is over all permutations σ of $\{1,2,3\}$. These can be proved by using Leibniz's rule, Hölder and Sobolev.

Using Leibniz's rule, the Correspondence Principle, Lemma 3.1.8, Gagliardo-Nirenberg (Lemma 3.1.14) and interpolation, the previous inequalities lead to the following inequalities for $k \ge 1$.

$$\begin{aligned} &\|s^{-1/2}\nabla_{t,x}\mathcal{O}(\psi_1, \nabla_x\psi_2)\|_{\mathcal{L}_s^{5/4+1,2}\dot{\mathcal{H}}_x^{k-1}} + \|s^{-1/2}\nabla_{t,x}\mathcal{O}(\nabla_x\psi_1, \psi_2)\|_{\mathcal{L}_s^{5/4+1,2}\dot{\mathcal{H}}_x^{k-1}} \\ &\qquad \le C\|\nabla_{t,x}\psi_1\|_{\mathcal{L}_s^{1/4+1/4,\infty}\mathcal{H}_x^k}\|\nabla_{t,x}\psi_2\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k}, \\ &\|\nabla_{t,x}\mathcal{O}(\psi_1,\psi_2,\psi_3)\|_{\mathcal{L}_s^{5/4+1,2}\dot{\mathcal{H}}_x^{k-1}} \\ &\qquad \le C\|\nabla_{t,x}\psi_1\|_{\mathcal{L}_s^{1/4+1/4,\infty}\mathcal{H}_x^k}\|\nabla_{t,x}\psi_2\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k}\|\nabla_{t,x}\psi_3\|_{\mathcal{L}_s^{1/4+1/4,\infty}\mathcal{H}^k}. \end{aligned}$$

Note the extra weight of $s^{1/4}$ for ψ_1, ψ_3. Put $\psi_1 = A, \psi_2 = F_s, \psi_3 = A$, and apply Lemma 4.5.2 (with $\ell > 0, p = \infty, q = 2$ and $X = L^2_x$) for $\|A\|$. Then for $k \ge 1$, we have

$$\begin{aligned} \sup_i \|\nabla_{t,x}({}^{(F_{si})}\mathcal{N})\|_{\mathcal{L}_s^{5/4+1,2}\dot{\mathcal{H}}_x^{k-1}} \le& C(\|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k} + \|\partial_{t,x}\underline{A}\|_{H^k_x})\|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k} \\ &+ C(\|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k} + \|\partial_{t,x}\underline{A}\|_{H^k_x})^2\|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^k}. \end{aligned} \tag{4.5.19}$$

Combining this with the obvious bound $\|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,\infty}\mathcal{H}_x^1} + \|\nabla_{t,x}F_s\|_{\mathcal{L}_s^{5/4,2}\mathcal{H}_x^2} \le \mathcal{F}$, we obtain (4.5.16) from the second part of Theorem 3.1.10.

Step 2: Proof of (2). We proceed in a similar fashion. The multilinear estimates are more complicated. On the other hand, as we are aiming to control derivatives of F_{si} only up to order 25 whereas $\underline{\mathcal{A}}$ controls derivatives of $\underline{A}_i$ up to order 30, we can be relaxed on the number of derivatives falling on $\underline{A}_i$.

For $\epsilon > 0$, we claim that the following estimate for ${}^{(F_{si})}\mathcal{N}$ holds for $1 \leq k \leq 24$.

$$\sup_i \|{}^{(F_{si})}\mathcal{N}\|_{\mathcal{L}_s^{5/4+1,2}\dot{\mathcal{S}}^k} \leq \epsilon \|F_s\|_{\mathcal{L}_s^{5/4,2}\dot{\mathcal{S}}^{k+2}} + \mathcal{B}_{\epsilon,k,\underline{\mathcal{A}}}(\|F_s\|_{\mathcal{L}_s^{5/4,2}\widehat{\mathcal{S}}^{k+1}})\|F_s\|_{\mathcal{L}_s^{5/4,2}\widehat{\mathcal{S}}^{k+1}} \tag{4.5.20}$$

where $\mathcal{B}_{\epsilon,k,\mathcal{A}}(r) > 0$ is non-decreasing in $r > 0$. Then for $\epsilon > 0$ sufficiently small, the second part of Theorem 3.1.10 can be applied. Combined with the obvious bound $\|F_s\|_{\mathcal{P}^{5/4}\dot{\mathcal{S}}^2} \leq \mathcal{F}$, we obtain a bound for $\|F_s\|_{\mathcal{P}^{5/4}\mathcal{S}^{21}}$ which can be computed by (3.1.14). This leads to (4.5.17), as desired.

Let us now prove the claim. We shall begin by establishing the following multilinear estimates for $\dot{S}^k$:

$$\left\{\begin{aligned} &\|\phi_1\partial_x\phi_2\|_{\dot{S}^1} \leq T^{1/2}\|\phi_1\|_{\dot{S}^{3/2}}\|\phi_2\|_{\dot{S}^{5/2}} + \|\phi_1\|_{\dot{S}^{3/2}\cap L^\infty_{t,x}}\|\phi_2\|_{\dot{S}^2},\\ &\|\phi_1\phi_2\phi_3\|_{\dot{S}^1} \leq T^{1/2}\sum_\sigma \|\phi_{\sigma(1)}\|_{L^\infty_{t,x}}\|\phi_{\sigma(2)}\|_{\dot{S}^{3/2}}\|\phi_{\sigma(3)}\|_{\dot{S}^{3/2}}\\ &\qquad\qquad + \sum_\sigma \|\phi_{\sigma(1)}\|_{\dot{S}^1}\|\phi_{\sigma(2)}\|_{\dot{S}^1}\|\phi_{\sigma(3)}\|_{\dot{S}^2}. \end{aligned}\right. \tag{4.5.21}$$

where the sum $\sum_\sigma$ is over all permutations σ of $\{1,2,3\}$.

For the first inequality of (4.5.21), it suffices to prove that $\|\phi_1\partial_x\phi_2\|_{L^\infty_t \dot{H}^1_x}$ and $T^{1/2}\|\Box(\phi_1\partial_x\phi_2)\|_{L^2_{t,x}}$ can be controlled by the right-hand side. Using Hölder and Sobolev, we can easily bound the former by $\leq C\|\phi_1\|_{L^\infty_t\dot{H}^{3/2}_x}\|\phi_2\|_{L^\infty_t\dot{H}^2_x}$, which is acceptable. For the latter, using Leibniz's rule for $\Box$, let us further decompose

$$T^{1/2}\|\Box(\phi_1\partial_x\phi_2)\|_{L^2_{t,x}} \leq 2T^{1/2}\|\partial_\mu\phi_1\partial_x\partial^\mu\phi_2\|_{L^2_{t,x}} + T^{1/2}\|\Box\phi_1\partial_x\phi_2\|_{L^2_{t,x}} + T^{1/2}\|\phi_1\partial_x\Box\phi_2\|_{L^2_{t,x}}.$$

Using Hölder and the $L^4_{t,x}$-Strichartz, we bound the first term by $\leq CT^{1/2}\|\phi_1\|_{\dot{S}^{3/2}}\|\phi_2\|_{\dot{S}^{5/2}}$, which is good. For the second term, let us use Hölder to put $\Box\phi_1$ in $L^2_tL^3_x$ and the other in $L^\infty_tL^6_x$. Then by Sobolev and the definition of $\dot{S}^k$, this is bounded by $\|\phi_1\|_{\dot{S}^{3/2}}\|\phi_2\|_{\dot{S}^2}$. Finally, for the third term, we use Hölder to estimate ϕ_1 in $L^\infty_{t,x}$ and $\partial_x\Box\phi_2$ in $L^2_{t,x}$, which leads to a bound $\leq \|\phi_1\|_{L^\infty_{t,x}}\|\phi_2\|_{\dot{S}^2}$. This prove the first inequality of (4.5.21).

The second inequality of (4.5.21) follows by a similar consideration, first dividing $\|\cdot\|_{\dot{S}^1}$ into $\|\cdot\|_{L^\infty_t\dot{H}^1_x}$ and $\|\Box(\cdot)\|_{L^2_{t,x}}$, and then using Leibniz's rule for $\Box$ to further split the latter. We leave the details to the reader.

Let us prove (4.5.20) by splitting ${}^{(F_{si})}\mathcal{N}$ into its quadratic part $s^{-1/2}\mathcal{O}(A,\nabla_xF_s)+s^{-1/2}\mathcal{O}(\nabla_xA,F_s)$

and its cubic part $\mathcal{O}(A, A, F_s)$. For the quadratic terms, we use the first inequality of (4.5.21), Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14. Then for $k \geq 1$ we obtain

$$\begin{aligned} \|s^{-1/2}\mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{S}}^k} \leq CT^{1/2} \sum_{p=0}^{k-1} \|\psi_1\|_{\mathcal{L}^{\ell_1,p_1}_s \dot{\mathcal{S}}^{3/2+p}} \|\psi_2\|_{\mathcal{L}^{\ell_2,p_2}_s \dot{\mathcal{S}}^{3/2+k-p}} \\ + C\|\psi_1\|_{\mathcal{L}^{\ell_1+1/8,p_1}_s \widehat{\mathcal{S}}^{k+1}} \|\psi_2\|_{\mathcal{L}^{\ell_2+1/8,p_2}_s \widehat{\mathcal{S}}^{k+1}} \end{aligned} \tag{4.5.22}$$

where $\ell_1 + \ell_2 = 3/2$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Note that we have obtained an extra weight of $s^{1/8}$ for each factor in the last term.

Let $1 \leq k \leq 24$, and apply (4.5.22) with $(\psi_1, \ell_1, p_1) = (A, 1/4, \infty)$, $(\psi_2, \ell_2, p_2) = (F_s, 2, 5/4)$ for $s^{-1/2}\mathcal{O}(A, \nabla_x F_s)$ and vice versa for $s^{-1/2}\mathcal{O}(\nabla_x A, F_s)$. We then apply Lemma 4.5.2 with $X = \dot{S}^1$, $p = \infty$ and $q = 2$ to control $\|A\|$ in terms of $\|F_s\|$ and $\|\underline{A}\|$ (here we use the extra weight of $s^{1/8}$). Next, we estimate $\|\underline{A}\|$ that arises by $\underline{\mathcal{A}}$, which is possible since we only consider $1 \leq k \leq 24$. As a result, we obtain the following inequality:

$$\begin{aligned} &\|s^{-1}\mathcal{O}(A, \nabla_x F_s) + s^{-1}\mathcal{O}(\nabla_x A, F_s)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{S}}^k} \\ &\leq CT^{1/2} \sum_{p=0}^{k} (\|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{3/2+p}} + \underline{\mathcal{A}})\|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{3/2+k-p}} + C(\|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}} + \underline{\mathcal{A}})\|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}}. \end{aligned}$$

The last term is acceptable. All summands of the first term on the right-hand side are also acceptable, except for the cases $p = 0, k$. Let us first treat the case $p = 0$. For $\epsilon > 0$, we apply Cauchy-Schwarz to estimate

$$\begin{aligned} &T^{1/2}(\|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{3/2}} + \underline{\mathcal{A}})\|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+3/2}} \\ &\quad \leq (\epsilon/2)\|F_s\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^{k+2}} + C_\epsilon T(\|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}} + \underline{\mathcal{A}})^2\|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}}, \end{aligned}$$

The case $p = k$ is similar. This proves (4.5.20) for the quadratic terms $s^{-1/2}\mathcal{O}(A, \nabla_x F_s) + s^{-1/2}\mathcal{O}(\nabla_x A, F_s)$.

Next, let us estimate the contribution of the cubic terms $\mathcal{O}(A, A, F_s)$. Starting from the second inequality of (4.5.21) and applying Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14, we obtain the following inequality:

$$\|\mathcal{O}(\psi_1, \psi_2, \psi_3)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{S}}^k} \leq CT^{1/2} \prod_{j=1,2,3} \|\psi_j\|_{\mathcal{L}^{\ell_j+1/12,p_j}_s \widehat{\mathcal{S}}^{k+1}} + C \prod_{j=1,2,3} \|\psi_j\|_{\mathcal{L}^{\ell_j+1/6,p_j}_s \widehat{\mathcal{S}}^{k+1}},$$

for $\ell_1 + \ell_2 + \ell_3 = 7/4$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2}$. Note the extra weight of $s^{1/12}$ and $s^{1/6}$ for each factor in the first and second terms on the right-hand side, respectively.

For $1 \leq k \leq 24$, let us put $(\psi_1, \ell_1, p_1) = (A, 1/4, \infty)$, $(\psi_2, \ell_2, p_2) = (A, 1/4, \infty)$ and $(\psi_3, \ell_3, p_3) = (F_s, 5/4, 2)$ in the last inequality, and furthermore apply Lemma 4.5.2 with $X = \dot{S}^1$, $p = \infty$ and $q = 2$ (which again uses the extra weights of powers of s) to control $\|A\|$ by $\|F_s\|$ and $\|\underline{A}\|$. Then estimating $\|\underline{A}\|$ by $\underline{\mathcal{A}}$ (which again is possible since $1 \leq k \leq 24$), we finally arrive at

$$\|\mathcal{O}(A, A, F_s)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{S}}^k} \leq C(1 + T^{1/2})(\|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}} + \underline{\mathcal{A}})^2 \|F_s\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1}}$$

which is acceptable. This proves (4.5.20). □

Remark 4.5.9. The fixed time parabolic estimate (4.5.16) will let us estimate $\underline{\mathcal{A}}$ in §4.7.1 in terms of $\mathcal{F}, \underline{\mathcal{A}}$, *despite* the fact that $\mathcal{F}$ controls a smaller number of derivatives (of F_{si}) than does $\underline{\mathcal{A}}$ (of $\underline{A}_i$). This is essentially due to the smoothing property of the parabolic equation satisfied by F_{si}. It will come in handy in §4.7.2 as well, since controlling only a small number of derivatives of F_{si} will suffice to control $\mathcal{F}$.

Accordingly, the space-time estimate (4.5.17) (to be used in §4.7.2) needs to be proved only for a finite range of k, which is taken to be smaller than the number of derivatives of $\underline{A}_i$ controlled by $\underline{\mathcal{A}}$. This allows us to estimate whatever $\|\underline{A}_i\|$ that arises by $\underline{\mathcal{A}}$; practically, we do not have to worry about the number of derivatives falling on $\underline{A}_i$. Moreover, we are also allowed to control (the appropriate space-time norm of) less and less derivatives for F_{s0} and w_i (indeed, see (4.5.36) and (4.5.47), respectively), as long as we control enough derivatives to carry out the analysis in §4.7.2 in the end. Again, this lets us forget about the number of derivatives falling on $\underline{A}_i$ and F_{si} (resp. $\underline{A}_i$, F_{si} and F_{s0}) while estimating the space-time norms of F_{s0} and w_i.

By essentially the same proof, the following difference analogue of Proposition 4.5.8 follows.

Proposition 4.5.10. *Suppose* $0 < T \leq 1$, *and that the caloric-temporal gauge condition holds.*

1. *Let* $t \in (-T, T)$. *Then for any* $k \geq 0$, *we have*

$$\begin{aligned} &\|\nabla_{t,x}(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{H}}^k_x(0,1]} + \|\nabla_{t,x}(\delta F_{si})(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k+1}_x(0,1]} \\ &\qquad \leq C_{k,\mathcal{F},\|\partial_{t,x}\underline{A}(t)\|_{H^k_x}} \cdot (\delta\mathcal{F} + \|\partial_{t,x}(\delta\underline{A})(t)\|_{H^k_x}), \end{aligned} \tag{4.5.23}$$

2. *For* $1 \leq k \leq 25$, *we have*

$$\|\delta F_{si}\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{S}}^k(0,1]} + \|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{S}}^k(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.24}$$

4.5.3 Estimates for F_{si} via integration

We also need some estimates for F_{si} without any derivatives, which we state below. The idea of the proof is to simply integrate the parabolic equation $\partial_s F_{si} = \triangle F_{si} + {}^{(F_{si})}\mathcal{N}$ backwards from $s = 1$.

Proposition 4.5.11. *Suppose $0 < T \leq 1$, and that the caloric-temporal gauge condition holds.*

1. *Let $t \in (-T, T)$. Then we have*

$$\|F_{si}(t)\|_{\mathcal{L}^{5/4,\infty}_s \mathcal{L}^2_x(0,1]} + \|F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}}). \tag{4.5.25}$$

2. *We have*

$$\|F_{si}\|_{\mathcal{L}^{5/4,\infty}_s \mathcal{L}^4_{t,x}(0,1]} + \|F_{si}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_{t,x}(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}}). \tag{4.5.26}$$

Proof. In the proof, all norms will be taken on the interval $s \in (0, 1]$. Let us start with the equation

$$\partial_s F_{si} = \triangle F_{si} + {}^{(F_{si})}\mathcal{N}.$$

Using the fundamental theorem of calculus, we obtain for $0 < s \leq 1$ the identity

$$F_{si}(s) = \underline{F}_{si} - \int_s^1 s' \triangle F_{si}(s') \, \frac{\mathrm{d}s'}{s'} - \int_s^1 s' ({}^{(F_{si})}\mathcal{N}(s')) \, \frac{\mathrm{d}s'}{s'}. \tag{4.5.27}$$

To prove (4.5.25) and (4.5.26), let us either fix $t \in (-T, T)$ and take the $\mathcal{L}^{5/4,p}_s \mathcal{L}^2_x$ norm of both sides or just take the $\mathcal{L}^{5/4,p}_s \mathcal{L}^4_{t,x}$ norm, respectively. We shall estimate the contribution of each term on the right-hand side of (4.5.27) separately.

For the first term on the right-hand side of (4.5.27), note the obvious estimates $\|\underline{F}_{si}(t)\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^2_x} \leq C_p \|\underline{F}_{si}(t)\|_{L^2_x}$ and $\|\underline{F}_{si}\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_{t,x}} \leq C_p \|\underline{F}_{si}\|_{L^4_{t,x}}$. Writing out $\underline{F}_{si} = \mathcal{O}(\partial^{(2)}_x \underline{A}) + \mathcal{O}(\underline{A}, \partial_x \underline{A}) + \mathcal{O}(\underline{A}, \underline{A}, \underline{A})$, we see that

$$\sup_{t \in (-T,T)} \|\underline{F}_{si}(t)\|_{L^2_x} + \|\underline{F}_{si}\|_{L^4_{t,x}} \leq C\underline{\mathcal{A}} + C\underline{\mathcal{A}}^2 + C\underline{\mathcal{A}}^3,$$

which is acceptable.

For the second term on the right-hand side of (4.5.27), let us apply Lemma 4.5.1 with $p = 2, \infty$

and $q = 2$ to estimate

$$\| \int_s^1 s' \triangle F_{si}(t,s') \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^2_x} \leq \| \int_s^1 (s/s')^{5/4} (s')^{5/4} \| \nabla^{(2)}_x F_{si}(t,s') \|_{\mathcal{L}^2_x(s')} \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^p_s}$$

$$\leq C_p \| F_{si}(t) \|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^2_x} \leq C_p \mathcal{F}.$$

Similarly, for $p = 2, \infty$, we can prove $\| \int_s^1 s' \triangle F_{si}(s') \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_{t,x}} \leq C_p \| F_{si} \|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{2,4}_x} \leq C_p \mathcal{F}$. Therefore, the contribution of the second term is okay.

Finally, for the third term on the right-hand side of (4.5.27), let us first proceed as in the previous case to reduce

$$\| \int_s^1 s' ({}^{(F_{si})}\mathcal{N}(t,s')) \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^2_x} \leq C_p \| {}^{(F_{si})}\mathcal{N}(t) \|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^2_x}.$$

Recall that ${}^{(F_{si})}\mathcal{N} = s^{-1/2} \mathcal{O}(A, \nabla_x F_s) + s^{-1/2} \mathcal{O}(\nabla_x A, F_s) + \mathcal{O}(A, A, F_s)$. Starting from the obvious inequalities

$$\| \phi_1 \partial_x \phi_2 \|_{L^2_x} \leq C \| \phi_1 \|_{\dot{H}^1_x} \| \phi_2 \|_{\dot{H}^{3/2}_x}, \quad \| \phi_1 \phi_2 \phi_3 \|_{L^2_x} \leq C \prod_{j=1,2,3} \| \phi_j \|_{\dot{H}^1_x},$$

and applying the Correspondence Principle, Lemma 3.1.8 and interpolation, we obtain

$$\| s^{-1/2} \mathcal{O}(A, \nabla_x F_s) + s^{-1/2} \mathcal{O}(\nabla_x A, F_s) \|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^2_x} \leq C \| \nabla_x A \|_{\mathcal{L}^{1/4+1/4,\infty}_s \mathcal{H}^1_x} \| \nabla_x F_s \|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^1_x}$$

$$\| \mathcal{O}(A, A, F_s) \|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^2_x} \leq C \| A \|^2_{\mathcal{L}^{1/4+1/4,\infty}_s \dot{\mathcal{H}}^1_x} \| F_s \|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^1_x}.$$

Note the extra weight of $s^{1/4}$ on each factor of A. This allows us to apply Lemma 4.5.2 (with $q = 2$) to estimate $\|A\|$ in terms of $\|F_s\|$ and $\|\underline{A}\|$. From the definition of $\mathcal{F}$ and $\underline{\mathcal{A}}$, it then follows that $\| {}^{(F_{si})}\mathcal{N}(t) \|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^2_x} \leq C(\mathcal{F} + \underline{\mathcal{A}})^2 + C(\mathcal{F} + \underline{\mathcal{A}})^3$ uniformly in $t \in (-T, T)$, which finishes the proof of (4.5.25).

Finally, as in the previous case, we have

$$\| \int_s^1 s' ({}^{(F_{si})}\mathcal{N}(s')) \frac{\mathrm{d}s'}{s'} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_{t,x}} \leq C_p \| {}^{(F_{si})}\mathcal{N}(s') \|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^4_{t,x}}.$$

Using the inequalities

$$\| \phi_1 \partial_x \phi_2 \|_{L^4_{t,x}} \leq C \| \phi_1 \|_{L^\infty_{t,x}} \| \phi_2 \|_{\dot{S}^{3/2}}, \quad \| \phi_1 \phi_2 \phi_3 \|_{L^4_{t,x}} \leq C T^{1/4} \prod_{j=1,2,3} \| \phi_j \|_{L^\infty_t L^{12}_x},$$

and proceeding as before using the Correspondence Principle, Lemmas 3.1.8, 3.1.14 and 4.5.2, it follows that $\|{}^{(F_{si})}\mathcal{N}\|_{\mathcal{L}^{5/4+1,2}_s \mathcal{L}^4_{t,x}} \leq C(\mathcal{F}+\underline{\mathcal{A}})^2 + C(\mathcal{F}+\underline{\mathcal{A}})^3$. This concludes the proof of (4.5.26). □

Again with essentially the same proof, the following difference analogue of Proposition 4.5.11 follows.

Proposition 4.5.12. *Suppose* $0 < T \leq 1$, *and that the caloric-temporal gauge condition holds.*

1. *Let* $t \in (-T, T)$. *Then we have*

$$\|\delta F_{si}(t)\|_{\mathcal{L}^{5/4,\infty}_s \mathcal{L}^2_x(0,1]} + \|\delta F_{si}(t)\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.28}$$

2. *We have*

$$\|\delta F_{si}\|_{\mathcal{L}^{5/4,\infty}_s \mathcal{L}^4_{t,x}(0,1]} + \|\delta F_{si}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_{t,x}(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.29}$$

4.5.4 Parabolic estimates for F_{s0}

In this subsection, we shall study the parabolic equation

$$\mathbf{D}_s F_{s0} - \mathbf{D}^\ell \mathbf{D}_\ell F_{s0} = -2[F_s{}^\ell, F_{0\ell}].$$

satisfied by $F_{s0} = -w_0$. Let us define

$${}^{(F_{s0})}\mathcal{N} := (\partial_s - \triangle)F_{s0} = {}^{(F_{s0})}\mathcal{N}_{\text{forcing}} + {}^{(F_{s0})}\mathcal{N}_{\text{linear}}$$

where

$$\begin{aligned}
{}^{(F_{s0})}\mathcal{N}_{\text{linear}} &= 2s^{-1/2}[A^\ell, \nabla_\ell F_{s0}] + s^{-1/2}[\nabla^\ell A_\ell, F_{s0}] + [A^\ell, [A_\ell, F_{s0}]], \\
{}^{(F_{s0})}\mathcal{N}_{\text{forcing}} &= 2s^{-1/2}[F_0{}^\ell, F_{s\ell}].
\end{aligned}$$

Our first proposition for F_{s0} is an *a priori* parabolic estimate for $\mathcal{E}(t)$, which requires a smallness assumption of some sort[8].

Proposition 4.5.13 (Estimate for $\mathcal{E}$)**.** *Suppose that the caloric-temporal gauge condition holds, and*

[8] In our case, as we normalized the s-interval to be $[0,1]$, we shall require directly that $\mathcal{F} + \underline{\mathcal{A}}$ is sufficiently small. On the other hand, we remark that this proposition can be proved just as well by taking the length of the s-interval to be sufficiently small.

furthermore that $\mathcal{F} + \underline{\mathcal{A}} < \delta_E$ where $\delta_E > 0$ is a sufficiently small constant. Then

$$\sup_{t \in (-T,T)} \mathcal{E}(t) \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.5.30}$$

where $C_{\mathcal{F},\underline{\mathcal{A}}} = C(\mathcal{F},\underline{\mathcal{A}})$ can be chosen to be continuous and non-decreasing with respect to both arguments.

Proof. Let us fix $t \in (-T,T)$. Define $E := |\partial_x|^{-1/2} F_{s0}$. From the parabolic equation for F_{s0}, we can derive the following parabolic equation for E:

$$(\partial_s - \triangle)E = s^{1/4} |\nabla_x|^{-1/2} ({}^{(F_{s0})}\mathcal{N}),$$

where $|\nabla_x|^a := s^{a/2} |\partial_x|^a$ is the p-normalization of $|\partial_x|^a$. The idea is to work with the new variable E, and then translate to the corresponding estimates for F_{s0} to obtain (4.5.30).

We begin by making two claims. First, for every small $\epsilon, \epsilon' > 0$, by taking $\delta_E > 0$ sufficiently small, the following estimate holds for $p = 1, 2$ and $0 < \underline{s} \leq 1$:

$$\| {}^{(F_{s0})}\mathcal{N} \|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{-1/2}_x (0,\underline{s}]} \leq \epsilon \|E\|_{\mathcal{P}^{3/4} \dot{\mathcal{H}}^2_x (0,1]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \big(\|s^{1/4-\epsilon'} E\|_{\mathcal{L}^{3/4,2}_s \dot{\mathcal{H}}^1_x (0,\underline{s}]} + (\mathcal{F} + \underline{\mathcal{A}})^2 \big). \tag{4.5.31}$$

Second, for $k = 1, 2$, the following estimate holds.

$$\| {}^{(F_{s0})}\mathcal{N} \|_{\mathcal{L}^{2,2}_s \dot{\mathcal{H}}^{k-1/2}_x (0,1]} \leq \epsilon \|E\|_{\mathcal{P}^{3/4} \dot{\mathcal{H}}^{k+2}_x (0,1]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|E\|_{\mathcal{P}^{3/4} \mathcal{H}^{k+1}_x (0,1]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.5.32}$$

Assuming these claims, we can quickly finish the proof. Note that $E = 0$ at $s = 0$, as $F_{s0} = 0$ there, and that the left-hand side of (4.5.31) is equal to $\|s^{1/4} |\nabla_x|^{-1/2} {}^{(F_{s0})}\mathcal{N}\|_{\mathcal{L}^{3/4+1,p}_s \mathcal{L}^2_x (0,\underline{s}]}$. Applying the first part of Theorem 3.1.10, we derive $\|E\|_{\mathcal{P}^{3/4} \mathcal{H}^2_x (0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2$. Using the preceding estimate and (4.5.31), an application of the second part of Theorem 3.1.10 then shows that $\|E\|_{\mathcal{P}^{3/4} \mathcal{H}^4_x (0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2$. Finally, as $E = s^{1/4} |\nabla_x|^{-1/2} F_{s0}$, it is easy to see that $\mathcal{E}(t) \leq \|E\|_{\mathcal{P}^{3/4} \mathcal{H}^4_x (0,1]}$, from which (4.5.30) follows.

To establish (4.5.31) and (4.5.32), we split ${}^{(F_{s0})}\mathcal{N}$ into ${}^{(F_{s0})}\mathcal{N}_{\mathrm{forcing}}$ and ${}^{(F_{s0})}\mathcal{N}_{\mathrm{linear}}$.

- *Case 1: The contribution of ${}^{(F_{s0})}\mathcal{N}_{\mathrm{forcing}}$.* In this case, we shall work on the whole interval $(0, 1]$. Let us start with the product inequality

$$\|\phi_1 \phi_2\|_{\dot{H}^{-1/2}_x} \leq C \|\phi_1\|_{\dot{H}^{1/2}_x} \|\phi_2\|_{\dot{H}^{1/2}_x},$$

which follows from Lemma 3.1.3. Using Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain for $0 \leq k \leq 2$

$$\|\mathcal{O}(\psi_1, \psi_2)\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{k-1/2}_x} \leq C \sum_{j=0}^{k} \|\psi_1\|_{\mathcal{L}^{3/4,r}_s \dot{\mathcal{H}}^{j+1/2}_x} \|\psi_2\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k-j+1/2}_x}.$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Let us put $\psi_1 = F_{0\ell}$, $\psi_2 = F_{s\ell}$.

In order to estimate $\|F_{0\ell}\|_{\mathcal{L}^{3/4,r}_s \dot{\mathcal{H}}^{1/2}_x}$ or $\|F_{0\ell}\|_{\mathcal{L}^{3/4,r}_s \dot{\mathcal{H}}^{j+1/2}_x}$ with $j > 0$, we apply (4.5.1) or (an interpolation of) (4.5.2) of Lemma 4.5.4, respectively. We then estimate $\|F_{s\ell}\|, \|\underline{A}_\ell\|$ which arise by $\mathcal{F}, \underline{\mathcal{A}}$, respectively. (We remark that this is possible as $0 \leq k \leq 2$.)

Next, to estimate $\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{1/2}_x}$, we first note, by interpolation, that it suffices to control $\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x}$ and $\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^1_x}$, to which we then apply Propositions 4.5.11 and 4.5.8, respectively. On the other hand, for $\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k-j+1/2}_x}$ with $j < k$, we simply apply (after an interpolation) Proposition 4.5.8. Observe that all of $\|\underline{A}\|$ which arise can be estimated by $\underline{\mathcal{A}}$. As a result, for $1 \leq p \leq 2$ and $0 \leq k \leq 2$, we obtain

$$\begin{aligned}\|{}^{(F_{s0})}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{k-1/2}_x(0,1]} \leq & C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}}) \sum_{j=0}^{k} (\|F_{s0}\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{1+j}_x(0,1]} + \|F_{s0}\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{3/2+j}_x(0,1]}) \\ & + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.\end{aligned}$$

As $E = |\partial_x|^{-1/2} F_{s0}$, note that

$$\|F_{s0}\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{1+j}_x(0,1]} + \|F_{s0}\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{3/2+j}_x(0,1]} = \|E\|_{\mathcal{L}^{3/4,2}_s \dot{\mathcal{H}}^{3/2+j}_x(0,1]} + \|E\|_{\mathcal{L}^{3/4,2}_s \dot{\mathcal{H}}^{2+j}_x(0,1]}.$$

Note furthermore that the right-hand side is bounded by $\|E\|_{\mathcal{P}^{3/4}\mathcal{H}^{2+j}_x(0,1]}$. Given $\epsilon > 0$, by taking $\delta_E > 0$ sufficiently small (so that $\mathcal{F} + \underline{\mathcal{A}}$ is sufficiently small), we obtain for $k = 0$, $p = 1, 2$

$$\|{}^{(F_{s0})}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{-1/2}_x(0,1]} \leq \epsilon \|E\|_{\mathcal{P}^{3/4}\mathcal{H}^2_x(0,1]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2,$$

and for $k = 1, 2$ (taking $p = 2$)

$$\|{}^{(F_{s0})}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{2,2}_s \dot{\mathcal{H}}^{k-1/2}_x(0,1]} \leq \epsilon \|E\|_{\mathcal{P}^{3/4}\dot{\mathcal{H}}^{k+2}_x(0,1]} + \epsilon \|E\|_{\mathcal{P}^{3/4}\mathcal{H}^{k+1}_x(0,1]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2,$$

both of which are acceptable.

- *Case 2: The contribution of* ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$. Let $\underline{s} \in (0, 1]$; we shall work on $(0, \underline{s}]$ in this case. We shall see that for this term, no smallness assumption is needed.

Let us start with the inequalities

$$\|\phi_1\phi_2\|_{\dot{H}_x^{-1/2}} \leq C\|\phi_1\|_{\dot{H}_x^{3/2}\cap L_x^\infty}\|\phi_2\|_{\dot{H}_x^{-1/2}},$$
$$\|\phi_1\phi_2\|_{\dot{H}_x^{-1/2}} \leq C\|\phi_1\|_{\dot{H}_x^{1/2}}\|\phi_2\|_{\dot{H}_x^{1/2}},$$
$$\|\phi_1\phi_2\phi_3\|_{\dot{H}_x^{-1/2}} \leq C\|\phi_1\|_{\dot{H}_x^1}\|\phi_2\|_{\dot{H}_x^{1/2}}\|\phi_3\|_{\dot{H}_x^1}.$$

The first inequality follows from Lemma 3.1.4, the second from Lemma 3.1.3, and the third inequality is an easy consequences of the Hardy-Littlewood-Sobolev fractional integration $L_x^{3/2} \subset \dot{H}_x^{-1/2}$, Hölder and Sobolev.

Let $\epsilon' > 0$. Using the preceding inequalities, along with the Correspondence Principle and Lemma 3.1.8, we obtain the following inequalities for $p = 1, 2$ on $(0, \underline{s}]$:

$$\left\{\begin{aligned} &\|s^{-1/2}\mathcal{O}(\psi_1, \nabla_x\psi_2)\|_{\mathcal{L}_s^{2,p}\dot{\mathcal{H}}_x^{-1/2}} + \|s^{-1/2}\mathcal{O}(\nabla_x\psi_1, \psi_2)\|_{\mathcal{L}_s^{2,p}\dot{\mathcal{H}}_x^{-1/2}} \\ &\quad \leq C\underline{s}^{\epsilon'}\|\psi_1\|_{\mathcal{L}_s^{1/4,\infty}(\dot{\mathcal{H}}_x^{3/2}\cap\mathcal{L}_x^\infty)}\|s^{1/4-\epsilon'}\,\psi_2\|_{\mathcal{L}_s^{1,2}\dot{\mathcal{H}}_x^{1/2}}, \\ &\|\mathcal{O}(\psi_1,\psi_2,\psi_3)\|_{\mathcal{L}_s^{2,p}\dot{\mathcal{H}}_x^{-1/2}} \leq C\underline{s}^{\epsilon'}\|\psi_1\|_{\mathcal{L}_s^{1/4+1/8,\infty}\dot{\mathcal{H}}_x^1}\|s^{1/4-\epsilon'}\,\psi_2\|_{\mathcal{L}_s^{1,2}\dot{\mathcal{H}}_x^{1/2}}\|\psi_3\|_{\mathcal{L}_s^{1/4+1/8,\infty}\dot{\mathcal{H}}_x^1}. \end{aligned}\right. \tag{4.5.33}$$

We remark that the factors of $\underline{s}^{\epsilon'}$, which can be estimated by ≤ 1, arise due to an application of Hölder for $\mathcal{L}_s^{\ell,p}$ (Lemma 3.1.8) in the case $p = 1$. Taking $\psi_1 = A$, $\psi_2 = F_{s0}$, $\psi_3 = A$ and using Lemmas 3.1.14, 4.5.2 and the fact that $\|s^{1/4-\epsilon'}F_{s0}\|_{\mathcal{L}_s^{1,2}\dot{\mathcal{H}}_x^{1/2}} = \|s^{1/4-\epsilon'}E\|_{\mathcal{L}_s^{3/4,2}\dot{\mathcal{H}}_x^1}$, we see that

$$\|^{(F_{s0})}\mathcal{N}_{\text{linear}}\|_{\mathcal{L}_s^{2,p}\dot{\mathcal{H}}_x^{-1/2}(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|s^{1/4-\epsilon'}\,E\|_{\mathcal{L}_s^{3/4,2}\dot{\mathcal{H}}_x^1(0,\underline{s}]}.$$

for $p = 1, 2$. Combining this with Case 1, (4.5.31) follows.

Proceeding similarly, but this time applying Leibniz's rule to (4.5.33), choosing $p = 2$ and $\underline{s} = 1$, we obtain for $k = 1, 2$

$$\|^{(F_{s0})}\mathcal{N}_{\text{linear}}\|_{\mathcal{L}_s^{2,2}\dot{\mathcal{H}}_x^{k-1/2}(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|E\|_{\mathcal{P}^{3/4}\mathcal{H}_x^{k+1}(0,1]},$$

(we estimated $s \leq 1$) from which, along with the previous case, (4.5.32) follows. □

Our next proposition for F_{s0} states that once we have a control of $\mathcal{E}(t)$, we can control higher derivatives of F_{s0} *without any smallness assumption.*

Proposition 4.5.14 (Parabolic estimates for F_{s0})**.** *Suppose* $0 < T \leq 1$*, and that the caloric-temporal* *gauge condition holds.*

1. *Let $t \in (-T, T)$. Then for $m \geq 4$, we have*

$$\begin{aligned} &\|F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \\ &\qquad \leq C_{\mathcal{F}, \|\partial_{t,x}\underline{A}(t)\|_{H^{m-2}_x}} \cdot \Big(\mathcal{E}(t) + (\mathcal{F} + \|\partial_{t,x}\underline{A}(t)\|_{H^{m-2}_x})^2 \Big). \end{aligned} \tag{4.5.34}$$

In particular, for $1 \leq m \leq 31$, we have

$$\|F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + (\mathcal{F} + \underline{\mathcal{A}})^2). \tag{4.5.35}$$

2. *For $1 \leq m \leq 21$, we have*

$$\|F_{s0}\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{E} + (\mathcal{F} + \underline{\mathcal{A}})^2). \tag{4.5.36}$$

Statement 1 of the preceding proposition tells us that in order to control m derivatives of F_{s0} *uniformly* in s (rather than in the $\mathcal{L}^2_s$ sense), we need to control m derivatives of $\underline{A}_i$. This fact will be used in an important way to close the estimates for $\underline{\mathcal{A}}$ in §4.7.1. On the other hand, as in Proposition 4.5.8, the range of k in Statement 2 was chosen so that we can estimate whatever derivative of $\underline{A}$ which arises by $\underline{\mathcal{A}}$.

Proof. Step 1: Proof of (1). Fix $t \in (-T, T)$. We shall be working on the whole interval $(0, 1]$.

Note that (4.5.35) follows immediately from (4.5.34) and the definition of $\mathcal{E}(t)$, as $\|\partial_{t,x}\underline{A}_i\|_{L^\infty_t H^{29}_x} \leq \underline{\mathcal{A}}$. In order to prove (4.5.34), we begin by claiming that the following estimate holds for $k \geq 2$:

$$\|{}^{(F_{s0})}\mathcal{N}\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \leq C_{\mathcal{F}, \|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot \|\nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x} + C_{\mathcal{F}, \|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot (\mathcal{F} + \|\partial_{t,x}\underline{A}(t)\|_{H^k_x})^2. \tag{4.5.37}$$

Assuming the claim, we may apply the second part of Theorem 3.1.10, along with the bound $\|F_{s0}\|_{\mathcal{P}^1 \mathcal{H}^3_x} \leq \mathcal{E}(t)$, to conclude (4.5.34).

To prove (4.5.37), we estimate the contributions of ${}^{(F_{s0})}\mathcal{N}_{\text{forcing}}$ and ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$ separately.

- *Case 1.1: The contribution of ${}^{(F_{s0})}\mathcal{N}_{\text{forcing}}$.* We start with the simple inequality $\|\phi_1\phi_2\|_{\dot{H}^2_x} \leq C\|\phi_1\|_{\dot{H}^2_x}\|\phi_2\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + \|\phi_1\|_{\dot{H}^{3/2}_x \cap L^\infty_x}\|\phi_2\|_{\dot{H}^2_x}$. Applying Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14, we get

$$\|\mathcal{O}(\psi_1,\psi_2)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \leq C \|\nabla_x \psi_1\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{H}^{k-1}_x} \|\nabla_x \psi_2\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k-1}_x},$$

for $k \geq 2$.

Let us put $\psi_1 = F_{0\ell}$, $\psi_2 = F_{s\ell}$, and apply Lemma 4.5.4 to control $\|\nabla_x F_{0\ell}\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{H}^{k-1}_x}$ in terms of $\|F_s\|$, $\|\underline{A}\|$ and $\|F_{s0}\|$. Then we apply Proposition 4.5.8 to estimate $\|F_s\|$ in terms of $\mathcal{F}$ and $\|\underline{A}\|$. At this point, one may check that all $\|\underline{A}\|$, $\|F_{s0}\|$ that have arisen may be estimated by $\|\partial_{t,x}\underline{A}\|_{H^k_x}$ and $\|\nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x}$, respectively. As a result, for $k \geq 2$, we obtain

$$\|{}^{(F_{s0})}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \leq C_{\mathcal{F},\|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot \|\nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x} + C_{\mathcal{F},\|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot (\mathcal{F} + \|\partial_{t,x}\underline{A}\|_{H^k_x})\, \mathcal{F},$$

which is good enough for (4.5.37).

- *Case 1.2: The contribution of* ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$. We shall begin with the following inequalities, which follow from Hölder and Sobolev:

$$\begin{cases} \|\phi_1 \partial_x \phi_2\|_{L^2_x} + \|\partial_x \phi_1 \phi_2\|_{L^2_x} \leq C \|\phi_1\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \|\phi_2\|_{\dot{H}^1_x}, \\ \|\phi_1 \phi_2\|_{L^2_x} \leq C \|\phi_1\|_{\dot{H}^{1/2}_x} \|\phi_2\|_{\dot{H}^1_x}, \quad \|\phi_1\phi_2\phi_3\|_{L^2_x} \leq C \|\phi_1\|_{\dot{H}^1_x} \|\phi_2\|_{\dot{H}^1_x} \|\phi_3\|_{\dot{H}^1_x}. \end{cases} \tag{4.5.38}$$

Applying Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14, we obtain

$$\begin{cases} \|s^{-1/2}\mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} + \|s^{-1/2}\mathcal{O}(\nabla_x\psi_1, \psi_2)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \\ \qquad \leq C \|\nabla_x \psi_1\|_{\mathcal{L}^{1/4+1/4,\infty}_s \mathcal{H}^k_x} \|\nabla_x \psi_2\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x}, \\ \|\mathcal{O}(\psi_1,\psi_2,\psi_3)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{H}}^k_x} \leq C \|\nabla_x\psi_1\|_{\mathcal{L}^{1/4+1/4,\infty}_s \mathcal{H}^k_x} \|\nabla_x \psi_2\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x} \|\nabla_x \psi_3\|_{\mathcal{L}^{1/4+1/4,\infty}_s \mathcal{H}^{k+1}_x}. \end{cases}$$

for $k \geq 1$.

Note the extra weight of $s^{1/4}$ on ψ_1, ψ_3. Let us put $\psi_1 = A, \psi_2 = F_{s0}, \psi_3 = A$, and apply Lemma 4.5.2 to control $\|A\|$ in terms of $\|F_s\|$ and $\|\underline{A}\|$. Then using Proposition 4.5.8, we can control $\|F_s\|$ by $\mathcal{F}$ and $\|\underline{A}\|$. Observe that all of $\|\underline{A}\|$ which have arisen can be estimated by $\|\partial_{t,x}\underline{A}\|_{H^k_x}$. As a result, we obtain the estimate

$$\|{}^{(F_{s0})}\mathcal{N}_{\text{linear}}\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \leq C_{\mathcal{F},\|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot \|\nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x},$$

for $k \geq 1$. Combining this with the previous case, (4.5.37) follows.

Step 2: Proof of (2). Let $0 \leq k \leq 19$, where the number k corresponds to the number of times the equation $(\partial_s - \triangle)F_{s0} = {}^{(F_{s0})}\mathcal{N}$ is differentiated. We remark that its range has been chosen to be small enough so that every norm of F_{si} and $\underline{A}_i$ that arises in the argument below can be controlled by $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})$ (by Propositions 4.5.8 and 4.5.11) and $\underline{\mathcal{A}}$, respectively.

We claim that for $\epsilon' > 0$ small enough, $0 \leq k \leq 19$ an integer, $1 \leq p \leq 2$ and $0 < \underline{s} \leq 1$, the following estimate holds:

$$\|{}^{(F_{s0})}\mathcal{N}\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|s^{1/2-\epsilon'} \nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \mathcal{H}^k_x(0,\underline{s}]} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\mathcal{F} + \underline{\mathcal{A}}). \quad (4.5.39)$$

Assuming (4.5.39), and taking $k = 0$, $p = 1, 2$, we can apply the first part of Theorem 3.1.10 to obtain (4.5.36) in the cases $m = 1, 2$. Then taking $1 \leq k \leq 19$ and $p = 2$, we can apply the second part of Theorem 3.1.10, along with the bound (4.5.36) in the case $m = 2$ that was just established, to conclude the rest of (4.5.36).

As before, in order to prove (4.5.39), we treat the contributions of ${}^{(F_{s0})}\mathcal{N}_{\text{forcing}}$ and ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$ separately.

- *Case 2.1: The contribution of* ${}^{(F_{s0})}\mathcal{N}_{\text{forcing}}$. We claim that the following estimate holds for $0 \leq k \leq 19$ and $1 \leq p \leq 2$:

$$\|{}^{(F_{s0})}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\mathcal{F} + \underline{\mathcal{A}}). \quad (4.5.40)$$

Note in particular that the right-hand side does not involve $\|\nabla_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \mathcal{H}^k_x}$. This is because we can use (4.5.35) to estimate whatever factor of $\|F_{s0}\|$ that arises in this case.

In what follows, we work on the whole s-interval $(0, 1]$. Starting from Hölder's inequality $\|\phi_1 \phi_2\|_{L^2_{t,x}} \leq \|\phi_1\|_{L^4_{t,x}} \|\phi_2\|_{L^4_{t,x}}$ and using Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain

$$\|\mathcal{O}(\psi_1, \psi_2)\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x} \leq C \|\psi_1\|_{\mathcal{L}^{3/4,r}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x} \|\psi_2\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x}.$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Let us put $\psi_1 = F_{0\ell}$, $\psi_2 = F_{s\ell}$ and use Lemma 4.5.4 to control $\|F_{0\ell}\|$ in terms of $\|F_{s0}\|$, $\|F_{s\ell}\|$ and $\|\underline{A}_\ell\|$. Then thanks to the assumption $0 \leq k \leq 19$, we can use (4.5.35), the second part of Proposition 4.5.8 and the definition of $\underline{\mathcal{A}}$ to control $\|F_{s0}\|$, $\|F_{s\ell}\|$ and $\|\underline{A}_\ell\|$ have arisen by $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + (\mathcal{F} + \underline{\mathcal{A}})^2)$, $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{F}$ and $\underline{\mathcal{A}}$, respectively.

On the other hand, to control $\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x}$, we first use Strichartz to estimate

$$\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_t \mathcal{W}^{k,4}_x} \leq C\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_{t,x}} + C\|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \widehat{\mathcal{S}}^{k+1/2}}$$

and then use Propositions 4.5.11 and 4.5.8 to estimate the first and the second terms by $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F}+\underline{\mathcal{A}})$ and $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{F}$, respectively. As a result, we obtain (4.5.40) for $1 \leq p \leq 2$.

- *Case 2.2: The contribution of* ${}^{(F_{s0})}\mathcal{N}_{\mathrm{linear}}$. Let $0 < \underline{s} \leq 1$; we shall work on the interval $(0, \underline{s}]$ in this case. Let us begin with the following estimates, which follow immediately from (4.5.38) by square integrating in t and using Hölder:

$$\begin{aligned}
&\|\phi_1 \partial_x \phi_2\|_{L^2_{t,x}} + \|\partial_x \phi_1 \phi_2\|_{L^2_{t,x}} \leq C\|\phi_1\|_{L^\infty_t (\dot{H}^{3/2}_x \cap L^\infty_x)} \|\phi_2\|_{L^2_t \dot{H}^1_x}, \\
&\|\phi_1 \phi_2 \phi_3\|_{L^2_{t,x}} \leq C\|\phi_1\|_{L^\infty_t \dot{H}^1_x} \|\phi_2\|_{L^2_t \dot{H}^1_x} \|\phi_3\|_{L^\infty_t \dot{H}^1_x}.
\end{aligned}$$

Using Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we obtain the following inequalities for $\epsilon' > 0$ small, $0 \leq k \leq 19$ and $1 \leq p \leq q \leq \infty$:

$$\begin{aligned}
&\|s^{-1/2} \mathcal{O}(\psi_1, \nabla_x \psi_2)\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x} + \|s^{-1/2} \mathcal{O}(\nabla_x \psi_1, \psi_2)\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x} \\
&\qquad \leq C\underline{s}^{\epsilon'} \Big(\sum_{j=0}^{k} \|\nabla^{(j)}_x \psi_1\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t (\dot{\mathcal{H}}^{3/2}_x \cap \mathcal{L}^\infty_x)} \Big) \|s^{1/4-\epsilon'} \nabla_x \psi_2\|_{\mathcal{L}^{1,q}_s \mathcal{L}^2_t \mathcal{H}^k_x}, \\
&\|\mathcal{O}(\psi_1, \psi_2, \psi_3)\|_{\mathcal{L}^{5/4+1,2}_s \dot{\mathcal{H}}^k_x} \\
&\qquad \leq C\underline{s}^{\epsilon'} \|\nabla_x \psi_1\|_{\mathcal{L}^{1/4+1/8,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^k_x} \|s^{1/4-\epsilon'} \nabla_x \psi_2\|_{\mathcal{L}^{1,q}_s \mathcal{L}^2_t \mathcal{H}^k_x} \|\nabla_x \psi_3\|_{\mathcal{L}^{1/4+1/8,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^k_x}.
\end{aligned}$$

The factors $\underline{s}^{\epsilon'}$ have arisen from applications of Hölder for $\mathcal{L}^{\ell,p}_s$ (Lemma 3.1.8); we estimate them by ≤ 1. Let us put $\psi_1 = A$, $\psi_2 = F_{s0}$ and $\psi_3 = A$, and apply Lemma 4.5.2 to control $\|A\|$ in terms of $\|F_s\|$ and $\underline{\mathcal{A}}$ (the latter thanks to the range of k). Then we apply Proposition 4.5.8 to control $\|F_s\|$ in terms of $\mathcal{F}$ and $\underline{\mathcal{A}}$ (again using the restriction of the range of k). As a result, we arrive at

$$\|{}^{(F_{s0})}\mathcal{N}_{\mathrm{linear}}\|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \|s^{1/4-\epsilon'} \nabla_x F_{s0}\|_{\mathcal{L}^{1,q}_s \mathcal{L}^2_t \mathcal{H}^k_x(0,\underline{s}]},$$

for $\epsilon' > 0$ small, $0 \leq k \leq 19$ and $1 \leq p \leq q \leq \infty$. Taking $q = 2$ and combining with the previous case, we obtain (4.5.39) □

The difference analogues of Propositions 4.5.13 and 4.5.14 can be proved in a similar manner, using the non-difference versions which have been just established. We give their statements below, omitting the proof.

Proposition 4.5.15 (Estimate for $\delta\mathcal{E}$). *Suppose that the caloric-temporal gauge condition holds, and furthermore that $\mathcal{F} + \underline{\mathcal{A}} < \delta_E$ where $\delta_E > 0$ is sufficiently small. Then*

$$\sup_{t\in(-T,T)} \delta\mathcal{E}(t) \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.41}$$

Proposition 4.5.16 (Parabolic estimates for δF_{s0}). *Suppose $0 < T \leq 1$, and that the caloric-temporal gauge condition holds. Then the following statements hold.*

1. *Let $t \in (-T, T)$. Then for $m \geq 4$, we have*

$$\begin{aligned} &\|\delta F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \\ &\quad \leq C_{\mathcal{F},\|\partial_{t,x}\underline{A}(t)\|_{H^{m-2}_x}} \cdot \delta\mathcal{E}(t) \\ &\qquad + C_{\mathcal{F},\|\partial_{t,x}\underline{A}(t)\|_{H^{m-2}_x}} \cdot (\mathcal{E}(t) + \mathcal{F} + \|\partial_{t,x}\underline{A}(t)\|_{H^{m-2}_x}) \\ &\qquad\quad \times (\delta\mathcal{F} + \|\partial_{t,x}(\delta\underline{A})(t)\|_{H^{m-2}_x}). \end{aligned} \tag{4.5.42}$$

In particular, for $1 \leq m \leq 31$, we have

$$\begin{aligned} &\|\delta F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \\ &\quad \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \delta\mathcal{E}(t) + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \end{aligned} \tag{4.5.43}$$

2. *For $1 \leq m \leq 21$, we have*

$$\begin{aligned} &\|\delta F_{s0}\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m}_x(0,1]} \\ &\quad \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \delta\mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{F} + \delta\underline{\mathcal{A}}). \end{aligned} \tag{4.5.44}$$

4.5.5 Parabolic estimates for w_i

Here, we shall study the parabolic equation (4.1.8) satisfied by w_i, i.e.

$$\mathbf{D}_s w_i - \mathbf{D}^\ell \mathbf{D}_\ell w_i = 2[F_i{}^\ell, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell} + \mathbf{D}_\ell F_{i\mu}].$$

Let us define

$${}^{(w_i)}\mathcal{N} := (\partial_s - \triangle) w_i = {}^{(w_i)}\mathcal{N}_{\text{forcing}} + {}^{(w_i)}\mathcal{N}_{\text{linear}}$$

where

$$
\begin{aligned}
{}^{(w_i)}\mathcal{N}_{\mathrm{linear}} &= 2s^{-1/2}[A^\ell, \nabla_\ell w_i] + s^{-1/2}[\nabla^\ell A_\ell, w_i] + [A^\ell, [A_\ell, w_i]] + 2[F_i{}^\ell, w_\ell],\\
{}^{(w_i)}\mathcal{N}_{\mathrm{forcing}} &= 2[F_{0\ell}, \mathbf{D}_0 F_i{}^\ell + \mathbf{D}^\ell F_{i0}].
\end{aligned}
$$

The following proposition proves parabolic estimates for w_i that we shall need in the sequel.

Proposition 4.5.17 (Parabolic estimates for w_i)**.** *Suppose* $0 < T \leq 1$*, and that the caloric-temporal gauge condition holds.*

1. *Let* $t \in (-T, T)$*. For* $1 \leq m \leq 30$ *we have*

$$
\|w_i(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|w_i(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.5.45}
$$

In the case $m = 31$*, on the other hand, we have the following estimate.*

$$
\begin{aligned}
&\|w_i(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{30}_x(0,1]} + \|w_i(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{31}_x(0,1]}\\
&\qquad \leq C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}},\|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}} + \|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x})^2.
\end{aligned} \tag{4.5.46}
$$

2. *For* $1 \leq m \leq 16$*, we have*

$$
\|w_i\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|w_i\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.5.47}
$$

Furthermore, for $0 \leq k \leq 14$*, we have the following estimate for* ${}^{(w_i)}\mathcal{N}$*.*

$$
\|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{k}_x(0,1]} + \|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{k}_x(0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.5.48}
$$

Remark 4.5.18. Note that Statement 1 of Proposition 4.5.17 does not require a smallness assumption, as opposed to Proposition 4.5.13. Moreover, in comparison with Proposition 4.5.14, we need m derivatives of $\underline{A}$ (i.e. one more derivative) to estimate m derivatives of w uniformly in s.

Proof. Step 1: Proof of (1), for $1 \leq m \leq 3$*.* Fix $t \in (-T, T)$. Let us define $v_i := |\partial_x|^{-1/2} w_i$. From the parabolic equation for w_i, we derive the following parabolic equation for v_i:

$$
(\partial_s - \triangle) v_i = s^{1/4} |\nabla_x|^{-1/2} ({}^{(w_i)}\mathcal{N}),
$$

where the right-hand side is evaluated at t. Note that $\|w_i\|_{\mathcal{L}^{1,p}_s \dot{\mathcal{H}}^k_x} = \|v_i\|_{\mathcal{L}^{3/4,p}_s \dot{\mathcal{H}}^{k+1/2}_x}$. The idea, as in the proof of Proposition 4.5.13, is to derive estimates for v_i and then to translate to the corresponding estimates for w_i using the preceding observation.

We shall make two claims: First, for $0 < \underline{s} \leq 1$ and $1 \leq p \leq 2$, the following estimate holds.

$$\begin{aligned}\sup_i \|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{-1/2}_x (0,\underline{s}]} \leq & C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot \|s^{1/4-\epsilon'} v\|_{\mathcal{L}^{3/4,2}_s \dot{\mathcal{H}}^1_x (0,\underline{s}]} \\ & + C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})^2.\end{aligned} \tag{4.5.49}$$

Second, for $k = 1, 2$, the following estimate holds.

$$\begin{aligned}\sup_i \|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,2}_s \dot{\mathcal{H}}^{k-1/2}_x (0,1]} \leq & C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot \|v\|_{\mathcal{P}^{3/4}\mathcal{H}^{k+1}_x (0,1]} \\ & + C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})^2.\end{aligned} \tag{4.5.50}$$

Note that $\|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^k_x (0,\underline{s}]} = \|s^{1/4}|\nabla_x|^{-1/2}({}^{(w_i)}\mathcal{N})\|_{\mathcal{L}^{3/4+1,p}_s \dot{\mathcal{H}}^{k+1/2}_x (0,\underline{s}]}$. Assuming (4.5.49) and using the preceding observation, we can apply the first part of Theorem 3.1.10 to v_i (note furthermore that $v_i = 0$ at $s = 0$), from which we obtain a bound on $\|v\|_{\mathcal{P}^{3/4}\mathcal{H}^2_x}$. Next, assuming (4.5.50) and applying the second part of Theorem 3.1.10 to v_i, we can also control $\|v\|_{\mathcal{P}^{3/4}\mathcal{H}^4_x}$. Using the fact that $v_i = s^{1/4}|\nabla_x|^{-1/2} w_i$, (4.5.45) now follows.

We are therefore left with the task of establishing (4.5.49) and (4.5.50). For this purpose, we divide ${}^{(w_i)}\mathcal{N} = {}^{(w_i)}\mathcal{N}_{\text{forcing}} + {}^{(w_i)}\mathcal{N}_{\text{linear}}$, and treat each of them separately.

- *Case 1.1: Contribution of* ${}^{(w_i)}\mathcal{N}_{\text{forcing}}$. In this case, we work on the whole interval $(0, 1]$. We start with the inequality

$$\|\phi_1 \phi_2\|_{\dot{H}^{-1/2}_x} \leq \|\phi_1\|_{\dot{H}^1_x} \|\phi_2\|_{L^2_x},$$

which follows from Lemma 3.1.3. Using Leibniz's rule, the Correspondence Principle and Lemma 3.1.8, we arrive at the following inequality for $k \geq 0$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$:

$$\|\mathcal{O}(\psi_1, \psi_2)\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{k-1/2}_x} \leq C \|\nabla_x \psi_1\|_{\mathcal{L}^{3/4,r}_s \mathcal{H}^k_x} \|\psi_2\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^k_x}.$$

Let us restrict to $0 \leq k \leq 2$ and put $\psi_1 = F_{0\ell}$, $\psi_2 = \mathbf{D}^\ell F_{0i} + \mathbf{D}_0 F^\ell{}_i$. In order to estimate $\|\nabla_x F_{0\ell}\|_{\mathcal{L}^{3/4,r}_s \mathcal{H}^k_x}$ and $\|\mathbf{D}^\ell F_{0i} + \mathbf{D}_0 F^\ell{}_i\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^k_x}$, we apply Lemmas 4.5.4 (with $p = r$) and 4.5.6 (with $p = 2$), respectively, from which we obtain an estimate of $\|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{k-1/2}_x}$ in terms of $\|F_{s0}\|$, $\|F_s\|$ and $\|\underline{A}\|$. The latter two types of terms can be estimated by $\mathcal{F}$ and $\underline{\mathcal{A}}$, respectively. Moreover, using Propositions 4.5.14, $\|F_{s0}(t)\|$ can be estimated by $\mathcal{E}(t)$, $\mathcal{F}$ and $\underline{\mathcal{A}}$. As a result, for $0 \leq k \leq 2$ and

$1 \leq p \leq 2$, we obtain

$$\sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{k-1/2}_x(0,1]} \leq C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})^2.$$

which is good enough for (4.5.49) and (4.5.50).

- *Case 1.2: Contribution of* ${}^{(w_i)}\mathcal{N}_{\text{linear}}$. Note that ${}^{(w_i)}\mathcal{N}_{\text{linear}}$ has the same schematic form as ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$. Therefore, the same proof as in Case 2 of the proof of Proposition 4.5.13 gives us the estimates

$$\sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^{-1/2}_x(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|s^{1/4-\epsilon'} v\|_{\mathcal{L}^{3/4,2}_s \dot{\mathcal{H}}^1_x(0,\underline{s}]},$$

for $p = 1, 2$, $0 < \underline{s} \leq 1$ and arbitrarily small $\epsilon' > 0$, and

$$\sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{2,2}_s \dot{\mathcal{H}}^{k-1/2}_x(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \|v\|_{\mathcal{P}^{3/4}\mathcal{H}^{k+1}_x(0,1]},$$

for $k = 1, 2$. Combined with the previous case, we obtain (4.5.49) and (4.5.50).

Step 2: Proof of (1), for $m \geq 4$. By working with v_i instead of w_i, we were able to prove the *a priori* estimate (4.5.45) for low m by an application of Theorem 3.1.10. The drawback of this approach, as in the case of F_{s0}, is that the estimate that we derive is not good enough in terms of the necessary number of derivatives of $\underline{A}$. In order to prove (4.5.45) for higher m, and (4.5.46) as well, we revert back to the parabolic equation for w_i.

We claim that the following estimate holds for $k \geq 2$:

$$\begin{aligned} \sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x(0,1]} \leq & C_{\mathcal{F},\|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot \|\nabla_x w\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x(0,1]} \\ & + C_{\mathcal{E}(t),\mathcal{F},\|\partial_{t,x}\underline{A}(t)\|_{H^{k+1}_x}} (\mathcal{E}(t) + \mathcal{F} + \|\partial_{t,x}\underline{A}(t)\|_{H^{k+1}_x})^2. \end{aligned} \tag{4.5.51}$$

Assuming the claim, let us first finish the proof of (1). Note that for $0 \leq k \leq 29$, we have $\|\partial_{t,x}\underline{A}\|_{H^k_x} \leq \underline{\mathcal{A}}$. Therefore, every norm $\|\partial_{t,x}\underline{A}\|$ arising in (4.5.51) for $2 \leq k \leq 28$ can be estimated by $\underline{\mathcal{A}}$. Using this, along with the estimate (4.5.45) for $1 \leq m \leq 3$ which has been established in Step 1, we can apply the second part of Theorem 3.1.10 to conclude (4.5.45) for all $4 \leq m \leq 30$.

Note, on the other hand, that for $k = 30$ we only have $\|\partial_{t,x}\underline{A}\|_{H^{30}_x} \leq \underline{\mathcal{A}} + \|\partial_0 \underline{A}\|_{\dot{H}^{30}_x}$. From

(4.5.51), we therefore obtain the estimate

$$\sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^{29}_x(0,1]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|\nabla_x w\|_{\mathcal{L}^{1,2}_s \mathcal{H}^{29}_x(0,1]} + C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}},\|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x}} (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}} + \|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x})^2.$$

Combining this with the case $k = 30$ of (4.5.45), an application of the second part of Theorem 3.1.10 gives (4.5.46).

We are therefore only left to prove (4.5.51). As usual, we shall treat ${}^{(w_i)}\mathcal{N}_{\text{forcing}}$ and ${}^{(w_i)}\mathcal{N}_{\text{linear}}$ separately, and work on the whole interval $(0, 1]$ in both cases.

- *Case 2.1: Contribution of* ${}^{(w_i)}\mathcal{N}_{\text{forcing}}$. As in Case 1.1 in the proof of Proposition 4.5.14, we begin with the inequality $\|\phi_1\phi_2\|_{\dot{H}^2_x} \leq C\|\phi_1\|_{\dot{H}^2_x}\|\phi_2\|_{\dot{H}^{3/2}_x \cap L^\infty_x} + C\|\phi_1\|_{\dot{H}^{3/2}_x \cap L^\infty_x}\|\phi_2\|_{\dot{H}^2_x}$ and apply Leibniz's rule, the Correspondence Principle, Lemma 3.1.8 and Lemma 3.1.14. As a result, for $k \geq 2$, we obtain

$$\|\mathcal{O}(\psi_1, \psi_2)\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x} \leq C\|\nabla_x \psi_1\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{H}^{k-1}_x} \|\nabla_x \psi_2\|_{\mathcal{L}^{5/4,2}_s \mathcal{H}^{k-1}_x}$$

As in Case 1.1, we put $\psi_1 = F_{0\ell}$, $\psi_2 = \mathbf{D}^\ell F_{0i} + \mathbf{D}_0 F^\ell{}_i$, and apply Lemmas 4.5.4 (with $p = \infty$) and 4.5.6 (with $p = 2$), by which we obtain an estimate of $\|{}^{(w_i)}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}^{2,p}_s \dot{\mathcal{H}}^k_x}$ in terms of $\|F_{s0}\|$, $\|F_{si}\|$ and $\|\partial_{t,x}\underline{A}\|$. Using Proposition 4.5.14 and Proposition 4.5.8 in order, we can estimate $\|F_{s0}\|$ and $\|F_{si}\|$ in terms of $\mathcal{E}(t)$, $\mathcal{F}$ and $\|\partial_{t,x}\underline{A}\|$. At this point, one may check that all $\|\partial_{t,x}\underline{A}\|$ that have arisen can be estimated by $\|\partial_{t,x}\underline{A}(t)\|_{H^{k+1}_x}$. As a result, we obtain the following estimate for $k \geq 2$:

$$\sup_i \|{}^{(w_i)}\mathcal{N}(t)\|_{\mathcal{L}^{2,2}_s \dot{\mathcal{H}}^k_x(0,1]} \leq C_{\mathcal{E}(t),\mathcal{F},\|\partial_{t,x}\underline{A}(t)\|_{H^{k+1}_x}} (\mathcal{E}(t) + \mathcal{F} + \|\partial_{t,x}\underline{A}(t)\|_{H^{k+1}_x})^2,$$

which is good.

- *Case 2.2: Contribution of* ${}^{(w_i)}\mathcal{N}_{\text{linear}}$. As ${}^{(w_i)}\mathcal{N}_{\text{linear}}$ looks schematically the same as ${}^{(F_{s0})}\mathcal{N}_{\text{linear}}$, Step 1.2 of the proof of Proposition 4.5.14 immediately gives

$$\sup_i \|{}^{(w_i)}\mathcal{N}_{\text{linear}}\|_{\mathcal{L}^{1+1,2}_s \dot{\mathcal{H}}^k_x(0,1]} \leq C_{\mathcal{F},\|\partial_{t,x}\underline{A}\|_{H^k_x}} \cdot \|\nabla_x w\|_{\mathcal{L}^{1,2}_s \mathcal{H}^k_x(0,1]},$$

for $k \geq 1$. Combined with the previous case, this proves (4.5.51), as desired.

Step 3: Proof of (2). Let $0 \leq k \leq 14$, where k corresponds to the number of times the equation $(\partial_s - \triangle)w_i = {}^{(w_i)}\mathcal{N}$ is differentiated. The range has been chosen so that Proposition 4.5.14 can be applied to estimate every norm of F_{s0} which arises in terms of $\mathcal{E}$, $\|F_s\|$ and $\|\underline{\mathcal{A}}\|$, and furthermore so that all $\|F_s\|$ and $\|\underline{A}\|$ that arise can be estimated by $C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{F}$ (by Proposition 4.5.8) and $\underline{\mathcal{A}}$, respectively.

We claim that for $\epsilon' > 0$ small enough, $0 \leq k \leq 14$ an integer, $1 \leq p \leq q \leq \infty$ and $0 < \underline{s} \leq 1$, the following estimate holds:

$$\sup_i \|{}^{(w_i)}\mathcal{N}\|_{\mathcal{L}_s^{1+1,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^k(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \|s^{1/4-\epsilon'}\nabla_x w\|_{\mathcal{L}_s^{1,q}\mathcal{L}_t^2\mathcal{H}_x^k(0,\underline{s}]} + C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}+\mathcal{F}+\underline{\mathcal{A}})^2. \quad (4.5.52)$$

Assuming the claim, let us prove (2). Taking $k=0$, $p=1,2$ and $q=2$, we may apply the first part of Theorem 3.1.10 (along with the fact that $w=0$ at $s=0$) to obtain (4.5.47) in the cases $m=1,2$. Combining this with (4.5.52) in the cases $1 \leq k \leq 14$, $p=q=2$ and $\underline{s}=1$, we can apply the second part of Theorem 3.1.10 to obtain the rest of (4.5.47). Finally, considering (4.5.52) with $0 \leq k \leq 14$ with $p=2,\infty$, $q=\infty$ and $\underline{s}=1$, and estimating $\|\nabla_x w\|_{\mathcal{L}_s^{1,\infty}\mathcal{H}_x^k(0,1]}$ in the first term on the right-hand side by (4.5.47), we obtain (4.5.48), which finishes the proof of Statement 2.

It therefore only remain to prove (4.5.52), for which we split ${}^{(w_i)}\mathcal{N} = {}^{(w_i)}\mathcal{N}_{\text{forcing}} + {}^{(w_i)}\mathcal{N}_{\text{linear}}$ as usual.

- *Case 3.1: Contribution of* ${}^{(w_i)}\mathcal{N}_{\text{forcing}}$. In this case, we work on the whole interval $(0,1]$.

Let us begin with the inequality $\|\phi_1\phi_2\|_{L^2_{t,x}} \leq \|\phi_1\|_{L^4_{t,x}}\|\phi_2\|_{L^4_{t,x}}$. Applying Leibniz's rule, the Correspondence Principle, Lemma 3.1.8, we obtain, for $k \geq 0$ and $1 \leq p \leq \infty$,

$$\|\mathcal{O}(\psi_1,\psi_2)\|_{\mathcal{L}_s^{1+1,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^k} \leq C\|\psi_1\|_{\mathcal{L}_s^{3/4,r_1}\mathcal{L}_t^4\mathcal{W}_x^{4,k}}\|\psi_2\|_{\mathcal{L}_s^{5/4,r_2}\mathcal{L}_t^4\mathcal{W}_x^{4,k}}$$

where $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}$. Since $p \geq 1$, we may choose r_1, r_2 so that $r_1, r_2 \geq 2$. As before, let us take $\psi_1 = F_{0\ell}$ and $\psi_2 = \mathbf{D}^\ell F_{0i} + \mathbf{D}_0 F^\ell_{\ i}$ and apply Lemma 4.5.4 (with $p=r_1$) and Lemma 4.5.6 (with $p=r_2$), respectively. Then we apply Proposition 4.5.14 and Proposition 4.5.8 in sequence, where we remark that both can be applied thanks to the restriction $0 \leq k \leq 14$. As a result, we obtain an estimate of $\|{}^{(w_i)}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}_s^{2,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^k}$ in terms of $\mathcal{E}, \mathcal{F}$ and $\|\partial_{t,x}\underline{A}\|$. One may then check that all terms that arise are at least quadratic in the latter three quantities, and furthermore that each $\|\partial_{t,x}\underline{A}\|$ which has arisen can be estimated by $\underline{\mathcal{A}}$, thanks again to the restriction $0 \leq k \leq 14$. In the end, we obtain, for $0 \leq k \leq 14$ and $1 \leq p \leq \infty$, the following estimate:

$$\sup_i \|{}^{(w_i)}\mathcal{N}_{\text{forcing}}\|_{\mathcal{L}_s^{1+1,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^k(0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}+\mathcal{F}+\underline{\mathcal{A}})^2.$$

- *Case 3.2: Contribution of* ${}^{(w_i)}\mathcal{N}_{\mathrm{linear}}$. As before, we utilize the fact that ${}^{(w_i)}\mathcal{N}_{\mathrm{linear}}$ looks schematically the same as ${}^{(F_{s0})}\mathcal{N}_{\mathrm{linear}}$. Consequently, Step 2.2 of the proof of Proposition 4.5.14 implies

$$\sup_i \| {}^{(w_i)}\mathcal{N}_{\mathrm{linear}} \|_{\mathcal{L}^{1+1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,\underline{s}]} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \| s^{1/4-\epsilon'} \nabla_x w \|_{\mathcal{L}^{1,r}_s \mathcal{L}^2_t \mathcal{H}^k_x(0,\underline{s}]},$$

for $\epsilon' > 0$ small, $0 \leq k \leq 14$, $1 \leq p \leq r \leq \infty$ and $0 < \underline{s} \leq 1$. Combined with the previous case, we obtain (4.5.52). □

Again, by essentially the same proof, the following difference analogue of Proposition 4.5.17 follows.

Proposition 4.5.19 (Parabolic estimates for δw_i)**.** *Suppose* $0 < T \leq 1$, *and that the caloric-temporal gauge condition holds.*

1. *Let* $t \in (-T, T)$. *For* $1 \leq m \leq 30$ *we have*

$$\|\delta w_i(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta w_i(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E}(t) + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.53}$$

In the case $m = 31$, *on the other hand, we have the following estimate.*

$$\begin{aligned} &\|\delta w_i(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^{30}_x(0,1]} + \|\delta w_i(t)\|_{\mathcal{L}^{1,2}_s \dot{\mathcal{H}}^{31}_x(0,1]} \\ &\leq C_{\mathcal{E}(t),\mathcal{F},\underline{\mathcal{A}},\|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x}} \cdot (\mathcal{E}(t) + \mathcal{F} + \underline{\mathcal{A}} + \|\partial_0 \underline{A}(t)\|_{\dot{H}^{30}_x}) \\ &\quad \times (\delta\mathcal{E}(t) + \delta\mathcal{F} + \delta\underline{\mathcal{A}} + \|\partial_0(\delta\underline{A})(t)\|_{\dot{H}^{30}_x}) \end{aligned} \tag{4.5.54}$$

2. *For* $1 \leq m \leq 16$, *we have*

$$\|\delta w_i\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x(0,1]} + \|\delta w_i\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m}_x(0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.55}$$

Furthermore, for $0 \leq k \leq 14$, *we have the following estimate for* ${}^{(\delta w_i)}\mathcal{N} := (\partial_s - \triangle)(\delta w_i)$.

$$\| {}^{(\delta w_i)}\mathcal{N} \|_{\mathcal{L}^{2,\infty}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,1]} + \| {}^{(\delta w_i)}\mathcal{N} \|_{\mathcal{L}^{2,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^k_x(0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.5.56}$$

4.6 Proofs of Propositions 4.4.1 - 4.4.4

In this section, we shall sketch the proofs of Propositions 4.4.1 - 4.4.4.

Proof of Proposition 4.4.1. We shall give a proof of the non-difference estimate (4.4.1), leaving the similar case of the difference estimate (4.4.2) to the reader.

In what follows, we work on the time interval $I = (-T, T)$. Recalling the definition of $\overline{\mathcal{A}}_0$, we need to estimate $\|\overline{A}_0\|_{L^\infty_t \dot{H}^{1/2}_x}$, $\|\overline{A}_0\|_{L^\infty_t \dot{H}^1_x}$, $\|\overline{A}_0\|_{L^1_t(\dot{H}^{3/2}_x \cap L^\infty_x)}$ and $\|\overline{A}_0\|_{L^1_t \dot{H}^2_x}$ by the right-hand side of (4.4.1).

Fix $t \in I$. Using $\partial_s A_0 = F_{s0}$, the first two terms can be estimated simply by $C\mathcal{E}$ as follows.

$$\begin{aligned}\|\overline{A}_0(t)\|_{\dot{H}^{1/2}_x} + \|\overline{A}_0(t)\|_{\dot{H}^1_x} \leq & \int_0^1 (s')^{1/2}(s')\|F_{s0}(t,s')\|_{\dot{\mathcal{H}}^{1/2}_x(s')} \frac{\mathrm{d}s'}{s'} \\ & + \int_0^1 (s')^{1/4}(s')\|F_{s0}(t,s')\|_{\dot{\mathcal{H}}^1_x(s')} \frac{\mathrm{d}s'}{s'} \\ \leq & C\mathcal{E}.\end{aligned}$$

For the next term, using Hölder in time, it suffices to estimate $\|\overline{A}_0\|_{L^2_t(\dot{H}^{3/2} \cap L^\infty_x)}$. Using (4.5.36) of Proposition 4.5.14, along with Gagliardo-Nirenberg, interpolation and Sobolev, these are estimated as follows.

$$\begin{aligned}\|\overline{A}_0\|_{L^2_t(\dot{H}^{3/2} \cap L^\infty_x)} \leq & \int_0^1 (s')^{1/4}(s')\|F_{s0}(s')\|_{\mathcal{L}^2_t(\dot{\mathcal{H}}^{3/2}_x \cap \mathcal{L}^\infty_x)(s')} \frac{\mathrm{d}s'}{s'} \\ \leq & C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.\end{aligned}$$

Unfortunately, the same argument applied to the term $\|\overline{A}_0\|_{L^2_t L^2_x}$ fails by a logarithm. In this case, we make use of the equations $\partial_s A_0 = F_{s0}$ and the parabolic equation for F_{s0}. Indeed, let us begin by writing

$$\begin{aligned}\triangle \overline{A}_0 = & -\int_0^1 \triangle F_{s0}(s')\,\mathrm{d}s' = -\int_0^1 \partial_s F_{s0}(s')\,\mathrm{d}s' + \int_s^1 {}^{(F_{s0})}\mathcal{N}(s')\,\mathrm{d}s' \\ = & \underline{F}_{s0} + \int_s^1 s'({}^{(F_{s0})}\mathcal{N})(s') \frac{\mathrm{d}s'}{s'},\end{aligned}$$

where on the last line, we used the fact that $F_{s0}(s=0) = -w_0(s=0) = 0$. Taking the $L^2_{t,x}$ norm of the above identity and applying triangle and Minkowski, we obtain

$$\|\triangle \overline{A}_0\|_{L^2_{t,x}} \leq \|\underline{F}_{s0}\|_{L^2_{t,x}} + \int_s^1 s'\|{}^{(F_{s0})}\mathcal{N}(s')\|_{L^2_{t,x}} \frac{\mathrm{d}s'}{s'}.$$

The first term can be estimated using (4.5.36), whereas the last term can be estimated by putting together (4.5.39) (in the proof of Proposition 4.5.14) and (4.5.36). As a consequence, we obtain

$$\|\triangle \overline{A}_0\|_{L^2_{t,x}} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.$$

By $\|\overline{A}_0\|_{L^2_t \dot{H}^2_x} \leq C\|\triangle \overline{A}_0\|_{L^2_{t,x}}$ (which holds since $\overline{A}_0(t) \in H^\infty_x$ for every t) and Hölder in time, the desired $L^1_t L^2_x$-estimate follows. This completes the proof of (4.4.1). □

Proof of Proposition 4.4.2. Again, we shall only treat the non-difference case, as the difference case follows by essentially the same arguments.

The goal is to estimate $\sup_i \sup_{0 \leq s \leq 1} \|A_i(s)\|_{\dot{\mathcal{S}}^1}$ in terms of $\mathcal{F} + \underline{\mathcal{A}}$. Note that, proceeding naively, one can easily prove the bound

$$\|A_i(s)\|_{\dot{\mathcal{S}}^1} \leq \int_s^1 s' \|F_{si}(s')\|_{\dot{\mathcal{S}}^1} \frac{\mathrm{d}s'}{s'} + \|\underline{A}_i\|_{\dot{\mathcal{S}}^1} \leq |\log s|^{1/2} \mathcal{F} + \underline{\mathcal{A}}. \tag{4.6.1}$$

The essential reason for having a logarithm is that we have an *absolute integral* of $\|F_{si}(s')\|_{\dot{\mathcal{S}}^1}$ in the inequality, whereas $\mathcal{F}$ only controls its *square integral.* The idea then is to somehow replace this absolute integral with a square integral, using the structure of the Yang-Mills system.

We start with the equation satisfied by A_i under the condition $A_s = 0$.

$$\partial_s A_i = \triangle A_i - \partial^\ell \partial_i A_\ell + {}^{(A_i)}\mathcal{N}', \tag{4.6.2}$$

where

$${}^{(A_i)}\mathcal{N}' = \mathcal{O}(A, \partial_x A) + \mathcal{O}(A, A, A).$$

Fix $t \in (-T, T)$. Let us take $\partial_{t,x}$ of (4.6.2), take the bi-invariant inner product[9] with $\partial_{t,x} A_i$ and integrate over $\mathbb{R}^3 \times [s, 1]$, for $0 < s \leq 1$. Summing up in i and performing integration by parts, we obtain the following identity.

$$\begin{aligned}\frac{1}{2}\sum_i \int |\partial_{t,x} A_i(s)|^2 \, \mathrm{d}x =& \frac{1}{2}\sum_i \int |\partial_{t,x} \underline{A}_i|^2 \, \mathrm{d}x - \sum_i \int_s^1 \int s' (\partial_{t,x}({}^{(A_i)}\mathcal{N}'), \partial_{t,x} A_i)(s') \, \mathrm{d}x \, \frac{\mathrm{d}s'}{s'} \\ &+ \sum_{i,\ell} \int_s^1 \int s' |\partial_\ell \partial_{t,x} A_i(s')|^2 \, \mathrm{d}x \frac{\mathrm{d}s'}{s'} - \sum_\ell \int_s^1 s' |\partial_{t,x} \partial_\ell A_\ell(s')|^2 \, \mathrm{d}x \, \frac{\mathrm{d}s'}{s'}.\end{aligned}$$

Take the supremum over $0 \leq s \leq 1$, and apply Cauchy-Schwarz and Hölder to deal with the second term on the right-hand side. Then taking the supremum over $t \in (-T, T)$ and applying

[9] In fact, for the purpose of this argument, it is possible to use any inner product on $\mathfrak{g}$ for which Leibniz's rule holds, so that integration by parts works.

Minkowski, we easily arrive at the following inequality.

$$\sup_{0\leq s\leq 1} \|\partial_{t,x} A(s)\|_{L^\infty_t L^2_x} \leq C\|\partial_{t,x}\underline{A}\|_{L^\infty_t L^2_x} + C\Big(\int_0^1 s\|\partial_x\partial_{t,x}A(s)\|^2_{L^\infty_t L^2_x}\frac{\mathrm{d}s}{s}\Big)^{1/2} + C\sup_i \int_0^1 s\|\partial_{t,x}({}^{(A_i)}\mathcal{N}')(s)\|_{L^\infty_t L^2_x}\frac{\mathrm{d}s}{s}.$$

Similarly, taking $\Box$ of (4.6.2), multiplying by $\Box A_i$, integrating over $(-T,T)\times\mathbb{R}^3\times[s,1]$ etc, we can also prove

$$\sup_{0\leq s\leq 1} \|\Box A(s)\|_{L^2_{t,x}} \leq C\|\Box\underline{A}\|_{L^2_{t,x}} + C\Big(\int_0^1 s\|\partial_x\Box A(s)\|^2_{L^2_{t,x}}\frac{\mathrm{d}s}{s}\Big)^{1/2} + C\sup_i \int_0^1 s\|\Box({}^{(A_i)}\mathcal{N}')(s)\|_{L^2_{t,x}}\frac{\mathrm{d}s}{s}.$$

Combining the last two inequalities and recalling the definition of the norm $\dot{S}^k$, we get

$$\sup_{0\leq s\leq 1}\|A(s)\|_{\dot{S}^1} \leq C\|\underline{A}\|_{\dot{S}^1} + C\Big(\int_0^1 s\|A(s)\|^2_{\dot{S}^2}\frac{\mathrm{d}s}{s}\Big)^{1/2} + C\sup_i\int_0^1 s\|{}^{(A_i)}\mathcal{N}'(s)\|_{\dot{S}^1}\frac{\mathrm{d}s}{s}.$$

Applying Lemma 4.5.2 (with $p=q=2$) to the second term on the right-hand side, we finally arrive at the following inequality.

$$\sup_{0\leq s\leq 1}\|A(s)\|_{\dot{S}^1} \leq C\|\underline{A}\|_{\dot{S}^1} + C(\|F_s\|_{\mathcal{L}^{5/4,2}_s\dot{\mathcal{S}}^2} + \|\underline{A}\|_{\dot{S}^2}) + C\sup_i\int_0^1 s\|{}^{(A_i)}\mathcal{N}'(s)\|_{\dot{S}^1}\frac{\mathrm{d}s}{s}. \qquad (4.6.3)$$

All terms on the right-hand side except the last term can be controlled by $C(\mathcal{F}+\underline{\mathcal{A}})$. Therefore, all that is left to show is that the last term on the right-hand side of (4.6.3) is okay. To this end, we claim

$$\sup_i\int_0^1 s\|{}^{(A_i)}\mathcal{N}'(s)\|_{\dot{S}^1}\frac{\mathrm{d}s}{s} \leq C_{\mathcal{F}+\underline{\mathcal{A}}}\cdot(\mathcal{F}+\underline{\mathcal{A}})^2.$$

Recalling the definition of the $\dot{S}^1$ norm, we must bound the contribution of $\|\partial_{t,x}({}^{(A_i)}\mathcal{N}')(s)\|_{L^\infty_t L^2_x}$ and $T^{1/2}\|\Box({}^{(A_i)}\mathcal{N}')(s)\|_{L^2_{t,x}}$. We shall only treat the latter (which is slightly more complicated), leaving the former to the reader.

Using the product rule for $\Box$, we compute the schematic form of $\Box({}^{(A_i)}\mathcal{N}')$ as follows.

$$\Box^{(A_i)}\mathcal{N}' = \mathcal{O}(\partial^\mu A, \partial_x\partial_\mu A) + \mathcal{O}(A,\partial^\mu A,\partial_\mu A) + \mathcal{O}(\Box A,\partial_x A) + \mathcal{O}(A,\partial_x\Box A) + \mathcal{O}(A,A,\Box A).$$

Let us treat each type in order. Terms of the first type are the most dangerous, in the sense that there is absolutely no extra s-weight to spare. Using Hölder, Cauchy-Schwarz and Strichartz, we have

$$\begin{aligned}
&\int_0^1 sT^{1/2} \|\mathcal{O}(\partial^\mu A(s), \partial_x \partial_\mu A(s))\|_{L^2_{t,x}} \frac{\mathrm{d}s}{s} \\
\leq& CT^{1/2} \Big(\int_0^1 s^{1/2} \|A(s)\|^2_{\dot{\mathcal{S}}^{3/2}} \frac{\mathrm{d}s}{s} \Big)^{1/2} \Big(\int_0^1 s^{3/2} \|A(s)\|^2_{\dot{\mathcal{S}}^{5/2}} \frac{\mathrm{d}s}{s} \Big)^{1/2}.
\end{aligned}$$

Using Lemma 4.5.2, the last line can be estimated by $C(\mathcal{F} + \underline{\mathcal{A}})^2$, which is acceptable.

Terms of the second type can be treated similarly using Hölder, Strichartz and Lemma 4.5.2, being easier due to the presence of extra s-weights. We estimate these terms as follows.

$$\begin{aligned}
&\int_0^1 sT^{1/2} \|\mathcal{O}(A, \partial^\mu A(s), \partial_\mu A(s))\|_{L^2_{t,x}} \frac{\mathrm{d}s}{s} \\
\leq& T^{1/2} \int_0^1 s^{1/4} \Big(s^{1/4} \|A(s)\|_{L^\infty_{t,x}} \Big) \Big(s^{1/4} \|\partial^\mu A(s)\|_{L^4_{t,x}} \Big) \Big(s^{1/4} \|\partial_\mu A(s)\|_{L^4_{t,x}} \Big) \frac{\mathrm{d}s}{s} \\
\leq& CT^{1/2} (\mathcal{F} + \underline{\mathcal{A}})^3.
\end{aligned}$$

The remaining terms all involve the d'Alembertian $\Box$. For these terms, using Hölder, we always put the factor with $\Box$ in $L^2_{t,x}$ and estimate by the $\dot{\mathcal{S}}^k$ norm, whereas the other terms are put in $L^\infty_{t,x}$. We shall always have some extra s-weight, and thus it is not difficult to show that

$$\int_0^1 sT^{1/2} \|\mathcal{O}(\Box A(s), \partial_x A(s)) + \mathcal{O}(A(s), \partial_x \Box A(s))\|_{L^2_{t,x}} \frac{\mathrm{d}s}{s} \leq C(\mathcal{F} + \underline{\mathcal{A}})^2,$$

$$\int_0^1 sT^{1/2} \|\mathcal{O}(A(s), A(s), \Box A(s))\|_{L^2_{t,x}} \frac{\mathrm{d}s}{s} \leq C(\mathcal{F} + \underline{\mathcal{A}})^3.$$

As desired, we have therefore proved

$$\sup_i \int_0^1 sT^{1/2} \|\Box({}^{(A_i)}\mathcal{N}')(s)\|_{L^2_{t,x}} \frac{\mathrm{d}s}{s} \leq C_{\mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2. \qquad \Box$$

Proof of Proposition 4.4.3. This is an immediate consequence of Propositions 4.5.13 and 4.5.15. $\Box$

Proof of Proposition 4.4.4. In fact, this proposition is a triviality in view of the simple definitions of the quantities $\mathcal{F}, \underline{\mathcal{A}}, \delta\mathcal{F}, \delta\underline{\mathcal{A}}$ and the fact that $A_{\mathbf{a}}$, $A'_{\mathbf{a}}$ are regular solutions to (HPYM). $\Box$

4.7 Hyperbolic estimates : Proofs of Theorems C and D

The purpose of this section is to prove Theorems C and D, which are based on analyzing the wave-type equations (4.1.5) and (4.1.6) for $\underline{A}_i$ and F_{si}, respectively. Note that the system of equations for $\underline{A}_i$ is nothing but the Yang-Mills equations with source in the temporal gauge. The standard way of solving this system (see [9]) is by deriving a wave equation for $F_{\mu\nu}$; due to a technical point, however, we shall take a slightly different route, which will be explained further in §4.7.1. The wave equation (2.1.6) for F_{si}, on the other hand, shares many similarities with that for A_i in the Coulomb gauge, as discussed in the Introduction. In particular, one can recover the null structure for the most dangerous bilinear interaction $[A^\ell, \partial_\ell F_{si}]$, which is perhaps the most essential analytic structure of (HPYM) in the caloric-temporal gauge which makes the whole proof work.

Throughout this section, we shall work with regular solutions $A_{\mathbf{a}}, A'_{\mathbf{a}}$ to (HPYM) on $(-T,T) \times \mathbb{R}^3 \times [0,1]$.

4.7.1 Proof of Theorem C

Recall that at $s = 1$, the connection coefficients $\underline{A}_\mu = A_\mu(s=1)$ satisfy the hyperbolic Yang-Mills equation with source, i.e.

$$\underline{\mathbf{D}}^\mu \underline{F}_{\nu\mu} = \underline{w}_\nu \text{ for } \nu = 0,1,2,3. \tag{4.7.1}$$

Furthermore, we have the temporal gauge condition $\underline{A}_0 = 0$.

Recall that $(\partial \times B)_i := \sum_{j,k} \epsilon_{ijk} \partial_j B_k$, where ϵ_{ijk} was the Levi-Civita symbol. In the proposition below, we record the equation of motion of $\underline{A}_i$, which are obtained simply by expanding (4.7.1) in terms of $\underline{A}_i$.

Proposition 4.7.1 (Equations for $\underline{A}_i$)**.** *The Yang-Mills equation with source* (4.7.1) *is equivalent to the following system of equations.*

$$\partial_0(\partial^\ell \underline{A}_\ell) = -[\underline{A}^\ell, \partial_0 \underline{A}_\ell] + \underline{w}_0, \tag{4.7.2}$$

$$\Box \underline{A}_i - \partial_i(\partial^\ell \underline{A}_\ell) = -2[\underline{A}^\ell, \partial_\ell \underline{A}_i] + [\underline{A}_i, \partial^\ell \underline{A}_\ell] + [\underline{A}^\ell, \partial_i \underline{A}_\ell] - [\underline{A}^\ell, [\underline{A}_\ell, \underline{A}_i]] - \underline{w}_i. \tag{4.7.3}$$

Taking the curl (i.e. $\partial \times \cdot$) of (4.7.3)*, we obtain the following wave equation for $\partial \times \underline{A}$.*

$$\begin{aligned}\Box(\partial \times \underline{A})_i = & -\partial \times (2[\underline{A}^\ell, \partial_\ell \underline{A}_i] + [\underline{A}_i, \partial^\ell \underline{A}_\ell] + [\underline{A}^\ell, \partial_i \underline{A}_\ell]) \\ & -\partial \times ([\underline{A}^\ell, [\underline{A}_\ell, \underline{A}_i]]) - (\partial \times \underline{w})_i.\end{aligned} \tag{4.7.4}$$

Remark 4.7.2. The usual procedure of solving (4.7.1) in temporal gauge consists of first deriving the hyperbolic equation for $\underline{F}_{\nu\mu}$, using the Bianchi identity and (4.7.1). Then one couples these equations with the transport equation

$$\underline{F}_{0i} = \partial_0 \underline{A}_i,$$

(which follows just from the definition of $\underline{F}_{0i}$ and the temporal gauge condition $\underline{A}_0 = 0$) and solves the system altogether. This is indeed the approach of Eardley-Moncrief [9] and Klainerman-Machedon [15]. A drawback to this approach, however, is that it requires taking a t-derivative when deriving hyperbolic equations for $\underline{F}_{0i}$. In particular, one has to estimate $\partial_0 \underline{w}_i$, which complicates matters in our setting.

The equations that we stated in Proposition 4.7.1 is the basis for a slightly different approach, which avoids taking ∂_0 at the expense of using a little bit of Hodge theory. We remark that such an approach had been taken by Tao [33], but with greater complexity than here as the paper was concerned with lower regularity (but small data) solutions to (YM).

We are now ready to prove Theorem C.

Proof of Theorem C. In the proof, we shall work on the time interval $(-T, T)$, where $0 < T \leq 1$. We shall give a rather detailed proof of (4.4.5). The difference analogue (4.4.6) can be proved in an analogous manner, whose details we leave to the reader.

Let us begin with a few product estimates.

$$\|\mathcal{O}(\underline{A}, \partial_0 \underline{A})\|_{L^\infty_t \dot{H}^m_x} \leq C\underline{\mathcal{A}}^2, \qquad \text{for } 0 \leq m \leq 29, \tag{4.7.5}$$

$$\|\mathcal{O}(\underline{A}, \partial_x \underline{A})\|_{L^\infty_t \dot{H}^m_x} \leq C\underline{\mathcal{A}}^2, \qquad \text{for } 0 \leq m \leq 30, \tag{4.7.6}$$

$$\|\mathcal{O}(\underline{A}, \underline{A}, \underline{A})\|_{L^\infty_t \dot{H}^m_x} \leq C\underline{\mathcal{A}}^3, \qquad \text{for } 0 \leq m \leq 31. \tag{4.7.7}$$

Each of these can be proved by Leibniz's rule, Hölder and Sobolev, as well as the fact that $\|\partial_x \underline{A}\|_{L^\infty_t H^{30}_x} + \|\partial_0 \underline{A}\|_{L^\infty_t H^{29}_x} \leq \underline{\mathcal{A}}$. Using the same techniques, we can also prove the following weaker version of (4.7.5) in the case $m = 30$:

$$\|\mathcal{O}(\underline{A}, \partial_0 \underline{A})\|_{L^\infty_t \dot{H}^{30}_x} \leq C\underline{\mathcal{A}}^2 + C\|\underline{A}\|_{L^\infty_{t,x}} \|\partial_x^{(30)} \partial_0 \underline{A}\|_{L^\infty_t L^2_x}. \tag{4.7.8}$$

Next, observe that $\|\underline{w}_0\|_{L^\infty_t \dot{H}^m_x} \leq \sup_{t \in (-T,T)} \|F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^m_x}$, where the latter can be controlled by (4.5.35) for $0 \leq m \leq 30$. Combining this with (4.7.2), (4.7.5), we obtain the following estimate for $0 \leq m \leq 29$:

$$\|\partial_0 (\partial^\ell \underline{A}_\ell)\|_{L^\infty_t \dot{H}^m_x} \leq C_{\mathcal{F}, \underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.7.9}$$

In the case $m = 30$, replacing the use of (4.7.5) by (4.7.8), we have

$$\|\partial_0(\partial^\ell \underline{A}_\ell)\|_{L^\infty_t \dot{H}^{30}_x} \leq C\underline{\mathcal{A}}\|\partial_x^{(30)}\partial_0 \underline{A}\|_{L^\infty_t L^2_x} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.$$

Recall the simple div-curl identity $\sum_{i,j} \|\partial_i B_j\|^2 = \frac{1}{2}\|\partial \times B\|^2_{L^2_x} + \|\partial^\ell B_\ell\|^2$ with $B = \underline{A}(t)$. Using furthermore (4.7.9) with $m = 29$ and the fact that $\underline{\mathcal{A}}$ controls $\|\partial_x^{(29)}\partial_0(\partial \times \underline{A})\|_{L^\infty_t L^2_x}$, we obtain the following useful control on $\|\partial_x^{(30)}\partial_0 \underline{A}\|_{L^\infty_t L^2_x}$:

$$\|\partial_x^{(30)}\partial_0 \underline{A}\|_{L^\infty_t L^2_x} \leq C\underline{\mathcal{A}} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.7.10}$$

Therefore, (4.7.9) holds in the case $m = 30$ as well, i.e.

$$\|\partial_0(\partial^\ell \underline{A}_\ell)\|_{L^\infty_t \dot{H}^{30}_x} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.$$

Integrating (4.7.9) with respect to t from $t = 0$, we obtain for $0 \leq m \leq 30$

$$\|\partial^\ell \underline{A}_\ell\|_{L^\infty_t \dot{H}^m_x} \leq \mathcal{I} + T\Big(C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2\Big). \tag{4.7.11}$$

Next, observe that $\|\underline{w}_i\|_{L^\infty_t \dot{H}^m_x} \leq \sup_{t \in (-T,T)} \|w_i(t)\|_{\mathcal{L}^{1,\infty}_s \dot{\mathcal{H}}^m_x}$. Combining this observation with (4.5.45) and (4.5.46) from Proposition 4.5.17, as well as (4.7.10) to control $\|\partial_0 \underline{A}\|_{\dot{H}^{30}_x}$, we have the following estimates for $0 \leq m \leq 30$:

$$\|\underline{w}_i\|_{L^\infty_t \dot{H}^m_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.7.12}$$

We are now ready to finish the proof. Let $i = 1, 2, 3$ and $1 \leq m \leq 30$. By the energy inequality and Hölder, we have

$$\|\underline{A}_i\|_{\dot{S}^m} \leq C\|\partial_{t,x}\underline{A}(t=0)\|_{\dot{H}^{m-1}_x} + CT\|\Box \underline{A}_i\|_{L^\infty_t \dot{H}^{m-1}_x}.$$

The first term is controlled by $C\mathcal{I}$. To control the second term, apply (4.7.3), (4.7.6), (4.7.7), (4.7.12). Furthermore, use (4.7.11) to control the contribution of $\partial_i \partial^\ell \underline{A}_\ell$. As a result, we obtain

$$\|\underline{A}_i\|_{\dot{S}^m} \leq C\mathcal{I} + T\Big(C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2\Big), \tag{4.7.13}$$

for $i = 1, 2, 3$ and $1 \leq m \leq 30$.

Similarly, by the energy inequality and Hölder, we have

$$\|(\partial \times \underline{A})_i\|_{\dot{S}^{30}} \leq C\|\partial_{t,x}(\partial \times \underline{A})(t=0)\|_{\dot{H}^{29}_x} + CT\|\Box(\partial \times \underline{A})_i\|_{L^\infty_t \dot{H}^{30}_x}.$$

The first term is again controlled by $C\mathcal{I}$. To control the second term, we apply (4.7.4), (4.7.6), (4.7.7), (4.7.12); note that this time we do not need an estimate for $\partial^\ell \underline{A}_\ell$. We conclude

$$\|(\partial \times \underline{A})_i\|_{\dot{S}^{30}} \leq C\mathcal{I} + T\, C_{\mathcal{E},\mathcal{F},\underline{A}} \cdot (\mathcal{E} + \mathcal{F} + \underline{A})^2. \tag{4.7.14}$$

Finally, using the div-curl identity, (4.7.11) and (4.7.14), we have

$$\|\underline{A}\|_{\dot{H}^{31}_x} \leq C\mathcal{I} + T\Big(C_{\mathcal{F},\underline{A}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{A}} \cdot (\mathcal{F} + \underline{A})^2\Big).$$

This concludes the proof. □

4.7.2 Proof of Theorem D

Let us recall the hyperbolic equation (2.1.6) satisfied by F_{si}:

$$\mathbf{D}^\mu \mathbf{D}_\mu F_{si} = 2[{F_s}^\mu, F_{i\mu}] - \mathbf{D}^\ell \mathbf{D}_\ell w_i + \mathbf{D}_i \mathbf{D}^\ell w_\ell - {}^{(w_i)}\mathcal{N}.$$

Note that we have rewritten $2[{F_i}^\ell, w_\ell] + 2[F^{\mu\ell}, \mathbf{D}_\mu F_{i\ell} + \mathbf{D}_\ell F_{i\mu}] = {}^{(w_i)}\mathcal{N}$ for convenience.

Let us begin by rewriting the wave equation for F_{si} in a form more suitable for our analysis. Writing out the covariant derivatives in (2.1.6), we obtain the following semi-linear wave equation for F_{si}.

$$\Box F_{si} = {}^{(F_{si})}\mathcal{M}_{\text{quadratic}} + {}^{(F_{si})}\mathcal{M}_{\text{cubic}} + {}^{(F_{si})}\mathcal{M}_w,$$

where

$$\begin{aligned}
{}^{(F_{si})}\mathcal{M}_{\text{quadratic}} :=& -2[A^\ell, \partial_\ell F_{si}] + 2[A_0, \partial_0 F_{si}] \\
&+ [\partial_0 A_0, F_{si}] - [\partial^\ell A_\ell, F_{si}] - 2[{F_i}^\ell, F_{s\ell}] + 2[F_{i0}, F_{s0}], \\
{}^{(F_{si})}\mathcal{M}_{\text{cubic}} :=& [A_0, [A_0, F_{si}]] - [A^\ell, [A_\ell, F_{si}]] \\
{}^{(F_{si})}\mathcal{M}_w :=& -\mathbf{D}^\ell \mathbf{D}_\ell w_i + \mathbf{D}_i \mathbf{D}^\ell w_\ell - {}^{(w_i)}\mathcal{N}.
\end{aligned}$$

The semi-linear equation for the difference $\delta F_{si} := F_{si} - F'_{si}$ is then given by

$$\Box \delta F_{si} = {}^{(\delta F_{si})}\mathcal{M}_{\mathrm{quadratic}} + {}^{(\delta F_{si})}\mathcal{M}_{\mathrm{cubic}} + {}^{(\delta F_{si})}\mathcal{M}_{w},$$

where ${}^{(\delta F_{si})}\mathcal{M}_{\mathrm{quadratic}} := {}^{(F_{si})}\mathcal{M}_{\mathrm{quadratic}} - {}^{(F'_{si})}\mathcal{M}_{\mathrm{quadratic}}$, ${}^{(\delta F_{si})}\mathcal{M}_{\mathrm{cubic}} := {}^{(F_{si})}\mathcal{M}_{\mathrm{cubic}} - {}^{(F'_{si})}\mathcal{M}_{\mathrm{cubic}}$ and ${}^{(\delta F_{si})}\mathcal{M}_{w} := {}^{(F_{si})}\mathcal{M}_{w} - {}^{(F'_{si})}\mathcal{M}_{w}$.

It is here that we shall use the null structure of (HPYM). As discussed earlier, for the purpose of proving Theorem D, we shall not need the full null structure uncovered in §2.2, but only for the term $[A^{\ell}, \partial_{\ell} F_{si}]$, or more precisely, $[(A^{\mathrm{df}})^{\ell}, \partial_{\ell} F_{si}]$.

Estimates for quadratic terms

We shall begin the proof of Theorem D by estimating the contribution of quadratic terms.

Lemma 4.7.3 (Estimates for quadratic terms)**.** *Assume* $0 < T \leq 1$. *For* $1 \leq m \leq 10$ *and* $p = 2, \infty$, *the following estimates hold.*

$$\sup_i \| {}^{(F_{si})}\mathcal{M}_{\mathrm{quadratic}} \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.7.15}$$

$$\sup_i \| {}^{(\delta F_{si})}\mathcal{M}_{\mathrm{quadratic}} \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.7.16}$$

Proof. We shall give a rather detailed proof of (4.7.15). The other estimate (4.7.16) may be proved by first using Leibniz's rule for δ to compute ${}^{(\delta F_{si})}\mathcal{M}_{\mathrm{quadratic}}$, and then proceeding in an analogous fashion. We shall omit the proof of the latter.

Let $1 \leq m \leq 10$ and $p = 2$ or ∞. We shall work on the whole s-interval $(0, 1]$. Let us begin with an observation that in order to prove (4.7.15), it suffices to prove that each of the following can be bounded by $C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$:

$$\left\{ \begin{aligned} & \|s^{-1/2}[(A^{\mathrm{cf}})^{\ell}, \nabla_{\ell} \nabla^{(m-1)}_x F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}}, \quad \|s^{-1/2}[(A^{\mathrm{df}})^{\ell}, \nabla_{\ell} \nabla^{(m-1)}_x F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}}, \\ & \sum_{j=1}^{m-1} \|s^{-1/2}[\nabla^{(j)}_x A^{\ell}, \nabla_{\ell} \nabla^{(m-1-j)}_x F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}}, \\ & \|s^{-1/2}[A_0, \nabla_0 F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x}, \quad \|s^{-1/2}[\nabla_0 A_0, F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x}, \\ & \|[F_{i0}, F_{s0}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x}, \quad \|[F_i{}^{\ell}, F_{s\ell}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x}. \end{aligned} \right.$$

Here, A^{cf} and A^{df}, called the *curl-free* and the *divergence-free* parts of A, respectively, constitute the *Hodge decomposition* of A, i.e. $A_i = A_i^{\mathrm{cf}} + A_i^{\mathrm{df}}$. They are defined by the formulae

$$A^{\mathrm{cf}} := -(-\triangle)^{-1}\partial_i\partial^\ell A_\ell, \quad A^{\mathrm{df}} := (-\triangle)^{-1}(\partial \times (\partial \times A))_i.$$

Let us treat each of them in order.

- *Case 1 : Proof of* $\|s^{-1/2}[(A^{\mathrm{cf}})^\ell, \nabla_\ell \nabla_x^{(m-1)} F_{si}]\|_{\mathcal{L}_s^{2,p}\mathcal{L}_{t,x}^2} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E}+\mathcal{F}+\underline{\mathcal{A}})^2$.

We claim that the following estimate for A^{cf} holds.

$$\|A^{\mathrm{cf}}(s)\|_{\mathcal{L}_s^{1/4,\infty}\mathcal{L}_t^2\mathcal{L}_x^\infty} \leq C\underline{\mathcal{A}} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F}+\underline{\mathcal{A}})^2. \tag{4.7.17}$$

Note, on the other hand, that $\|\nabla_\ell \nabla_x^{(m-1)} F_{si}\|_{\mathcal{L}_s^{5/4,p}\mathcal{L}_t^\infty\mathcal{L}_x^2} \leq \mathcal{F}$ for $1 \leq m \leq 10$. Assuming the claim, the desired estimate then follows immediately by Hölder.

The key to our proof of (4.7.17) is the covariant Coulomb condition satisfied by F_{si}

$$\mathbf{D}^\ell F_{s\ell} = 0,$$

which was proved in Chapter 2. Writing out the covariant derivative $\mathbf{D}^\ell = \partial^\ell + A^\ell$ and using the relation $F_{s\ell} = \partial_s A_\ell$, we arrive at the following *improved transport equation* for $\partial^\ell A_\ell$.

$$\partial_s\big(\partial^\ell A_\ell(s)\big) = -[A^\ell(s), F_{s\ell}(s)]. \tag{4.7.18}$$

Observe furthermore that $\|A_\ell^{\mathrm{cf}}(s)\|_{\mathcal{L}_s^{1/4,\infty}\mathcal{L}_t^2\mathcal{L}_x^\infty} = \sup_{0<s\leq 1}\|A^{\mathrm{cf}}(s)\|_{L_t^2 L_x^\infty}$. Our goal, therefore, is to estimate the latter by using (4.7.18).

Using the fundamental theorem of calculus and Minkowski, we obtain, for $1 \leq r \leq \infty$, the inequality

$$\sup_{0<s\leq 1}\|\partial^\ell A_\ell(s)\|_{L_t^2 L_x^r} \leq \|\partial^\ell \underline{A}_\ell\|_{L_t^2 L_x^r} + \int_0^1 \|[A^\ell(s), F_{s\ell}(s)]\|_{L_t^2 L_x^r}\,\mathrm{d}s. \tag{4.7.19}$$

Let us recall that $(A^{\mathrm{cf}})_i = (-\triangle)^{-1}\partial_i\partial^\ell A_\ell$ by Hodge theory. It then follows that $\partial_i(A^{\mathrm{cf}})_j = R_i R_j(\partial^\ell A_\ell)$, where R_i, R_j are Riesz transforms. By elementary harmonic analysis [32], for $1 < r < \infty$, we have the inequality

$$\|\partial_x A^{\mathrm{cf}}\|_{L_t^2 L_x^r} \leq C_r \|\partial^\ell A_\ell\|_{L_t^2 L_x^r}.$$

On the other hand, using Sobolev and Gagliardo-Nirenberg, we have

$$\|A^{\mathrm{cf}}\|_{L_t^2 L_x^\infty} \leq C\|\partial_x A^{\mathrm{cf}}\|_{L_{t,x}^2}^{1/3}\|\partial_x A^{\mathrm{cf}}\|_{L_t^2 L_x^4}^{2/3}.$$

As a result of these two inequalities, it suffices to bound the $L^2_{t,x}$ and $L^2_t L^4_x$ norms of $\partial^\ell A_\ell(s)$ using (4.7.19). For the first term on the right-hand side of (4.7.19), we obviously have

$$\|\partial^\ell \underline{A}_\ell\|_{L^2_{t,x}} + \|\partial^\ell \underline{A}_\ell\|_{L^2_t L^4_x} \leq CT^{1/2}(\|\partial^\ell \underline{A}_\ell\|_{L^\infty_t L^2_x} + \|\partial^\ell \underline{A}_\ell\|_{L^\infty_t L^4_x}) \leq C\underline{\mathcal{A}}.$$

by Hölder in time. Next, note that the second term on the right-hand side of (4.7.19) is equal to $\|[A^\ell, F_{s\ell}]\|_{\mathcal{L}^{\ell_r,1}_s \mathcal{L}^2_t \mathcal{L}^r_x}$, where $\ell_r = \frac{5}{4} + \frac{3}{2r}$. In the case $r = 2$, we estimate this, using Lemma 4.5.2 and Proposition 4.5.11, as follows.

$$\|[A^\ell, F_{s\ell}]\|_{\mathcal{L}^{2,1}_s \mathcal{L}^2_{t,x}} \leq CT^{1/2}\|s^{1/4}\|_{\mathcal{L}^2_s}\|A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_{t,x}}\|F_s\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^\infty_t \mathcal{L}^2_x} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.$$

In the other case $r = 4$, we proceed similarly, again using Lemma 4.5.2 and Proposition 4.5.11.

$$\|[A^\ell, F_{s\ell}]\|_{\mathcal{L}^{13/8,1}_s \mathcal{L}^2_t \mathcal{L}^4_x} \leq C\|s^{1/8}\|_{\mathcal{L}^2_s}\|A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^4_t \mathcal{L}^\infty_x}\|F_s\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^4_{t,x}} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2.$$

Combining these estimates, we obtain (4.7.17).

- *Case 2 : Proof of* $\|s^{-1/2}[(A^{\mathrm{df}})^\ell, \nabla_\ell \nabla^{(m-1)}_x F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

In this case, we *cannot* estimate A^{df} in $L^2_t L^\infty_x$. Here, we need to look more closely into the exact form of the nonlinearity, and recover a *null form*, à la Klainerman [12], Christodoulou [5] and Klainerman-Machedon [13]. We remark that this is the only place where we utilize the null form estimate in our proof.

For $B = B_i$ $(i = 1, 2, 3)$, ϕ smooth and $B_i, \phi \in \dot{S}^1$, we claim that the following estimate holds for $0 < s \leq 1$:

$$\|[(B^{\mathrm{df}})^\ell, \partial_\ell \phi]\|_{L^2_{t,x}} \leq C(\sup_k \|B_k\|_{\dot{S}^1})\|\phi\|_{\dot{S}^1}. \tag{4.7.20}$$

Assuming the claim, by the Correspondence Principle, we then obtain the estimate

$$\|s^{-1/2}[(\mathcal{T}^{\mathrm{df}})^\ell, \nabla_\ell \psi]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}} \leq C(\sup_k \|\mathcal{T}_k\|_{\mathcal{L}^{1/4,\infty}_s \dot{\mathcal{S}}^1})\|\psi\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{S}}^1},$$

for smooth $\mathcal{T} = \mathcal{T}_i(s)$ (i=1,2,3) ψ such that the right-hand side is finite. Let us take $\mathcal{T} = A$, $\psi = \nabla^{(m-1)}_x F_{si}$. By Proposition 4.4.2, we have $\|A\|_{\mathcal{L}^{1/4,\infty}_s \dot{\mathcal{S}}^1} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})$, whereas by definition $\|\nabla^{(m-1)}_x F_{si}\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{S}}^1} \leq C\mathcal{F}$ for $1 \leq m \leq 10$. The desired estimate therefore follows.

Now, it is only left to prove (4.7.20). The procedure that we are about to describe is standard, due to Klainerman-Machedon [14], [15]. We reproduce the argument here for the sake of completeness.

Let us first assume that B_i is Schwartz in x for every t, s. Then simple Hodge theory tells us that $B_i^{\mathrm{df}} = (\partial \times V)_i$, where

$$V_i(x) := (-\triangle)^{-1}(\partial \times B)_i(x) = \frac{1}{4\pi} \int \Big(B(y) \times \frac{(x-y)}{|x-y|^3} \Big)_i \mathrm{d}y,$$

where we suppressed the variables t, s. Substituting $(B^{\mathrm{df}})^\ell = (\partial \times V)^\ell$ on the left-hand side of (4.7.20), we have

$$\begin{aligned} \| \sum_{j,k,\ell} \epsilon_{\ell j k} [\partial_j V_k(s), \partial_\ell \psi(s)] \|_{L^2_{t,x}} \leq & \frac{1}{2} \sum_{j,k,\ell} \| Q_{j\ell}(V_k(s), \psi)(s) \|_{L^2_{t,x}} \\ \leq & C (\sup_k \| V_k(s) \|_{\dot{S}^2}) \| \psi(s) \|_{\dot{S}^1}, \end{aligned}$$

where we remind the reader that $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi$, and on the last line we used (4.2.9) of Proposition 4.2.2 (null form estimate). Since $\partial_j V_k = (-\triangle)^{-1} \partial_j (\partial \times B)_i$, and $\| \cdot \|_{\dot{S}^1}$ is an L^2_x-type norm, we see that

$$\sup_k \| V_k(s) \|_{\dot{S}^2} = \sup_{j,k} \| \partial_j V_k(s) \|_{\dot{S}^1} \leq C \sup_k \| B_k(s) \|_{\dot{S}^1},$$

from which (4.7.20) follows, under the additional assumption that B_i are Schwartz in x. Then, using the quantitative estimate (4.7.20), it is not difficult to drop the Schwartz assumption by approximation.

- *Case 3 : Proof of* $\sum_{j=1}^{m-1} \| s^{-1/2} [\nabla_x^{(j)} A^\ell, \nabla_\ell \nabla_x^{(m-1-j)} F_{si}] \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}} \leq C_{\mathcal{E}, \mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

By the Hölder inequality $L^4_{t,x} \cdot L^4_{t,x} \subset L^2_{t,x}$, the Correspondence Principle and Hölder for $\mathcal{L}^{\ell,p}_s$ (Lemma 3.1.8), we immediately obtain the estimate

$$\sum_{j=1}^{m-1} \| s^{-1/2} [\nabla_x^{(j)} A^\ell, \nabla_\ell \nabla_x^{(m-1-j)} F_{si}] \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_{t,x}} \leq C \sum_{j=1}^{m-1} \| A \|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{j,4}_x} \| F_{si} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{m-j,4}_x}.$$

Let us apply Lemma 4.5.2 to $\| A \|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{j,4}_x}$; as $1 \leq j \leq m-1 \leq 9$, this can be estimated by $C(\mathcal{F} + \underline{\mathcal{A}})$. On the other hand, as $1 \leq m-j \leq m-1 \leq 9$, $\| F_{si} \|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^4_t \dot{\mathcal{W}}^{m-j,4}_x}$ can be controlled by $C\mathcal{F}$ via Strichartz. The desired estimate follows.

- *Case 4 : Proof of* $\| s^{-1/2} [A_0, \nabla_0 F_{si}] \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \leq C_{\mathcal{E}, \mathcal{F}, \underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

By Leibniz's rule, the Hölder inequality $L^2_t L^\infty_x \cdot L^\infty_t L^2_x \subset L^2_{t,x}$, the Correspondence Principle and

Hölder for $\mathcal{L}^{\ell,p}_s$ (Lemma 3.1.8), we have

$$\|s^{-1/2}[A_0, \nabla_0 F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \leq C \sum_{j=0}^{m-1} \|\nabla^{(j)}_x A_0\|_{\mathcal{L}^{0+1/4,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \|\nabla_0 F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \dot{\mathcal{H}}^{m-1-j}_x}.$$

Thanks to the extra weight of $s^{1/4}$ and the fact that $0 \leq j \leq m-1 \leq 9$, we can easily prove $\|\nabla^{(j)}_x A_0\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$ via Lemma 4.5.3, Gagliardo-Nirenberg (Lemma 3.1.14) and Proposition 4.5.14. On the other hand, as $0 \leq m-1-j \leq 9$, we have $\|\nabla_0 F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \dot{\mathcal{H}}^{m-1-j}_x} \leq C\mathcal{F}$. The desired estimate then follows.

- *Case 5 : Proof of* $\|s^{-1/2}[\nabla_0 A_0, F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

We claim that the following estimate for $\nabla_0 A_0$ holds for $0 \leq j \leq 9$.

$$\|\nabla^{(j)}_x \nabla_0 A_0\|_{\mathcal{L}^{0,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2. \tag{4.7.21}$$

Assuming the claim, let us prove the desired estimate. As in the previous case, we have

$$\|s^{-1/2}[\nabla_0 A_0, F_{si}]\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \leq C \sum_{j=0}^{m-1} \|\nabla^{(j)}_x \nabla_0 A_0\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \|F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \dot{\mathcal{H}}^{m-1-j}_x}.$$

The factor $\|\nabla^{(j)}_x \nabla_0 A_0\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x}$ can be controlled by (4.7.21). For the other factor, we divide into two cases: For $1 \leq j \leq m-1 \leq 9$, we have $\|F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \dot{\mathcal{H}}^{m-1-j}_x} \leq C\mathcal{F}$, whereas for $j = 0$ we use Proposition 4.5.11. The desired estimate then follows.

To prove the claim, we begin with the formula $\partial_0 A_0 = -\int_s^1 \partial_0 F_{s0}(s') \, \mathrm{d}s'$. Proceeding as in the proofs of the Lemmas 4.5.2 and 4.5.3, we obtain the estimate

$$\|\nabla^{(j)}_x \nabla_0 A_0\|_{\mathcal{L}^{0,\infty}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \leq C \|\nabla^{(j)}_x \nabla_0 F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \mathcal{L}^\infty_x}.$$

In order to estimate the right-hand side, recall the identity $\partial_0 F_{s0} = \partial^\ell w_\ell + [A^\ell, w_\ell] + [A_0, F_{s0}]$ from Chapter 2. It therefore suffices to prove

$$\|\nabla^{(j)}_x \nabla^\ell w_\ell\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} + \|\nabla^{(j)}_x [A^\ell, w_\ell]\|_{\mathcal{L}^{3/2,2}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} + \|\nabla^{(j)}_x [A_0, F_{s0}]\|_{\mathcal{L}^{3/2,2}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2,$$

for $0 \leq j \leq 9$.

By Gagliardo-Nirenberg (Lemma 3.1.14) and Proposition 4.5.17, we have $\|\nabla^{(j)}_x \nabla^\ell w_\ell\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \mathcal{L}^\infty_x} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$ for $0 \leq j \leq 9$.

Next, by Leibniz's rule, Hölder, the Correspondence Principle and Lemma 3.1.8, we obtain

$$\|\nabla_x^{(j)}[A^\ell, w_\ell]\|_{\mathcal{L}_s^{3/2,2}\mathcal{L}_t^2\mathcal{L}_x^\infty} \leq C \sum_{j'=0}^{j} \|\nabla_x^{(j')} A\|_{\mathcal{L}_s^{1/4+1/4,\infty}\mathcal{L}_{t,x}^\infty} \|\nabla_x^{(j-j')} w\|_{\mathcal{L}_s^{1,2}\mathcal{L}_t^2\mathcal{L}_x^\infty}.$$

Note the extra weight of $s^{1/4}$ on the first factor. As $0 \leq j' \leq 9$, by Lemma 4.5.2, Gagliardo-Nirenberg (Lemma 3.1.14) and Proposition 4.5.8, we have $\|\nabla_x^{(j')} A\|_{\mathcal{L}_s^{1/2,\infty}\mathcal{L}_{t,x}^\infty} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})$. On the other hand, $\|\nabla_x^{(j-j')} w\|_{\mathcal{L}_s^{1,2}\mathcal{L}_t^2\mathcal{L}_x^\infty} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$ by Gagliardo-Nirenberg (Lemma 3.1.14) and Proposition 4.5.17.

Finally, we can show $\|\nabla_x^{(j)}[A_0, F_{s0}]\|_{\mathcal{L}_s^{3/2,2}\mathcal{L}_t^2\mathcal{L}_x^\infty} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$ by proceeding similarly, with applications of Propositions 4.5.8 and 4.5.17 replaced by Proposition 4.5.14. We leave the details to the reader.

- *Case 6 : Proof of* $\|[F_{i0}, F_{s0}]\|_{\mathcal{L}_s^{2,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^{m-1}} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

By Leibniz's rule, Hölder, the Correspondence Principle and Lemma 3.1.8, we have

$$\|[F_{i0}, F_{s0}]\|_{\mathcal{L}_s^{2,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^{m-1}} \leq C \sum_{j=0}^{m-1} \|\nabla_x^{(j)} F_{i0}\|_{\mathcal{L}_s^{3/4,2}\mathcal{L}_{t,x}^\infty} \|F_{s0}\|_{\mathcal{L}_s^{5/4,\infty}\mathcal{L}_t^2\dot{\mathcal{H}}_x^{m-1-j}}.$$

Using Gagliardo-Nirenberg (Lemma 3.1.14) and Lemma 4.5.4, combined with Propositions 4.5.14, 4.5.8, we can prove the following estimate for the first factor (for $0 \leq j \leq 9$):

$$\|\nabla_x^{(j)} F_{i0}\|_{\mathcal{L}_s^{3/4,2}\mathcal{L}_{t,x}^\infty} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}}).$$

For the second factor, we simply apply Proposition 4.5.14 to conclude $\|F_{s0}\|_{\mathcal{L}_s^{5/4,\infty}\mathcal{L}_t^2\dot{\mathcal{H}}_x^{m-1-j}} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2$ for $0 \leq m-1-j \leq 9$, which is good.

- *Case 7 : Proof of* $\|[F_i{}^\ell, F_{s\ell}]\|_{\mathcal{L}_s^{2,p}\mathcal{L}_t^2\dot{\mathcal{H}}_x^{m-1}} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2$.

In this case, we simply expands out $F_{i\ell} = \partial_i A_\ell - \partial_\ell A_i + [A_i, A_\ell]$. Note that the first two terms give additional terms of the form already handled in Step 3, whereas the last term will give us cubic terms which can simply be estimated by using Hölder and Sobolev. For more details, we refer to the proof of Lemma 4.7.4 below. □

Estimates for cubic terms

The contribution of cubic terms are much easier to handle compared to quadratic terms. Indeed, we have the following lemma.

Lemma 4.7.4 (Estimates for cubic terms)**.** *Assume* $0 < T \leq 1$*. For* $1 \leq m \leq 10$ *and* $p = 2, \infty$*, the following estimates hold.*

$$\sup_i \|^{(F_{si})}\mathcal{M}_{\mathrm{cubic}}\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^3, \tag{4.7.22}$$

$$\sup_i \|^{(\delta F_{si})}\mathcal{M}_{\mathrm{cubic}}\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2 (\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.7.23}$$

Proof. As before, we give a proof of (4.7.22), leaving the similar proof of the difference version (4.7.23) to the reader.

Let $1 \leq m \leq 10$ and $p = 2$ or ∞. As before, we work on the whole interval $(0, 1]$. We begin with the obvious inequality

$$\|\phi_1 \phi_2 \phi_3\|_{L^2_{t,x}} \leq C T^{1/2} \prod_{i=1,2,3} \|\phi_i\|_{L^\infty_t \dot{H}^1_x},$$

which follows from Hölder and Sobolev. By Leibniz's rule, the Correspondence Principle and Hölder for $\mathcal{L}^{\ell,p}_s$ (Lemma 3.1.8), we obtain

$$\begin{aligned} \sup_i \|^{(F_{si})}\mathcal{M}_{\mathrm{cubic}}\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \leq & C T^{1/2} \|\nabla_x A\|^2_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \|\nabla_x F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \\ & + C T^{1/2} \|\nabla_x A_0\|^2_{\mathcal{L}^{0+1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \|\nabla_x F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x}. \end{aligned}$$

Note the obvious bound $\|\nabla_x F_s\|_{\mathcal{L}^{5/4,p}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \leq C\mathcal{F}$. Applying Lemma 4.5.3 to $\|A_0\|$ (using the extra weight of $s^{1/4}$) and Proposition 4.5.14, we also obtain $\|\nabla_x A_0\|_{\mathcal{L}^{0+1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot \mathcal{E} + C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})^2$. Finally, we split $\|\nabla_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x}$ into $\|A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t \dot{\mathcal{H}}^1_x}$ and $\|\nabla^{(2)}_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-2}_x}$ (where the latter term does not exist in the case $m = 1$). For the former we apply Proposition 4.4.2, whereas for the latter we apply Lemma 4.5.2. We then conclude $\|\nabla_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_t \mathcal{H}^{m-1}_x} \leq C_{\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{F} + \underline{\mathcal{A}})$. Combining all these estimates, (4.7.22) follows. □

Estimates for terms involving w_i

Finally, the contribution of $^{(F_{si})}\mathcal{M}_w$ is estimated by the following lemma.

Lemma 4.7.5 (Estimates for terms involving w_i)**.** *Assume* $0 < T \leq 1$*. For* $1 \leq m \leq 10$ *and* $p = 2, \infty$*, the following estimates hold.*

$$\sup_i \|^{(F_{si})}\mathcal{M}_w\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2, \tag{4.7.24}$$

$$\sup_i \| {}^{(\delta F_{si})}\mathcal{M}_w \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x (0,1]} \leq C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})(\delta\mathcal{E} + \delta\mathcal{F} + \delta\underline{\mathcal{A}}). \tag{4.7.25}$$

Proof. As before, we shall only give a proof of (4.7.24), leaving the similar proof of (4.7.25) to the reader.

Let $1 \leq m \leq 10$ and $p = 2$ or ∞. We work on the whole interval $(0, 1]$. Note that, schematically,

$$ {}^{(F_{si})}\mathcal{M}_w = \partial^\ell \partial_\ell w_i - \partial_i \partial^\ell w_\ell + {}^{(w_i)}\mathcal{N} + \mathcal{O}(A, \partial_x w) + \mathcal{O}(\partial_x A, w) + \mathcal{O}(A, A, w).$$

By Leibniz's rule, the Correspondence Principle (from the Hölder inequality $L^\infty_{t,x} \cdot L^2_{t,x} \subset L^2_{t,x}$) and Lemma 3.1.8, we obtain the following estimate.

$$\begin{aligned}
&\| {}^{(F_{si})}\mathcal{M}_w \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x} \\
&\quad \leq C \|w\|_{\mathcal{L}^{1,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m+1}_x} + C \| {}^{(w_i)}\mathcal{N} \|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m+1}_x} + C \sum_{j=0}^{m} \|\nabla^{(j)}_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_{t,x}} \|w\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-j}_x} \\
&\qquad + C \sum_{j,j' \geq 0, j+j' \leq m-1} \|\nabla^{(j)}_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_{t,x}} \|\nabla^{(j')}_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_{t,x}} \|w\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1-j-j'}_x}
\end{aligned} \tag{4.7.26}$$

By Lemma 4.5.2, combined with Proposition 4.5.8, the following estimate holds for $0 \leq j \leq 10$.

$$\|\nabla^{(j)}_x A\|_{\mathcal{L}^{1/4,\infty}_s \mathcal{L}^\infty_{t,x}} \leq C_{\mathcal{F},\underline{\mathcal{A}}} (\mathcal{F} + \underline{\mathcal{A}}). \tag{4.7.27}$$

Now (4.7.24) follows from (4.7.26), (4.7.27) and (4.5.47), (4.5.48) of Proposition 4.5.17, thanks to the restriction $1 \leq m \leq 10$. □

Completion of the proof

We are now prepared to give a proof of Theorem D.

Proof of Theorem D. Let us begin with (4.4.7). Recalling the definition of $\mathcal{F}$, it suffices to show

$$\|F_{si}\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{S}}^m (0,1]} \leq C\mathcal{I} + T^{1/2} C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}} \cdot (\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2,$$

for $i = 1, 2, 3$, $p = 2, \infty$ and $1 \leq m \leq 10$. Starting with the energy inequality and applying the Correspondence Principle, we obtain

$$\|F_{si}\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{S}}^m} \leq C \|\nabla_{t,x} F_{si}(t=0)\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^{m-1}_x} + C T^{1/2} \|\Box F_{si}\|_{\mathcal{L}^{2,p}_s \mathcal{L}^2_t \dot{\mathcal{H}}^{m-1}_x}.$$

The first term on the right-hand side is estimated by $C\mathcal{I}$. For the second term, as $\Box F_{si} = {}^{(F_{si})}\mathcal{M}_{\mathrm{quadratic}} + {}^{(F_{si})}\mathcal{M}_{\mathrm{cubic}} + {}^{(F_{si})}\mathcal{M}_{w}$, we may apply Lemmas 4.7.3 – 4.7.5 (estimates (4.7.15), (4.7.22) and (4.7.24), in particular) to conclude

$$\|\Box F_{si}\|_{\mathcal{L}^{5/4+1,p}_s \mathcal{L}^1_t \dot{\mathcal{H}}^{m-1}_x} \leq T^{1/2} C_{\mathcal{E},\mathcal{F},\underline{\mathcal{A}}}(\mathcal{E} + \mathcal{F} + \underline{\mathcal{A}})^2,$$

which is good.

The proof of (4.4.8) is basically identical, this time controlling the initial data term by $C\delta\mathcal{I}$ and using (4.7.16), (4.7.23), (4.7.25) in place of (4.7.15), (4.7.22), (4.7.24). $\Box$

Remark 4.7.6. Recall that in [15], one has to recover two types of null forms, namely $Q_{ij}(|\partial_x|^{-1}A, A)$ and $|\partial_x|^{-1}Q_{ij}(A, A)$, in order to prove H^1_x local well-posedness in the Coulomb gauge. An amusing observation is that we did not need to uncover the second type of null forms in our proof. Note, however, that we do *see* this null form in the caloric-temporal as well; see §2.2.

Chapter 5

Proof of the Main GWP Theorem

In this chapter, we shall prove the Main GWP Theorem for (YM). Again, we restrict to the case $d = 3$.

To begin with, in §5.1, we shall reduce the proof of Main GWP Theorem to that for Theorems E, F and G, which correspond to Steps 1, 2 and 3 in §1.6. We hope that by reading this section first, the reader will get a better idea about the overall argument in this chapter.

Next, in §5.2, we shall develop covariant techniques to deal with covariant parabolic equations. As an application, we shall derive in §5.3 *covariant parabolic estimates* for curvature components of solutions to (dYMHF) and (cYMHF); the key results are Propositions 5.3.2 and 5.3.4, respectively.

The rest of this chapter will be devoted to proofs of Theorems E, F and G. In §5.4, we shall apply the covariant parabolic estimates established in §5.3 to study the Yang-Mills heat flows in the caloric gauge. In particular, an improved local well-posedness for (cYMHF) and (dYMHF) in the caloric gauge (using only the smallness of the $\mathbf{B}[\overline{F}]$ or $\mathbf{E}[\overline{\mathbf{F}}]$, respectively) will be established; see Theorems 5.4.1 and 5.4.4. The former immediately gives an alternative proof of *finite energy global well-posedness* of (YMHF) (Corollary 5.4.3), first proved by Råde [27]. The latter will be an essential ingredient for the proof of Theorem E in §5.5. In §5.6, we shall prove Theorem F using the covariant parabolic estimates in §5.3. Finally, in §5.7, we shall prove Theorem G. The proof will be similar to that of Theorem B (Time dynamics of (HPYM) in the caloric-temporal gauge; stated in §4.3.2 and proved in §4.4), except that we shall substitute the use of Proposition 4.4.3 (not applicable in the present setting) by Proposition 5.7.1 using the covariant parabolic estimates in §5.3.

The materials in this chapter had been previously published in [24].

5.1 Reduction of the Main GWP Theorem to Theorems E, F and G

In this section, we shall reduce the Main GWP Theorem to Theorems E, F and G. As discussed earlier, these theorems correspond to Steps 1, 2 and 3 introduced in §1.6, respectively.

Our first theorem, Theorem E, may be viewed as a strengthening of Theorem A (Transformation to caloric-temporal gauge; see §4.3.2). Morally, the statement of the latter theorem was as follows: Given a regular solution $A^\dagger_\mu$ to the Yang-Mills equations in the temporal gauge on $I \times \mathbb{R}^3$ with $\|A^\dagger\|_{L^\infty_t \dot{H}^1_x(I)}$ sufficiently small, there exists a regular gauge transform V on $I \times \mathbb{R}^3$ and regular solution $A_{\mathbf{a}}$ to (HPYM) on $I \times \mathbb{R}^3 \times [0,1]$ such that $A_\mu(s=0)$ is the gauge transformation of $A^\dagger_\mu$ by V, i.e.

$$A_\mu(s=0) = V(A^\dagger_\mu)V^{-1} - \partial_\mu V V^{-1}$$

and $A_{\mathbf{a}}$ is in the *caloric-temporal gauge* $A_s = 0, \underline{A}_0 := A_0(s=1) = 0$. Moreover, V, $F_{si} := \partial_s A_i$ and $\underline{A}_i := A_i(s=1)$ at $t=0$ obey estimates in terms of the initial data $\mathring{\mathcal{I}} := \|\mathring{A}\|_{\dot{H}^1_x} + \|\mathring{E}\|_{L^2_x}$. The key improvement in Theorem E is that only the smallness of the *conserved energy*, instead of $\|A^\dagger\|_{L^\infty_t \dot{H}^1_x}$, is needed to draw the same conclusions.

Theorem E (Transformation to the caloric-temporal gauge, improved version)**.** *Consider a regular initial data set* $(\mathring{A}_i, \mathring{E}_i)$ *to* (YM) *which satisfies*

$$\|\mathring{A}\|_{\dot{H}^1_x} < \delta_P, \quad \mathbf{E}[\mathring{\mathbf{F}}] < \delta_C. \tag{5.1.1}$$

where $\delta_P, \delta_C > 0$ *are small absolute constants in Propositions 3.2.1 and 5.3.2, respectively. Let* $A^\dagger_i$ *be the corresponding regular solution to* (YM) *in the temporal gauge given by the Main LWP Theorem, which we assume to exist on* $(-T_0, T_0) \times \mathbb{R}^3$ *for some* $T_0 > 0$*. Then the following statements hold:*

1. *There exists a regular gauge transform* $V = V(t,x)$ *on* $(-T_0, T_0) \times \mathbb{R}^3$ *and a regular solution* $A_{\mathbf{a}} = A_{\mathbf{a}}(t,x,s)$ *to* (HPYM) *in the caloric-temporal gauge on* $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ *such that* $F_{\mu\nu}$ *is regular and*
$$A_\mu(s=0) = V A^\dagger_\mu V^{-1} - \partial_\mu V V^{-1}. \tag{5.1.2}$$

2. *With the notations* $\mathring{\mathcal{I}} := \|\mathring{A}\|_{\dot{H}^1_x} + \|\mathring{E}\|_{L^2_x}$ *and* $\mathring{V} := V(t=0)$*, the following estimates hold.*

$$\mathcal{I}(0) \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}, \quad \|\mathring{V}\|_{L^\infty_x} \leq C_{\mathring{\mathcal{I}}}, \quad \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^2_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{3/2}_x \cap L^\infty_x} \leq C_{\mathring{\mathcal{I}}} \cdot \mathring{\mathcal{I}}. \tag{5.1.3}$$

The identical estimate as the last holds for $\mathring{V}$ *replaced by* $\mathring{V}^{-1}$ *as well.*

We remind the reader that $\mathcal{I}(t)$ had been defined in §4.3.2.

The next theorem basically says that the conserved energy $\mathbf{E}[\mathbf{F}] := \mathbf{E}[\mathbf{F}(t)]$ can be used to control $\mathcal{I}(t)$ for every $t \in (-T_0, T_0)$. ; we refer the reader to Step 2 in §1.6 for the basic idea behind the theorem.

Theorem F (Fixed-time control by $\mathbf{E}$ in the caloric-temporal gauge)**.** *Let* $T_0 > 0$, *and consider a regular solution* $A_{\mathbf{a}}$ *to* (HPYM) *in the caloric-temporal gauge on* $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$ *satisfying* $\mathcal{I}(0) < \infty$ *and* $\mathbf{E}[\overline{\mathbf{F}}] < \infty$.

Then for $t \in (-T_0, T_0)$, $\mathcal{I}(t)$ *can be bounded in terms of the initial data and* T_0, *i.e.*

$$\sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq C_{\mathcal{I}(0), \mathbf{E}[\overline{\mathbf{F}}], T_0} < \infty. \tag{5.1.4}$$

From Theorems E and F, we obtain *a priori* estimates for $A_{\mathbf{a}}$ in the caloric-temporal gauge on each fixed-time slice $\{t\} \times \mathbb{R}^3 \times [0, 1]$. In order to estimate the gauge transform back to the temporal gauge, however, one needs to control the $\overline{\mathcal{A}}_0$-norm of $A_0(s = 0)$ (recall Lemma 4.3.6), and for this purpose it turns out that these fixed-time estimates are insufficient. In order to estimate $\overline{\mathcal{A}}_0$ we need to take advantage of the fact that the dynamic variables $F_{si}, \underline{A}_i$ satisfy wave equations, which is exactly what the next theorem achieves.

Theorem G (Short time estimates for (HPYM) in the caloric-temporal gauge)**.** *Let* $T_0 > 0$, *and consider a regular solution* $A_{\mathbf{a}}$ *to* (HPYM) *in the caloric-temporal gauge on* $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$. *Suppose furthermore that*

$$\mathbf{E}[\overline{\mathbf{F}}] < \delta_C, \qquad \sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq D, \tag{5.1.5}$$

where $D > 0$ *is an arbitrarily large finite number and* $\delta_C > 0$ *is an absolute small constant independent of* D.

Then there exists a number $d = d(D, \mathbf{E}[\overline{\mathbf{F}}])$, *which depends on* $D, \mathbf{E}[\overline{\mathbf{F}}]$ *in a non-increasing fashion, such that on every subinterval* $I_0 \subset I$ *of length* d, *the following estimate holds:*

$$\|\partial_{t,x} \overline{A}\|_{C_t(I_0, L^2_x)} + \overline{\mathcal{A}}_0(I_0) \leq C_{D, \mathbf{E}[\overline{\mathbf{F}}]}. \tag{5.1.6}$$

Remark 5.1.1. Theorem G is very similar to Theorem B (Time dynamics of (HPYM) in the caloric-temporal gauge; see §4.3.2). However, there is a little twist which necessitates the extra hypotheses $\mathbf{E}[\overline{\mathbf{F}}] < \delta_C$ in (5.1.5).

If we had only hyperbolic equations to analyze, then although $\mathcal{I}(t)$ may be large, we could have used the smallness of the time interval to close the estimates. In reality, however, among the equations of (HPYM) is an equation for F_{s0} which, unlike the other components F_{si}, is parabolic. As such, smallness of the time interval *cannot* be utilized to solve this equation in a perturbative manner. We remark that in the proof of Theorem B in §4.4, we did not have this problem as $\mathcal{I}(t)$ were assumed to be small while the length of the t- and s-intervals were ~ 1[1]; see Proposition 4.4.3.

What will save us in the present case is the fact that the parabolic equation for F_{s0} is *covariant*, and therefore can be analyzed using the covariant techniques presented in §5.2. The condition $\mathbf{E}[\overline{\mathbf{F}}] < \delta_C$ therefore provides the necessary smallness for this analysis; see Proposition 5.7.1. A rigorous proof will be given in §5.7.

We are now prepared to give a proof of the Main GWP Theorem, under the assumption that Theorems E, F and G are true.

Proof of the Main GWP Theorem, assuming Theorems E, F and G. To begin with, let us consider a regular initial data set; note that it has finite conserved energy, i.e. $\mathbf{E}(\mathring{\mathbf{F}}) < \infty$. Applying the Main LWP Theorem to $(\mathring{A}_i, \mathring{E}_i)$, there exists a unique regular solution to (YM) in the temporal gauge on some time interval centered at 0, which we shall denote by $A^\dagger_\mu$. We shall first show this solution exists globally in time.

For the purpose of contradiction, suppose that the solution $A^\dagger_\mu$ cannot be extended globally as a unique regular solution to (YM) in the temporal gauge. Then there exists a positive finite number $0 < T_0 < \infty$, which is the largest positive number for which the solution $A^\dagger_\mu$ can be extend as a regular solution on $(-T_0, T_0)$. We claim that there exists a finite positive constant $C = C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0}$, which depends only on the initial data and T_0, such that the following inequality holds.

$$\sup_{t \in (-T_0, T_0)} \|A^\dagger_\mu(t)\|_{\dot{H}^1_x} \leq C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0} < \infty. \tag{5.1.7}$$

Let us complete the proof of the Main GWP Theorem first, under the assumption that the claim is true. If the claim were true, then the solution may be extended as a unique regular solution to $(-T_0 - \epsilon, T_0 + \epsilon)$ for some $\epsilon > 0$ by the Main LWP Theorem, which is a contradiction. It follows that $T_0 = \infty$, and thus $A^\dagger_\mu$ can be extended globally in time as a unique regular solution to (YM) in the temporal gauge. Observe that the estimate (5.1.7) still holds for the global solution $A^\dagger_\mu$ for every $T_0 > 0$.

[1] By scaling, this is equivalent to having $\mathcal{I}(t)$ large and the lengths of both the s- and t-intervals small.

Next, Lemma 4.3.5 implies that an admissible initial data can be approximated by a sequence of regular initial data sets $(\mathring{A}_{(n)i}, \mathring{E}_{(n)i})$. Let us denote the corresponding unique global regular solutions by $A_{(n)\mu}$. Using the Main LWP Theorem repeatedly (with the help of (5.1.7)), the following statement may be proved: For every $T_0 > 0$, the sequence of regular solutions $A_{(n)\mu}$ restricted to the time interval $(-T_0, T_0)$ is a Cauchy sequence in the topology $C_t((-T_0, T_0), \dot{H}^1_x \cap L^3_x)$. Hence a limit A_μ exists on $(-T_0, T_0)$. Moreover, it is also possible to show that $\partial_t A_{(n)\mu} \to \partial_t A_\mu$ in $C_t((-T_0, T_0), L^2_x)$. Thus it follows that A_μ is an admissible solution to (YM) in the temporal gauge on $(-T_0, T_0)$. Uniqueness among the class of admissible solutions follows from the corresponding statement for regular solutions. As $T_0 > 0$ is arbitrary, A_μ is global, and the Main GWP Theorem follows.

We are only left to establish the claim, which is a rather straightforward application of Theorems E, F and G. First, by scaling, we may assume that $\|\mathring{A}\|_{\dot{H}^1_x} < \delta_P$ and $\mathbf{E}[\mathring{\mathbf{F}}] < \delta$, i.e. (5.1.1) holds. This allows us to apply Theorem E, from which we obtain a regular gauge transform V and a regular solution $A_{\mathbf{a}}$ to (HPYM) in the caloric-temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ such that (5.1.2) holds. Note that as $\overline{A}_\mu$ is a regular solution to the (YM), $\mathbf{E}[\overline{\mathbf{F}}(t)] = \mathbf{E}[\mathring{\mathbf{F}}]$ for all $t \in (-T_0, T_0)$. By Theorem F, along with the estimate for $\mathcal{I}(0)$ in (5.1.3), we see that

$$\sup_{t \in (-T_0, T_0)} \mathcal{I}(t) \leq C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0} < \infty.$$

To use Theorem G, let us cover $(-T_0, T_0)$ by subintervals of length d; the number of subintervals required can be bounded from above by, say, $10(T_0/d)$. Applying Theorem G on each subinterval, we are led to the estimate

$$\|\partial_{t,x} \overline{A}\|_{C_t(I_0, L^2_x)} + \sup_{s \in [0,1]} \overline{\mathcal{A}}_0(I_0) \leq C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0} < \infty.$$

The only remaining step is to transfer the above estimate to $A^\dagger_\mu$; for this purpose, observe from (5.1.2) that V satisfies $\partial_t V = V A_0(s=0)$. Using Lemma 4.3.6, along with the previous estimates for $\overline{\mathcal{A}}_0$ and $\mathring{V}$, we are led to the following estimates for the gauge transform V:

$$\begin{aligned} \|V - \mathrm{Id}\|_{L^\infty_t \dot{H}^{\gamma+1}_x} + \|V - \mathrm{Id}\|_{L^\infty_t (\dot{H}^{d/2}_x \cap L^\infty_x)} \leq & C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0}, \\ \|\partial_t (V - \mathrm{Id})\|_{L^\infty_t \dot{H}^\gamma_x} + \|\partial_t (V - \mathrm{Id})\|_{L^\infty_t \dot{H}^{(d-2)/2}_x} \leq & C_{\mathring{\mathcal{I}}, \mathbf{E}[\mathring{\mathbf{F}}], T_0}. \end{aligned}$$

Here, all norms have been taken over $(-T_0, T_0) \times \mathbb{R}^3$. Moreover, identical estimates for V^{-1} also hold. These estimates, applied to the formula (5.1.2) (with the help of Lemma A.3.1), implies (5.1.7) as desired. □

5.2 Preliminaries

Here, we shall collect some techniques which are applicable to the study of *covariant* parabolic equations. The use of such techniques, instead of those for handling the usual scalar heat equation, is the key analytic difference between this chapter and the last.

We shall begin with a well-known inequality which relates a covariant derivative with an ordinary derivative of the modulus.

Lemma 5.2.1 (Kato's inequality)**.** *Let σ be a $\mathfrak{g}$-valued function. Then*

$$|\partial_x|\sigma|| \leq |\mathbf{D}_x \sigma| \tag{5.2.1}$$

in the distributional sense.

Proof. Let $\epsilon > 0$. We compute

$$\partial_x \sqrt{(\sigma, \sigma) + \epsilon} = \frac{(\sigma, \mathbf{D}_x \sigma)}{\sqrt{(\sigma, \sigma) + \epsilon}} \leq |\frac{\sqrt{(\sigma, \sigma)}}{\sqrt{(\sigma, \sigma) + \epsilon}}| \cdot |\mathbf{D}_x \sigma| \leq |\mathbf{D}_x \sigma|.$$

Testing against a non-negative test function and taking $\epsilon \to 0$, we see that $\partial_x |\sigma| \leq |\mathbf{D}_x \sigma|$ in the distributional sense. Repeating the same argument to $-\partial_x \sqrt{(\sigma, \sigma) + \epsilon}$, we obtain (5.2.1). □

The following Sobolev inequalities for covariant derivatives are easy consequences of Kato's inequality.

Corollary 5.2.2 (Sobolev and Gagliardo-Nirenberg inequalities for covariant derivatives)**.** *For a $\mathfrak{g}$-valued function $\sigma \in H^\infty_x$, the following estimates hold.*

$$\|\sigma\|_{L^3_x} \leq C \|\sigma\|^{1/2}_{L^2_x} \|\mathbf{D}_x \sigma\|^{1/2}_{L^2_x}, \tag{5.2.2}$$

$$\|\sigma\|_{L^6_x} \leq C \|\mathbf{D}_x \sigma\|_{L^2_x}, \tag{5.2.3}$$

$$\|\sigma\|_{L^\infty_x} \leq C \|\mathbf{D}_x \sigma\|^{1/2}_{L^2_x} \|\mathbf{D}^{(2)}_x \sigma\|^{1/2}_{L^2_x}. \tag{5.2.4}$$

Next, consider an inhomogeneous covariant heat equation

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell) \sigma = \mathcal{N}. \tag{5.2.5}$$

Adapting the usual proof of the energy integral inequality (integration by parts) for the ordinary heat equation to (5.2.5), we obtain the following gauge-invariant version of the energy integral inequality.

Lemma 5.2.3 (Energy integral inequality)**.** *Let $\ell \in \mathbb{R}$, $(s_1, s_2] \subset (0, \infty)$ and suppose that σ and A_i are 'sufficiently nice[2]'. Then the following estimate holds.*

$$\begin{aligned} &\|\sigma\|_{\mathcal{L}^{\ell,\infty}_s \mathcal{L}^2_x(s_1,s_2]} + \|\mathcal{D}_x \sigma\|_{\mathcal{L}^{\ell,2}_s \mathcal{L}^2_x(s_1,s_2]} \\ &\qquad \leq C s_1^\ell \|\sigma(s_1)\|_{\mathcal{L}^2_x(s_1)} + C(\ell - 3/4)\|\sigma\|_{\mathcal{L}^{\ell,2}_s \mathcal{L}^2_x(s_1,s_2]} + C\|\mathcal{N}\|_{\mathcal{L}^{\ell+1,1}_s \mathcal{L}^2_x(s_1,s_2]}. \end{aligned} \tag{5.2.6}$$

Proof. We shall carry out a formal computation, discarding all boundary terms at the spatial infinity which arise; it is easy to verify that for 'sufficiently nice' σ and A_i, this can be made into a rigorous proof. See also the proof of Proposition 3.1.11.

Let $\overline{s} \in (s_1, s_2]$. Taking the bi-invariant inner product of the equation $(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell)\sigma = \mathcal{N}$ with $s^{2\ell-3/2}\sigma$ and integrating by parts over $(s_1, \overline{s}]$, we arrive at

$$\begin{aligned} &\frac{1}{2}\overline{s}^{2\ell-3/2} \int (\sigma,\sigma)(\overline{s}) \, \mathrm{d}x + \int_{s_1}^{\overline{s}} \int s^{2\ell-1/2} (\mathbf{D}^\ell \sigma, \mathbf{D}_\ell \sigma)(s) \, \mathrm{d}x \, \frac{\mathrm{d}s}{s} \\ &\quad = \frac{1}{2} s_1^{2\ell-3/2} \int (\sigma,\sigma)(s_1)\, \mathrm{d}x + (\ell - 3/4) \int_{s_1}^{\overline{s}} \int s^{2\ell-3/2} (\sigma,\sigma)(s) \, \mathrm{d}x \, \frac{\mathrm{d}s}{s} \\ &\qquad + \int_{s_1}^{\overline{s}} s^{2\ell-1/2} (\mathcal{N}(s), \sigma(s)) \, \mathrm{d}x \, \frac{\mathrm{d}s}{s}. \end{aligned}$$

Taking the supremum over $s_1 < \overline{s} \leq s_2$ and rewriting in terms of p-normalized norms, we obtain

$$\begin{aligned} \frac{1}{2}\|\sigma\|^2_{\mathcal{L}^{\ell,\infty}_s \mathcal{L}^2_x(s_1,s_2]} + \|\sigma\|^2_{\mathcal{L}^{\ell,2}_s \dot{\mathcal{H}}^1_x(s_1,s_2]} \leq & \frac{1}{2} s_1^{2\ell} \|\sigma(s_1)\|^2_{\mathcal{L}^2_x(s_1)} + (\ell - 3/4)\|\sigma\|^2_{\mathcal{L}^{\ell,2}_s \mathcal{L}^2_x(s_1,s_2]} \\ & + \|(\mathcal{N},\sigma)\|_{\mathcal{L}^{2\ell+1,1}_s \mathcal{L}^1_x(s_1,s_2]}. \end{aligned}$$

By Hölder, Lemma 3.1.8 and Cauchy-Schwarz, the last term can be estimated by

$$\|\mathcal{N}\|^2_{\mathcal{L}^{\ell+1,1}_s \mathcal{L}^2_x(s_1,s_2]} + \frac{1}{4}\|\sigma\|^2_{\mathcal{L}^{\ell,\infty}_s \mathcal{L}^2_x(s_1,s_2]},$$

where the latter can be absorbed into the left-hand side. Then taking the square root of both sides, we obtain (5.2.6). □

Proceeding as in the proof of Kato's inequality, we can derive the following parabolic inequality

[2] A sufficient condition for (5.2.6) to hold, which will be verifiable in applications below, is that σ is smooth and the left-hand side of (5.2.6) is finite.

for $|\sigma|$.

Lemma 5.2.4 (Bochner-Weitzenböck-type inequality)**.** *The following inequality holds in the distributional sense.*

$$(\partial_s - \triangle)|\sigma| \leq |\mathcal{N}|. \tag{5.2.7}$$

Proof. Let $\epsilon > 0$. We compute

$$\begin{aligned}
\partial_s \sqrt{(\sigma,\sigma)+\epsilon} &= \frac{1}{\sqrt{(\sigma,\sigma)+\epsilon}}(\sigma, \mathbf{D}_s \sigma), \\
\triangle \sqrt{(\sigma,\sigma)+\epsilon} &= \frac{1}{\sqrt{(\sigma,\sigma)+\epsilon}}\Big((\sigma, \mathbf{D}^\ell \mathbf{D}_\ell \sigma) + (\mathbf{D}^\ell \sigma, \mathbf{D}_\ell \sigma) - \frac{(\sigma, \mathbf{D}^\ell \sigma)(\sigma, \mathbf{D}_\ell \sigma)}{(\sigma,\sigma)+\epsilon}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\partial_s - \triangle)\sqrt{(\sigma,\sigma)+\epsilon} =& \frac{1}{\sqrt{(\sigma,\sigma)+\epsilon}}\Big((\sigma, \mathcal{N}) - (\mathbf{D}^\ell \sigma, \mathbf{D}_\ell \sigma) + \frac{(\sigma, \mathbf{D}^\ell \sigma)(\sigma, \mathbf{D}_\ell \sigma)}{(\sigma,\sigma)+\epsilon}\Big) \\
\leq & \frac{1}{\sqrt{(\sigma,\sigma)+\epsilon}}(\sigma, \mathcal{N}) \leq |\mathcal{N}|.
\end{aligned}$$

Testing against a non-negative test function and taking $\epsilon \to 0$, we obtain (5.2.7). □

The virtue of (5.2.7) is that it allows us to use estimates arising from the (standard) heat kernel. Before we continue, let us briefly recap the definition and basic properties of the heat kernel.

Let $e^{s\triangle}$ denote the solution operator for the free heat equation. It is an integral operator, defined by.

$$e^{s\triangle}\psi_0(x) = \frac{1}{\sqrt{4\pi s}^3} \int e^{-|x-y|^2/4s} \psi_0(y) \, \mathrm{d}y.$$

The kernel on the right hand side is called the *heat kernel* on $\mathbb{R}^3$. Using Young's inequality, it is easy to derive the following basic inequality for the heat kernel:

$$\|e^{s\triangle}\psi_0\|_{L^r_x} \leq C_{p,r}\, s^{-3/(2p)+3/(2r)} \|\psi_0\|_{L^p_x}, \tag{5.2.8}$$

where $1 \leq p \leq r$.

Now consider the initial value problem for the inhomogeneous heat equation $(\partial_s - \triangle)\psi = N$. Duhamel's principle tells us that this problem can be equivalently formulated in an integral form as follows:

$$\psi(s) = e^{s\triangle}\psi(s=0) + \int_0^s e^{(s-\overline{s})\triangle} N(\overline{s}) \, \mathrm{d}\overline{s}.$$

With these prerequisites, we are ready to derive a simple comparison principle for $|\sigma|$, along with a simple weak maximum principle; both statements are easily proved using basic properties of the heat kernel.

Corollary 5.2.5. *Let* $\overline{\sigma} := \sigma(s=0)$. *Then the following point-wise inequality holds.*

$$|\sigma|(x,s) \leq e^{s\triangle}|\overline{\sigma}|(x) + \int_0^s e^{(s-\overline{s})\triangle}|\mathcal{N}(\overline{s})|(x)\,\mathrm{d}\overline{s}, \tag{5.2.9}$$

Proof. This is an immediate consequence of (5.2.7), Duhamel's principle, and the fact that the heat kernel $K(x,y) = \frac{1}{(4\pi s)^{3/2}} e^{-|x-y|^2/4s}$ is everywhere positive. □

For later use, we need the following lemma for the Duhamel integral, whose proof utilizes the basic inequality (5.2.8) for the heat kernel.

Lemma 5.2.6. *The following estimate holds.*

$$\| \int_0^s e^{(s-\overline{s})\triangle}\mathcal{N}(\overline{s})\,\mathrm{d}\overline{s}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x(0,s_0]} \leq C\|\mathcal{N}\|_{\mathcal{L}^{1+1,2}_s \mathcal{L}^1_x(0,s_0]}. \tag{5.2.10}$$

Proof. Unwinding the definitions of p-normalized norms, (5.2.10) is equivalent to

$$\Big(\int_0^{s_0} s^{1/2}\| \int_0^s e^{(s-\overline{s})\triangle}\mathcal{N}(\overline{s})\,\mathrm{d}\overline{s}\|^2_{L^2_x} \frac{\mathrm{d}s}{s}\Big)^{1/2} \leq C\Big(\int_0^{s_0} s\|\mathcal{N}(s)\|^2_{L^1_x} \frac{\mathrm{d}s}{s}\Big)^{1/2}. \tag{5.2.11}$$

Let us put $f(s) = s^{1/2}\|\mathcal{N}(s)\|_{L^1_x}$; then it suffices to estimate the left-hand side of (5.2.11) by $C\|f\|_{\mathcal{L}^2_s(0,s_0]}$. By Minkowski and (5.2.8), we have

$$\| \int_0^s e^{(s-\overline{s})\triangle}\mathcal{N}(\overline{s})\,\mathrm{d}\overline{s}\|_{L^2_x} \leq C\int_0^s (s-\overline{s})^{-3/4}(\overline{s})^{1/2} f(\overline{s})\,\frac{\mathrm{d}\overline{s}}{\overline{s}}$$

Therefore the left-hand side of (5.2.11) is bounded from above by

$$C\Big(\int_0^{s_0} \Big(\int_0^s s^{1/4}(s-\overline{s})^{-3/4}(\overline{s})^{1/2} f(\overline{s})\,\frac{\mathrm{d}\overline{s}}{\overline{s}}\Big)^2 \frac{\mathrm{d}s}{s}\Big)^{1/2}. \tag{5.2.12}$$

Observe that

$$\sup_{s\in(0,s_0]} \int_0^s s^{1/4}(s-\overline{s})^{-3/4}(\overline{s})^{1/2}\,\frac{\mathrm{d}\overline{s}}{\overline{s}} \leq C, \quad \sup_{\overline{s}\in(0,s_0]} \int_{\overline{s}}^{s_0} s^{1/4}(s-\overline{s})^{-3/4}(\overline{s})^{1/2}\,\frac{\mathrm{d}s}{s} \leq C.$$

Therefore, by Schur's test, (5.2.12) is estimated by $\|f(s)\|_{\mathcal{L}^2_s(0,s_0]}$ as desired. □

Finally, we end this section with a simple lemma which is useful for substituting covariant derivatives by usual derivatives and vice versa.

Lemma 5.2.7. *For $k \geq 1$, and α be a multi-index of order k. Then the following schematic algebraic identities hold.*

$$\mathbf{D}_x^{(\alpha)}\sigma = \partial_x^{(\alpha)}\sigma + \sum_{\star} \mathcal{O}_\alpha(\partial_x^{(\ell_1)}A, \partial_x^{(\ell_2)}A, \cdots, \partial_x^{(\ell_j)}A, \partial_x^{(\ell)}\sigma), \tag{5.2.13}$$

$$\partial_x^{(\alpha)}\sigma = \mathbf{D}_x^{(\alpha)}\sigma + \sum_{\star} \mathcal{O}_\alpha(\partial_x^{(\ell_1)}A, \partial_x^{(\ell_2)}A, \cdots, \partial_x^{(\ell_j)}A, \mathbf{D}_x^{(\ell)}\sigma). \tag{5.2.14}$$

In both cases, the summation is over all $1 \leq j \leq k$ and $0 \leq \ell_1, \ldots, \ell_j, \ell \leq k-1$ such that

$$j + \ell_1 + \cdots \ell_j + \ell = k.$$

Proof. In the case $k = 1$, both (5.2.13) and (5.2.14) follow from the simple identity

$$\mathbf{D}_i\sigma = \partial_i\sigma + [A_i, \sigma].$$

The cases of higher k follow from a simple induction argument, using Leibniz's rule. We leave the easy detail to the reader. □

5.3 Analysis of covariant parabolic equations

The goal of this section is to analyze the covariant parabolic equations of (dYMHF) and (cYMHF) using the covariant techniques developed in §5.2. The key result for (dYMHF) is Proposition 5.3.2, which morally states that covariant parabolic estimates hold, i.e. any $\|\mathbf{D}_x^{(k)}\mathbf{F}(s)\|_{L^2_x}$ for $s > 0$ can be controlled by the Yang-Mills energy $\mathbf{E}$ with an appropriate weight of s. A parallel development for (cYMHF) using the magnetic energy $\mathbf{B}[F(t)] := \frac{1}{2}\sum_{i<j}\int |F_{ij}|^2 \mathrm{d}x$ will also be given; see Proposition 5.3.4.

5.3.1 Covariant parabolic equations of (dYMHF)

Let $I \subset \mathbb{R}$ be an interval, and consider a smooth solution $A_{\mathbf{a}}$ to the *dynamic Yang-Mills heat flow*

$$F_{s\mu} = \mathbf{D}^\ell F_{\ell\mu}, \tag{dYMHF}$$

on $I \times \mathbb{R}^3 \times [0,1]$.

Recall, from Chapter 2, the following parabolic equation satisfied by $F_{\mu\nu}$:

$$\mathbf{D}_s F_{\mu\nu} - \mathbf{D}^\ell \mathbf{D}_\ell F_{\mu\nu} = -2[F_\mu{}^\ell, F_{\nu\ell}].$$

We shall derive covariant parabolic equations satisfied by higher *covariant* derivatives of $\mathbf{F}$. Given a $\mathfrak{g}$-valued tensor B, we compute

$$\mathbf{D}_i \mathbf{D}_s B - \mathbf{D}_i \mathbf{D}^\ell \mathbf{D}_\ell B = \mathbf{D}_s \mathbf{D}_i B - \mathbf{D}^\ell \mathbf{D}_\ell \mathbf{D}_i B - 2[F_i{}^\ell, \mathbf{D}_\ell B].$$

Concisely, $[\mathbf{D}_i, \mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell] B = \mathbb{O}(F, \mathbf{D}_x B)$. Using this, it is not difficult to prove the following proposition.

Proposition 5.3.1 (Covariant parabolic equations of (dYMHF))**.** *Let $A_{\mathbf{a}}$ be a solution to* (dYMHF)*. Then the curvature 2-form $F_{\mu\nu}$ satisfies the following parabolic equation.*

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell) F_{\mu\nu} = -2[F_\mu{}^\ell, F_{\nu\ell}]. \tag{5.3.1}$$

The covariant derivatives of $F_{\mu\nu}$ satisfy the following schematic equation.

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell)(\mathbf{D}_x^{(k)} \mathbf{F}) = \sum_{j=0}^{k} \mathbb{O}(\mathbf{D}_x^{(j)} \mathbf{F}, \mathbf{D}_x^{(k-j)} \mathbf{F}). \tag{5.3.2}$$

Proceeding in the same manner for a solution A_a ($a = x^1, x^2, x^3, s$) to (cYMHF), we may derive the following equation for $\mathbf{D}_x^{(k)} F_{ij}$:

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell)(\mathbf{D}_x^{(k)} F) = \sum_{j=0}^{k} \mathbb{O}(\mathbf{D}_x^{(j)} F, \mathbf{D}_x^{(k-j)} F). \tag{5.3.3}$$

5.3.2 Estimates for the covariant parabolic equations

Let us fix a time $t \in I$. Let us denote the Yang-Mills energy of $\mathbf{F}(t)$ at $s = 0$ by $\mathbf{E}(t)$, i.e.

$$\mathbf{E}(t) := \mathbf{E}[\mathbf{F}(t, s=0)] = \sum_{\mu<\nu} \frac{1}{2} \| F_{\mu\nu}(t, s=0) \|_{L^2_x}^2.$$

Recall the notation $\mathcal{D}_i := s^{1/2} \mathbf{D}_i$. The following proposition, which is proved by applying covariant techniques to (5.3.2), is the analytic heart of this chapter.

Proposition 5.3.2 (Covariant parabolic estimates for $\mathbf{F}$). *Let $I \subset \mathbb{R}$ be an interval, and $t \in I$. Suppose that $A_{\mathbf{a}}$ is a solution to* (dYMHF) *on $I \times \mathbb{R}^3 \times [0,1]$ in $C^\infty_{t,s}(I \times [0,1], H^\infty_x)$. Then there exists $\delta_C > 0$ such that the following statement holds: If $\mathbf{E}(t) < \delta_C$, then for every integer $k \geq 1$, we have*

$$\|\mathcal{D}_x^{(k-1)}\mathbf{F}(t)\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x(0,1]} + \|\mathcal{D}_x^{(k)}\mathbf{F}(t)\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,1]} \leq C_{k,\mathbf{E}(t)} \cdot \sqrt{\mathbf{E}(t)}. \tag{5.3.4}$$

Proof. Fix $t \in I$, and let us start with the cases $k = 1, 2$. Let $\underline{s} \in (0,1]$. Applying the energy integral estimate (5.2.6) with $\ell = 3/4$ to (5.3.1) and and $\ell = 3/4 + 1/2$ to (5.3.2) for $\mathbf{D}_x\mathbf{F}$, we have

$$\|\mathbf{F}\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x(0,\underline{s}]} + \|\mathcal{D}_x\mathbf{F}\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\sqrt{\mathbf{E}} + C\|\mathbb{O}(\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]}.$$
$$\|\mathcal{D}_x\mathbf{F}\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x(0,\underline{s}]} + \|\mathcal{D}_x^{(2)}\mathbf{F}\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\|\mathcal{D}_x\mathbf{F}\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,\underline{s}]} + C\|\mathbb{O}(\mathcal{D}_x\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]}.$$

No term at $s = 0$ arises for the second estimate, as we have $\liminf_{s\to 0} s^{3/4}\|\mathcal{D}_x\mathbf{F}(s)\|_{\mathcal{L}^2_x(s)} = 0$ for a $\mathbf{F} \in C^\infty_s([0,1], H^\infty_x)$.

Combining the two inequalities, we obtain

$$\mathcal{B}_2(\underline{s}) \leq C\sqrt{\mathbf{E}} + C(\|\mathbb{O}(\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]} + \|\mathbb{O}(\mathcal{D}_x\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]}).$$

where

$$\mathcal{B}_2(\underline{s}) := \sum_{k=1,2} (\|\mathcal{D}_x^{(k-1)}\mathbf{F}\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x(0,\underline{s}]} + \|\mathcal{D}_x^{(k)}\mathbf{F}\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,\underline{s}]}).$$

Using Hölder and Corollary 5.2.2, we see that

$$\|\mathbb{O}(\sigma_1,\sigma_2)\|_{L^2_x} \leq C\|\sigma_1\|^{1/2}_{L^2_x}\|\mathbf{D}_x\sigma_1\|^{1/2}_{L^2_x}\|\mathbf{D}_x\sigma_2\|_{L^2_x}.$$

By the Correspondence Principle, Lemma 3.1.8 and the fact that $\underline{s} \leq 1$, we have

$$\|\mathbb{O}(\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\underline{s}^{1/4}\|\mathbf{F}\|^{1/2}_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x(0,\underline{s}]}\|\mathcal{D}_x\mathbf{F}\|^{3/2}_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\mathcal{B}_2(\underline{s})^2.$$

Similarly, we also have

$$\|\mathbb{O}(\mathcal{D}_x\mathbf{F},\mathbf{F})\|_{\mathcal{L}^{3/4+1,1}_s \mathcal{L}^2_x(0,\underline{s}]} \leq C\mathcal{B}_2(\underline{s})^2.$$

Therefore, we obtain a bound of the form $\mathcal{B}_2(\underline{s}) \leq C\sqrt{\mathbf{E}} + C\mathcal{B}_2(\underline{s})^2$, for every $\underline{s} \in (0,1]$. Then by a simple bootstrap argument, the bound $\mathcal{B}_2(1) \leq C\sqrt{\mathbf{E}}$ follows, which implies the desired estimate.

Let us turn to the case $k \geq 3$, which is proved by induction. Fix $k \geq 3$, and suppose, for the purpose of induction, that (5.3.4) holds for up to $k-1$. That is, defining

$$\mathcal{B}_{k-1} := \sum_{j=1}^{k-1} \Big[\|\mathcal{D}_x^{(j-1)} \mathbf{F}\|_{\mathcal{L}_s^{3/4,\infty} \mathcal{L}_x^2 (0,1]} + \|\mathcal{D}_x^{(j)} \mathbf{F}\|_{\mathcal{L}_s^{3/4,2} \mathcal{L}_x^2 (0,1]} \Big],$$

we shall assume that $\mathcal{B}_{k-1} \leq C_{k,\mathbf{E}} \cdot \sqrt{\mathbf{E}}$.

Applying the energy integral estimate (5.2.6) with $\ell = \frac{3}{4} + \frac{k-1}{2}$ to (5.3.2) for $\mathbf{D}_x^{(k-1)} \mathbf{F}$, we obtain

$$\|\mathcal{D}_x^{(k-1)} \mathbf{F}\|_{\mathcal{L}_s^{3/4,\infty} \mathcal{L}_x^2} + \|\mathcal{D}_x^{(k)} \mathbf{F}\|_{\mathcal{L}_s^{3/4,2} \mathcal{L}_x^2} \leq C \|\mathcal{D}_x^{(k-1)} \mathbf{F}\|_{\mathcal{L}_s^{3/4,2} \mathcal{L}_x^2} + C \sum_{j=0}^{k-1} \|\mathbb{O}(\mathcal{D}_x^{(j)} \mathbf{F}, \mathcal{D}_x^{(k-1-j)} \mathbf{F})\|_{\mathcal{L}_s^{3/4+1,1} \mathcal{L}_x^2}.$$

where we used the fact that $\liminf_{s \to 0} s^{3/4} \|\mathcal{D}_x^{(k-1)} \mathbf{F}(s)\|_{\mathcal{L}^2(s)} = 0$.

The first term is bounded by $\mathcal{B}_{k-1}$; therefore, (5.3.4) for k will follow once we establish

$$\sum_{j=0}^{k-1} \|\mathbb{O}(\mathcal{D}_x^{(j)} \mathbf{F}, \mathcal{D}_x^{(k-1-j)} \mathbf{F})\|_{\mathcal{L}_s^{3/4+1,1} \mathcal{L}_x^2} \leq C \mathcal{B}_{k-1}^2. \tag{5.3.5}$$

By Leibniz's rule, we see that (5.3.5) follows once we establish the estimates

$$\begin{cases} \|\mathbb{O}(\mathcal{D}_x G_1, \mathcal{D}_x G_2)\|_{\mathcal{L}_s^{3/4+1,1} \mathcal{L}_x^2} \leq C \mathcal{B}_2^2 \\ \|\mathbb{O}(G_1, \mathcal{D}_x^{(2)} G_2)\|_{\mathcal{L}_s^{3/4+1,1} \mathcal{L}_x^2} + \|\mathbb{O}(\mathcal{D}_x^{(2)} G_1, G_2)\|_{\mathcal{L}_s^{3/4+1,1} \mathcal{L}_x^2} \leq C \mathcal{B}_2^2, \end{cases} \tag{5.3.6}$$

for any $\mathfrak{g}$-valued 2-forms $G_i = G_i(x,s)$. Note that these roughly correspond to the case $k = 3$ of (5.3.5).

Using the Correspondence Principle, Lemma 3.1.8, and recalling the definition of $\mathcal{B}_{k-1}$, it suffices to prove the estimates

$$\begin{aligned} \|\mathbb{O}(\mathbf{D}_x \sigma_1, \mathbf{D}_x \sigma_2)\|_{L_x^2} \leq & C \|\mathbf{D}_x \sigma_1\|_{L_x^2}^{1/2} \|\mathbf{D}_x^{(2)} \sigma_1\|_{L_x^2}^{1/2} \|\mathbf{D}_x^{(2)} \sigma_2\|_{L_x^2}, \\ \|\mathbb{O}(\sigma_1, \mathbf{D}_x^{(2)} \sigma_2)\|_{L_x^2} \leq & C \|\mathbf{D}_x \sigma_1\|_{L_x^2}^{1/2} \|\mathbf{D}_x^{(2)} \sigma_1\|_{L_x^2}^{1/2} \|\mathbf{D}_x^{(2)} \sigma_2\|_{L_x^2}. \end{aligned}$$

The former is an easy consequence of Hölder, (5.2.2) and (5.2.3), whereas the latter is proved similarly by applying Hölder, (5.2.3) and (5.2.4). □

Recalling $F_{s\nu} = \mathbf{D}^\ell F_{\ell\nu}$, we obtain the following estimates for $F_{s\nu}$.

Corollary 5.3.3. *Under the same hypotheses as Proposition 5.3.2, the following estimates hold for* *every integer* $k \geq 0$.

$$\|\mathcal{D}_x^{(k)} \mathbf{F}_s\|_{\mathcal{L}_s^{5/4,\infty} \mathcal{L}_x^2(0,1]} + \|\mathcal{D}_x^{(k)} \mathbf{F}_s\|_{\mathcal{L}_s^{5/4,2} \mathcal{L}_x^2(0,1]} \leq C_{k,\mathbf{E}} \cdot \sqrt{\mathbf{E}}, \tag{5.3.7}$$

$$\|\mathcal{D}_x^{(k)} \mathbf{F}_s\|_{\mathcal{L}_s^{5/4,\infty} \mathcal{L}_x^\infty(0,1]} + \|\mathcal{D}_x^{(k)} \mathbf{F}_s\|_{\mathcal{L}_s^{5/4,2} \mathcal{L}_x^\infty(0,1]} \leq C_{k,\mathbf{E}} \cdot \sqrt{\mathbf{E}}. \tag{5.3.8}$$

Proof. The L^2-type estimate (5.3.7) follows immediately from Proposition 5.3.2 by the relation $F_{s\nu} = \mathbf{D}^\ell F_{\ell\nu}$. The L^∞-type estimate (5.3.8) then follows from (5.3.7) by (5.2.4) of Corollary 5.2.2 (covariant Gagliardo-Nirenberg) and the Correspondence Principle. □

The above discussion may be easily restricted to spatial connection 1-form A_i satisfying (cYMHF). Given a spatial 2-form $F = F_{ij}$ $(i, j = 1, 2, 3)$, let us define the *magnetic energy* $\mathbf{B}[F]$ by

$$\mathbf{B}[F] := \sum_{i<j} \frac{1}{2} \|F_{ij}(s=0)\|_{L_x^2}^2.$$

Repeating the proof of Proposition 5.3.2, the following proposition easily follows.

Proposition 5.3.4 (Covariant parabolic estimates for F)**.** *Let $\delta_C > 0$ be as in Proposition 5.3.2, and consider a solution (A_i, A_s) to the covariant Yang-Mills heat flow $F_{si} = \mathbf{D}^\ell F_{\ell i}$ in $C_s^\infty([0,1], H_x^\infty)$. Suppose furthermore that $\mathbf{B} := \mathbf{B}[F(s=0)] < \delta_C$. Then the following estimate holds for every integer $k \geq 1$:*

$$\|\mathcal{D}_x^{(k-1)} F\|_{\mathcal{L}_s^{3/4,\infty} \mathcal{L}_x^2(0,1]} + \|\mathcal{D}_x^{(k)} F\|_{\mathcal{L}_s^{3/4,2} \mathcal{L}_x^2(0,1]} \leq C_{k,\mathbf{B}} \cdot \sqrt{\mathbf{B}}. \tag{5.3.9}$$

5.4 Yang-Mills heat flows in the caloric gauge

The main focus of §5.4.1 will be to apply the covariant smoothing estimates proved in §5.3 to (cYMHF) in the caloric gauge, continuing the study in §3.6. Note that (cYMHF) in the caloric gauge is nothing but the original Yang-Mills heat flow (YMHF).

As a byproducts of the analysis, we shall obtain an alternative proof of global existence of a solution to the IVP for (YMHF) when the initial data is regular and possesses finite (magnetic) energy (Corollary 5.4.3). This result was first proved[3] by Råde in his paper [27], but employing a different method than here. For more discussion, we refer the reader to the remark after Theorem C below.

In §5.4.2, we shall develop a parallel study of *dynamic Yang-Mills heat flow* (dYMHF) under the

[3]The paper [27] deals with the Yang-Mills heat flow on *compact* 2- and 3-dimensional Riemannian manifolds, but the proof also applies to the case of $\mathbb{R}^3$.

caloric gauge condition $A_s = 0$. This will be the main ingredient for the proof of Theorem E in the next section.

Finally, in §5.4.3, we shall present some results which are useful for transferring estimates involving covariant derivatives (such as those obtained in §5.3) to the corresponding ones with usual derivatives. The key ingredient is (5.4.4), which is an L^∞_x-estimate for the connection coefficients $\mathbf{A} = A_\mu$.

5.4.1 Analysis of (YMHF)

Here, we consider (cYMHF) in the caloric gauge $A_s = 0$. As this system is simply the original Yang-Mills heat flow, we shall refer to it simply as (YMHF).

Theorem 5.4.1 (Improved local well-posedness for (YMHF))**.** *Consider an initial data set $\overline{A}_i \in H^\infty_x$. Suppose furthermore that $\overline{F}_{ij} := \partial_i \overline{A}_j - \partial_j \overline{A}_i + [\overline{A}_i, \overline{A}_j]$ belongs to L^2_x and the norm is sufficiently small, i.e.*

$$\mathbf{B}[\overline{F}] = \frac{1}{2} \sum_{i<j} \| \overline{F}_{ij} \|^2_{L^2_x} < \delta_C,$$

where $\delta_C > 0$ is the the positive number as in Proposition 5.3.2.

Then there exists a unique solution A_i to (YMHF) *with initial data $A_i(s=0) = \overline{A}_i$ on* $[0,1]$*, which belongs to $C^\infty_s([0,1], H^\infty)$.*

Remark 5.4.2. Other constituents of a local well-posedness statement, such as continuous dependence on the data, can be proved by a minor modification of the proof below. Also, the statement can be extended to a rougher class of initial data and solutions by an approximation argument. We shall not provide proofs for these as they are not needed in the sequel; we welcome the interested reader to fill in the details.

Proof. By Proposition 3.6.1 (with rescaling), there exists $s^\star > 0$ such that a unique solution A_i to the IVP for (YMHF) exists in $C^\infty_s([0, s^\star], H^\infty_x)$ and obeys

$$\sup_{0 \le s \le s^\star} \| A(s) \|_{L^6_x} \le C \sup_{0 \le s \le s^\star} \| A(s) \|_{\dot{H}^1_x} \le C \| \overline{A} \|_{\dot{H}^1_x}. \tag{5.4.1}$$

We remark that the first inequality holds by Sobolev embedding.

Let us denote by $s_{\max}$ the largest s-parameter for which A_i extends as a unique solution in $C^\infty_s([0, s_{\max}), H^\infty_x)$. We claim that under the hypothesis that $\mathbf{B}[\overline{F}] < \delta_C$, the following statement holds:

$$\text{If } s_{\max} \le 1 \text{ then } \sup_{s \in [0, s_{\max})} \|A_i\|_{\dot{H}^1_x} < \infty. \tag{5.4.2}$$

If this claim were true, then we may apply Proposition 3.6.1 to extend A_i past $s_{\max}$ if $s_{\max} \le 1$. Therefore, it would follow that $s_{\max} > 1$.

Let us establish (5.4.2). The first step is to show that $\|A(s)\|_{L^6_x}$ does not blow up on $[0, s_{\max})$. By (5.4.1), it suffices to restrict our attention to $s > s^\star$; therefore, $s \in (s^\star, s_{\max})$. Since $s_{\max} \le 1$, by Proposition 5.3.4 and Corollary 5.2.2, we see that

$$\|\partial_s A_i(s)\|_{L^6_x} = \|\mathbf{D}^\ell F_{\ell i}(s)\|_{L^6_x} \le s^{-1} C_{\mathbf{B}} \cdot \sqrt{\mathbf{B}}.$$

Integrating from $s = s^\star$ and using (5.4.1), we arrive at[4]

$$\sup_{s^\star < s < s_{\max}} \|A(s)\|_{L^6_x} \le \|\overline{A}\|_{\dot{H}^1_x} + C_{\mathbf{B}} \cdot |\log s^\star| \sqrt{\mathbf{B}} < \infty. \tag{5.4.3}$$

Next, let us show that $\|A_i(s)\|_{\dot{H}^1_x}$ does not blow up on $[0, s_{\max})$. Again, it suffices to consider $s \in (s^\star, s_{\max})$. Recall that $\mathbf{D}_x \mathbf{D}^\ell F_{\ell i} = \partial_x \mathbf{D}^\ell F_{\ell i} + [A, \mathbf{D}^\ell F_{\ell i}]$; thus by triangle and Hölder,

$$\|\partial_s A_i(s)\|_{\dot{H}^1_x} \le \|\mathbf{D}_x \mathbf{D}^\ell F_{\ell i}(s)\|_{L^2_x} + \|A(s)\|_{L^6_x} \|\mathbf{D}^\ell F_{\ell i}(s)\|_{L^3_x}.$$

Using Proposition 5.3.4 and Corollary 5.2.2, we obtain

$$\|\partial_s A_i(s)\|_{\dot{H}^1_x} \le s^{-1} C_{\mathbf{B}} \cdot \sqrt{\mathbf{B}} + s^{-3/4} C_{\mathbf{B}} \cdot \sqrt{\mathbf{B}} \Big(\sup_{s^\star < s < s_{\max}} \|A(s)\|_{L^6_x} \Big).$$

Recalling (5.4.2) and integrating from $s^\star$, we see that $\sup_{s^\star < s < s_{\max}} \|A_i\|_{\dot{H}^1_x} < \infty$, as desired. $\square$

For any initial data (in H^∞_x) with finite magnetic energy, we can use scaling to make $\mathbf{B}(s = 0) < \delta_C$; thus, Theorem 5.4.1 applies also to initial data with large magnetic energy. Furthermore, using the fact that the magnetic energy $\mathbf{B}(s)$ is non-increasing in s under the Yang-Mills heat flow (which is formally obvious, as the Yang-Mills heat flow is the gradient flow of $\mathbf{B}$; see [27]), we can in fact iterate Theorem 5.4.1 to obtain a unique global solution to the IVP, leading to an independent proof of the following classical result of [27].

Corollary 5.4.3 (Råde [27]). *Consider the IVP for* (YMHF) *with an initial data set* $\mathring{A}_i \in H^\infty_x$ *which possesses finite magnetic energy, i.e.* $\mathbf{B}[\overline{F}] := (1/2) \sum_{i<j} \|\overline{F}_{ij}\|_{L^2_x} < \infty$. *Then there exists a*

[4] Note that integrating from $s^\star$ allows us to bypass the issue of logarithmic divergence at $s = 0$.

unique global solution A_i to the IVP in $C_s^\infty([0,\infty), H_x^\infty)$.

5.4.2 Analysis of (dYMHF) in the caloric gauge

Here, we shall present an analogue of Theorem 5.4.1 for (dYMHF).

Theorem 5.4.4 (Local well-posedness for (dYMHF) in the caloric gauge)**.** *Let $I \subset \mathbb{R}$ be an interval and consider an initial data set $\overline{A}_\mu \in C_t^\infty(I, H_x^\infty)$. Suppose furthermore that the energy is uniformly small on I, i.e.*

$$\sup_{t\in I} \mathbf{E}[\overline{\mathbf{F}}(t)] = \sup_{t\in I} \frac{1}{2}\sum_{\mu<\nu} \|\overline{F}_{\mu\nu}(t)\|_{L_x^2}^2 < \delta_C,$$

where $\delta_C > 0$ is the the small constant in Proposition 5.3.2.

Then there exists a unique solution A_μ to (dYMHF) *in the caloric gauge with initial data $A_\mu(s = 0) = \overline{A}_\mu$ on $[0,1]$, which belongs to $C_{t,s}(I\times[0,1], H_x^\infty)$.*

The proof is analogous to that for Theorem 5.4.1, replacing the use of Proposition 3.6.1 by3.6.4. We shall omit the details.

5.4.3 Substitution of covariant derivatives by usual derivatives

At several points below, we shall need to transfer estimates for covariant derivatives to the corresponding estimates for usual derivatives. The purpose of this part is to develop a general technique for carrying out such procedures. Our starting point is the following proposition, which concerns estimates for the L_x^∞ norm of $\mathbf{A}$.

To state the following proposition, we need the following definition.

$${}^{(\underline{A})}\mathcal{I}(t) := \sum_{k=1}^{31} \|\partial_{t,x}\underline{A}(t)\|_{\dot{H}_x^{k-1}}.$$

In fact, this is a part of a larger norm $\mathcal{I}(t)$, whose definition had been given in §4.3.2.

Proposition 5.4.5. *Let $I \subset \mathbb{R}$ be an interval, $t \in I$, and consider a solution $A_\mu \in C_{t,s}^\infty(I\times[0,1], H_x^\infty)$ to* (dYMHF) *in the caloric gauge $A_s = 0$ on $I\times\mathbb{R}^3\times[0,1]$. Suppose that $\mathbf{E}(t) := \mathbf{E}[\overline{\mathbf{F}}(t)] < \delta_C$, where $\delta_C > 0$ is the small constant in Proposition 5.3.2. Then the following estimate holds for all $0 \le k \le 29$.*

$$\|\nabla_x^{(k)}\mathbf{A}(t)\|_{\mathcal{L}_s^{1/4,\infty}\mathcal{L}_x^\infty(0,1]} \le {}^{(\underline{A})}\mathcal{I}(t) + C_{k,\mathbf{E}(t),{}^{(\underline{A})}\mathcal{I}(t)} \cdot \sqrt{\mathbf{E}(t)}. \tag{5.4.4}$$

Proof. Henceforth, we shall fix $t \in I$ and omit writing t. By the caloric gauge condition $A_s = 0$, we have the relation $\partial_s A_\nu = F_{s\nu}$, where the latter can be controlled by Corollary 5.3.3. Observe furthermore that

$$\|\partial_x^{(k)} \underline{\mathbf{A}}\|_{L^\infty_x} \leq C\, {}^{(\underline{A})}\mathcal{I}$$

for $0 \leq k \leq 29$, by Sobolev (or Gagliardo-Nirenberg). Now, the idea is to use the fundamental theorem of calculus of control $\partial_x^{(k)} \mathbf{A}(s)$ for $0 < s \leq 1$.

We shall proceed by induction on k. Let us start with the case $k = 0$. By the fundamental theorem of calculus and Minkowski's inequality, we have

$$s^{1/4} \|\mathbf{A}(s)\|_{L^\infty_x} \leq s^{1/4} \|\underline{\mathbf{A}}\|_{L^\infty_x} + \int_s^1 (s/s')^{1/4} (s')^{5/4} \|\mathbf{F}_s(s')\|_{L^\infty_x} \frac{\mathrm{d}s'}{s'}.$$

As remarked earlier, the first term on the right-hand side may be estimated by ${}^{(\underline{A})}\mathcal{I}$ uniformly in $s \in (0, 1]$. For the second term, we apply (5.3.8) of Corollary 5.3.3 and estimate $(s')^{5/4} \|\mathbf{F}_s(s')\|_{L^\infty_x}$ by $C_{k,\mathbf{E}} \cdot \sqrt{\mathbf{E}}$. The case $k = 0$ of (5.4.4) follows, since

$$\sup_{0<s\leq 1} \int_s^1 (s/s')^{1/4} \frac{\mathrm{d}s'}{s'} \leq C < \infty.$$

Next, for the purpose of induction, assume that (5.4.4) holds for $0, 1, \ldots, k-1$, where $1 \leq k \leq 29$. Taking $\partial_x^{(k)}$ of $\partial_s A_\nu = F_{s\nu}$, using the fundamental theorem of calculus, Minkowski's inequality and multiplying both sides by $s^{1/4+k/2}$, we arrive at

$$s^{1/4} \|\nabla_x^{(k)} \mathbf{A}(s)\|_{L^\infty_x} \leq s^{1/4+k/2} \|\partial_x^{(k)} \underline{\mathbf{A}}\|_{L^\infty_x} + \int_s^1 (s/s')^{1/4+k/2} (s')^{5/4} \|\nabla_x^{(k)} \mathbf{F}_s(s')\|_{L^\infty_x} \frac{\mathrm{d}s'}{s'}.$$

Once we establish

$$\sup_{0<s\leq 1} \|s^{5/4} \nabla_x^{(k)} \mathbf{F}_s\|_{\mathcal{L}^\infty_x} \leq C_{k,\mathbf{E},{}^{(\underline{A})}\mathcal{I}} \cdot \sqrt{\mathbf{E}}, \tag{5.4.5}$$

then proceeding as in the previous case, (5.4.4) for k will follow, which completes the induction.

Fix $0 < s \leq 1$. Applying (5.2.14) of Lemma 5.2.7 and multiplying both sides by $s^{5/4+k/2}$, we see that

$$s^{5/4} \nabla_x^{(k)} \mathbf{F}_s(s) = s^{5/4} \mathcal{D}_x^{(k)} \mathbf{F}_s(s) + \sum_\star s^{j/4} \mathcal{O}(s^{1/4} \nabla_x^{(\ell_1)} A, \cdots, s^{1/4} \nabla_x^{(\ell_j)} A, s^{5/4} \mathcal{D}_x^{(\ell)} \mathbf{F}_s),$$

where the range of the summation is as specified in Lemma 5.2.7. Let us take the L^∞_x-norm of both sides; by the triangle inequality and (5.3.8) of Corollary 5.3.3, it suffices to control

$$s^{j/4} \|\mathcal{O}(s^{1/4} \nabla_x^{(\ell_1)} A, \cdots, s^{1/4} \nabla_x^{(\ell_j)} A, s^{5/4} \mathcal{D}_x^{(\ell)} \mathbf{F}_s)\|_{L^\infty_x}$$

for each summand of $\sum_\star$. Let us throw away the extra power $s^{j/4}$ (which is okay as $0 < s \leq 1$) [5]. Observe that $0 \leq \ell_1, \ldots, \ell_j \leq k-1$; therefore, by the induction hypothesis, we have

$$\|s^{1/4}\nabla_x^{(\ell_1)} A(s)\|_{L_x^\infty}, \cdots, \|s^{1/4}\nabla_x^{(\ell_j)} A(s)\|_{L_x^\infty} \leq C_{k,\mathbf{E},{}^{(\underline{A})}\mathcal{I}}.$$

Note furthermore that $\|s^{5/4}\mathcal{D}_x^{(\ell)}\mathbf{F}_s(s)\|_{L_x^\infty} \leq C_{\ell,\mathbf{E}} \cdot \sqrt{\mathbf{E}}$ by Corollary 5.3.3. Hence, by Hölder, each summand may be estimated by $C_{k,\mathbf{E},{}^{(\underline{A})}\mathcal{I}}$, and thus (5.4.5) follows. □

As a consequence, we obtain the following corollary which allows us to easily switch estimates for covariant derivatives to those for usual derivatives.

Corollary 5.4.6 (Substitution of covariant derivatives by usual derivatives)**.** *Assume that the hypotheses of Proposition 5.4.5 hold. Let σ be a $\mathfrak{g}$-valued function on $\{t\} \times \mathbb{R}^3 \times (0,1]$, $m \geq 0$ an integer, $b \geq 0, 1 \leq p, r \leq \infty$. Suppose that there exists $D > 0$ such that the estimate*

$$\|\mathcal{D}_x^{(k)}\sigma\|_{\mathcal{L}_s^{b,p}\mathcal{L}_x^r} \leq D \tag{5.4.6}$$

holds for $0 \leq k \leq m$. Then we have

$$\|\nabla_x^{(k)}\sigma\|_{\mathcal{L}_s^{b,p}\mathcal{L}_x^r} \leq C_{{}^{(\underline{A})}\mathcal{I}(t),\mathbf{E}(t)} \cdot D \tag{5.4.7}$$

for $0 \leq k \leq \min(m, 30)$.

Proof. We shall again omit t in this proof. The case $k = 0$ is obvious; we thus fix $1 \leq k \leq \min(m, 30)$. Using (5.2.14) of Lemma 5.2.7 to σ and multiplying by $s^{b+k/2}$, we get

$$s^b\nabla_x^{(k)}\sigma(s) = s^b\mathcal{D}_x^{(k)}\sigma(s) + \sum_\star s^{j/4}\mathcal{O}(s^{1/4}\nabla_x^{(\ell_1)}A(s), \cdots, s^{1/4}\nabla_x^{(\ell_j)}A(s), s^b\mathcal{D}_x^{(\ell)}\sigma(s)),$$

where the range of summation $\sum_\star$ is as specified in Lemma 5.2.7. Taking the $\mathcal{L}_s^p\mathcal{L}_x^r$ norm of both sides, applying triangle and using (5.4.6) to estimate $\|s^b\mathcal{D}_x^{(k)}\sigma(s)\|_{\mathcal{L}_s^p\mathcal{L}_x^r} = \|\mathcal{D}_x^{(k)}\sigma\|_{\mathcal{L}_s^{b,p}\mathcal{L}_x^r} \leq D$, we are left to establish

$$s^{j/4}\|\mathcal{O}(s^{1/4}\nabla_x^{(\ell_1)}A(s), \cdots, s^{1/4}\nabla_x^{(\ell_j)}A(s), s^b\mathcal{D}_x^{(\ell)}\sigma(s))\|_{\mathcal{L}_s^p\mathcal{L}_x^r} \leq C_{{}^{(\underline{A})}\mathcal{I}} \cdot D \tag{5.4.8}$$

for each summand in $\sum_\star$. Note that we have an extra power of $s^{j/4}$, which we can just throw away

[5] We gain an extra power of $s^{1/4}$ for each factor of A_i replacing ∂_i, thanks to the subcriticality of the problem at hand.

(as $0 < s \leq 1$). Let us use Hölder to put each $s^{1/4} \nabla_x^{(\ell_i)} A(s)$ in $\mathcal{L}^\infty_s \mathcal{L}^\infty_x$ and $s^b \mathcal{D}_x^{(\ell)} \sigma(s)$ in $\mathcal{L}^p_s \mathcal{L}^r_x$. Then using Proposition 5.4.5 (This is possible since $k \leq 30$) and (5.4.6) to control the respective norms, we obtain (5.4.8). □

5.5 Transformation to the caloric-temporal gauge: Proof of Theorem E

The purpose of this section is to prove Theorem E. The key idea is to complement Theorem A with the improved local well-posedness of (dYMHF) in the caloric gauge (Theorem 5.4.4).

Proof of Theorem E. We begin with a regular solution $A^\dagger_\mu$ to (YM) in the temporal gauge, defined on $(-T_0, T_0) \times \mathbb{R}^3$. Thanks to the regularity assumption, note that $A^\dagger_\mu \in C^\infty_t((-T_0, T_0), H^\infty_x)$.

Step 1. Construction of regular solution to (HPYM) *in caloric-temporal gauge.* Recall the hypothesis (5.1.1). By smoothness in t and conservation of energy, respectively, it follows that

$$\sup_{t \in (-\epsilon_0, \epsilon_0)} \|A^\dagger\|_{\dot{H}^1} < \delta_P, \qquad \sup_{t \in (-T_0, T_0)} \mathbf{E}[\mathbf{F}^\dagger(t)] < \delta_C, \tag{5.5.1}$$

for some small $\epsilon_0 > 0$. The second smallness condition allows us to apply Theorem 5.4.4, from which we obtain a unique solution $\widetilde{A}_{\mathbf{a}}$ to (dYMHF) in the caloric gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$, which belongs to $C_{t,s}((-T_0, T_0) \times [0, 1], H^\infty_x)$.

We shall apply a gauge transform $V = V(t, x, s)$ to $\widetilde{A}_{\mathbf{a}}$ to enforce the caloric-temporal gauge condition. Let us denote the resulting connection coeffients $A_{\mathbf{a}}$, i.e.

$$A_{\mathbf{a}} := V \widetilde{A}_{\mathbf{a}} V^{-1} - \partial_{\mathbf{a}} V V^{-1}.$$

In order for $A_{\mathbf{a}}$ to be in the caloric-temporal gauge, we need a gauge transform V which is A) independent of s (to keep $A_s = 0$) and B) makes $\underline{A}_0 = 0$. These two requirements are in fact equivalent (once one assumes enough regularity of V) to V solving the ODE

$$\begin{cases} \partial_t V = V \, \underline{\widetilde{A}}_0, \\ V(t = 0) = \mathring{V}, \end{cases}$$

where $\underline{\widetilde{A}}_0 := \widetilde{A}_0(s = 1)$ and $\mathring{V}$ is a gauge transform on $\mathbb{R}^3$, to be specified in Step 2 in accordance to Theorem A.

Step 2. Application of Theorem A.

The next step of the proof is to apply Theorem A to choose $\mathring{V}$ and furthermore obtain a quantitative estimate for ${}^{(\underline{A})}\mathcal{I}(0)$. Thanks to the first inequality of (5.5.1), we may apply Theorem A on the time interval $(-\epsilon_0, \epsilon_0)$. Let us mark the objects obtained from Theorem A with a prime, i.e. A'_μ, V' and $\mathring{V}'$. Consider $\widetilde{A}'_\mu = \widetilde{A}'_\mu(t,x,s)$ defined by

$$\widetilde{A}'_\mu := (V')^{-1} A'_\mu V' - \partial_\mu (V')^{-1} V',$$

where we remind the reader that $V' = V'(t,x)$, $(V')^{-1} = (V')^{-1}(t,x)$ are independent of s.

Note that $\widetilde{A}'_\mu$ is a solution to (dYMHF) in the caloric gauge in $C_{t,s}((-\epsilon_0,\epsilon_0)\times[0,1], H^\infty_x)$, as is $\widetilde{A}_\mu$. Moreover, their initial data sets coincide (both being $A^\dagger_\mu$). By the uniqueness lemma (Lemma 3.6.6), we conclude that $\widetilde{A}_\mu = \widetilde{A}'_\mu$ on $(-\epsilon_0,\epsilon_0)\times\mathbb{R}^3\times[0,1]$, i.e.

$$\widetilde{A}_\mu = (V')^{-1} A'_\mu V' - \partial_\mu (V')^{-1} V'$$

on $(-\epsilon_0,\epsilon_0)\times\mathbb{R}^3\times[0,1]$. As $\underline{A}'_0 = 0$, we also see that

$$\begin{cases} \partial_t V' = V' \underline{\widetilde{A}}_0, \\ V'(t=0) = \mathring{V}'. \end{cases}$$

on $(-\epsilon_0,\epsilon_0)\times\mathbb{R}^3$.

At this point, let us make the choice $\mathring{V} = \mathring{V}'$. Then the previous ODE is exactly that satisfied by V. Therefore, by uniqueness for ODE with smooth coefficients, $V = V'$ on $(-\epsilon_0,\epsilon_0)\times\mathbb{R}^3$, and hence we conclude that

$$A_\mu = A'_\mu$$

on $(-\epsilon_0,\epsilon_0)\times\mathbb{R}^3\times[0,1]$. From Theorem A, the quantitative estimates in (5.1.3) follow. Moreover, it is not difficult to show that V is a regular gauge transform on $(-T_0,T_0)\times\mathbb{R}^3$. It also follows that A_μ is a regular solution on $(-T_0,T_0)\times\mathbb{R}^3\times[0,1]$, since $\widetilde{A}_\mu$ were regular. This completes the proof of Theorem E. □

5.6 Fixed-time estimates by E : Proof of Theorem F

Our aim in this section is to prove Theorem F. To begin with, we shall split $\mathcal{I}(t) = {}^{(\underline{A})}\mathcal{I}(t) + {}^{(F_s)}\mathcal{I}(t)$, where

$$\begin{aligned}
{}^{(F_s)}\mathcal{I}(t) &:= \sum_{k=1}^{10} \Big[\|\nabla_{t,x} F_s(t)\|_{\mathcal{L}^{5/4,\infty}_s \dot{\mathcal{H}}^{k-1}_x} + \|\nabla_{t,x} F_s(t)\|_{\mathcal{L}^{5/4,2}_s \dot{\mathcal{H}}^{k-1}_x} \Big], \\
{}^{(\underline{A})}\mathcal{I}(t) &:= \sum_{k=1}^{31} \|\partial_{t,x} \underline{A}(t)\|_{\dot{H}^{k-1}_x}.
\end{aligned}$$

Theorem F will be reduced to establishing two inequalities, namely (5.6.1) and (5.6.6) of Propositions 5.6.1 and 5.6.2, respectively.

Throughout this section, we shall be concerned with a regular solution $A_{\mathbf{a}}$ to (HPYM) in the caloric-temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0, 1]$ $(T_0 > 0)$, which satisfies $\mathcal{I}(0) = {}^{(\underline{A})}\mathcal{I}(0) + {}^{(F_s)}\mathcal{I}(0) < \infty$ and $\mathbf{E}[\overline{\mathbf{F}}] < \infty$. By conservation of energy for (YM) along $s = 0$, we see that

$$\mathbf{E}[\mathbf{F}(t, s = 0)] = \mathbf{E}[\overline{\mathbf{F}}] \quad \forall t \in (-T_0, T_0).$$

We shall denote the common value of $\mathbf{E}[\mathbf{F}(t, s = 0)]$ by $\mathbf{E}$.

Proposition 5.6.1. *There exists*[6] $N > 0$ *such that for any* $t \in (-T_0, T_0)$, *we have*

$$ {}^{(\underline{A})}\mathcal{I}(t) \leq C_{{}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} \, (1 + |t|)^N. \tag{5.6.1}$$

Proof. By symmetry, it suffices to consider $t > 0$. The main idea is to use the relation

$$\underline{F}_{0i} = \partial_t \underline{A}_i, \tag{5.6.2}$$

which holds thanks to the fact that we are in the temporal gauge $\underline{A}_0 = 0$ along $s = 1$, and proceed as in the proof of Proposition 5.4.5.

We first estimate the L^∞_x norms. We claim that

$$\|\partial_x^{(k)} \underline{A}(t)\|_{L^\infty_x} \leq C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} \, (1 + t)^{k+1}. \tag{5.6.3}$$

for $0 \leq k \leq 29$.

[6] In the course of the proof, it will be clear that N may be chosen to depend only on the number of derivatives of $\underline{A}_i$ controlled. In our case, in which we control up to 31 derivatives of $\underline{A}_i$, we may choose $N = 32$.

Let us begin with the case $k = 0$ and proceed by induction. Note the inequality

$$\|\underline{A}_i(t)\|_{L^\infty_x} \leq \|\underline{A}_i(t=0)\|_{L^\infty_x} + \int_0^t \|\underline{F}_{0i}(t')\|_{L^\infty_x} \, \mathrm{d}t'.$$

Using Proposition 5.3.2, we may estimate the last term by $C_{\mathbf{E}}\, t$; from this, the $k = 0$ case of (5.6.3) follows.

Next, to carry out the induction, let us assume that (5.6.3) holds for $0, 1, \ldots, k-1$, where $1 \leq k \leq 29$. Taking $\partial_x^{(k)}$ of both sides of (5.6.2) and using the fundamental theorem of calculus, we obtain

$$\|\partial_x^{(k)} \underline{A}_i(t)\|_{L^\infty_x} \leq \|\partial_x^{(k)} \underline{A}_i(t=0)\|_{L^\infty_x} + \int_0^t \|\partial_x^{(k)} \underline{F}_{0i}(t')\|_{L^\infty_x} \, \mathrm{d}t'.$$

The first term is estimated by ${}^{(\underline{A})}\mathcal{I}(t=0)$, as $1 \leq k \leq 29$. For the second term, we apply (5.2.14) of Lemma 5.2.7. Then it suffices to estimate

$$\int_0^t \Big(\|\underline{\mathbf{D}}_x^{(k)} \underline{F}_{0i}(t')\|_{L^\infty_x} + \sum_\star \|\mathcal{O}(\partial_x^{(\ell_1)} \underline{A}, \ldots, \partial_x^{(\ell_j)} \underline{A}, \underline{\mathbf{D}}_x^{(\ell)} \underline{F}_{0i})(t')\|_{L^\infty_x} \Big) \, \mathrm{d}t'$$

(where $\underline{\mathbf{D}}_x := \partial_x + [\underline{A}_x, \cdot]$.) The range of the summation $\sum_\star$ is as in Lemma 5.2.7; in particular, $\ell_1, \ldots, \ell_j, \ell \leq k-1$. Let us use Hölder to estimate each factor in L^∞_x, and estimate the derivatives of $\underline{A}$ and $\underline{F}_{0i}$ by the induction hypothesis and Propostion 5.3.2, respectively. Then it is not difficult to see that the worst term (in terms of growth in t) is of the size

$$C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} \int_0^t (1+t')^k \, \mathrm{d}t' = C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} \, (1+t)^{k+1}.$$

Therefore, (5.6.3) for k follows. By induction, this establishes the claim.

With (5.6.3) in hand, we now proceed to prove

$$\|\partial_x^{(k)} \underline{A}(t)\|_{L^2_x} \leq C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} \, (1+t)^{k+1}. \tag{5.6.4}$$

for $1 \leq k \leq 31$.

Arguing as in the proof of (5.6.3), we arrive at the inequality

$$\begin{aligned} \|\partial_x^{(k)} \underline{A}_i(t)\|_{L^2_x} \leq & \|\partial_x^{(k)} \underline{A}_i(t=0)\|_{L^2_x} \\ & + \int_0^t \Big(\|\mathbf{D}_x^{(k)} \underline{F}_{0i}(t')\|_{L^2_x} + \sum_\star \|\mathcal{O}(\partial_x^{(\ell_1)} A, \ldots, \partial_x^{(\ell_j)} A, \mathbf{D}_x^{(\ell)} \underline{F}_{0i})(t')\|_{L^2_x} \Big) \, \mathrm{d}t'. \end{aligned}$$

It suffices to estimate the t'-integral. For $1 \leq k \leq 30$, let us use Hölder to estimate each $\partial_x^{(\ell_i)} A$ in L^∞_x and $\mathbf{D}_x^{(\ell)} \underline{F}_{0i}$ in L^2_x. Then we estimate these by (5.6.3) and Proposition 5.3.2, respectively, from which (5.6.4) follows immediately for $1 \leq k \leq 30$.

Next, proceeding similarly in the case $k = 31$, all terms are easily seen to be okay except

$$\|\mathcal{O}(\partial_x^{(30)} A, \underline{F}_{0i})(t')\|_{L^2_x},$$

for which we cannot use (5.6.3). In this case, however, we may put $\partial_x^{(30)} A$ in L^2_x and $\underline{F}_{0i}$ in L^∞_x. Then the former can be estimated by using the case $k = 30$ of (5.6.4) that we have just established, whereas the estimate for the latter follows from Proposition 5.3.2. It follows that this term is of size $C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} (1 + t')^{31}$. Integrating over $[0, t]$ gives the growth $C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} (1 + t)^{32}$.

Finally, we are left to prove estimates for $\partial_t \underline{A}_i$. For this purpose, we claim

$$\|\partial_x^{(k-1)} \partial_t \underline{A}(t)\|_{L^2_x} \leq C_{k, {}^{(\underline{A})}\mathcal{I}(0), \mathbf{E}} (1 + t)^{k-1}. \tag{5.6.5}$$

for $1 \leq k \leq 30$.

To prove (5.6.5), recall that $\partial_t \underline{A}_i = \underline{F}_{0i}$; therefore, the case $k = 1$ follows immediately from Proposition 5.3.2. For $k > 1$, we take $\partial_x^{(k-1)}$ and use Lemma 5.2.7 to substitute the usual derivatives by covariant derivatives. Then by (5.6.3), (5.6.4) and Proposition 5.3.2, (5.6.5) follows.

Combining (5.6.4) and (5.6.5), we obtain (5.6.1) with $N = 32$. □

Proposition 5.6.2. *For any $t \in (-T_0, T_0)$, we have*

$$^{(F_s)}\mathcal{I}(t) \leq C_{{}^{(\underline{A})}\mathcal{I}(t), \mathbf{E}} \cdot \sqrt{\mathbf{E}}. \tag{5.6.6}$$

Proof. Throughout the proof, the time $t \in (-T_0, T_0)$ will be fixed and thus be omitted.

Recalling the definition of $^{(F_s)}\mathcal{I}$, establishing (5.6.6) reduces to proving

$$\|\nabla_x F_s\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^{k-1}_x} \leq C_{k, {}^{(\underline{A})}\mathcal{I}, \mathbf{E}} \cdot \sqrt{\mathbf{E}}, \tag{5.6.7}$$

$$\|\nabla_0 F_s\|_{\mathcal{L}^{5/4,p}_s \dot{\mathcal{H}}^{k-1}_x} \leq C_{k, {}^{(\underline{A})}\mathcal{I}, \mathbf{E}} \cdot \sqrt{\mathbf{E}} \tag{5.6.8}$$

for $1 \leq k \leq 10$ and $p = 2, \infty$.

The estimate (5.6.7) is an easy consequence of (5.3.7) of Corollary 5.3.3 and Corollary 5.4.6. On the other hand, to prove (5.6.8), we use the formula

$$\mathbf{D}_0 F_{si} = \mathbf{D}^\ell \mathbf{D}_\ell F_{0i} + \mathbf{D}_i F_{s0} - 2[F_0{}^\ell, F_{i\ell}] - [A_0, F_{si}],$$

which is an easy consequence of the Bianchi identity and the parabolic equation for $\mathbf{D}_s F_{0i}$. Taking $\mathbf{D}_x^{(k-1)}$ of both sides and using Proposition 5.3.2, Corollary 5.3.3 and Proposition 5.4.5, we obtain

$$\|\mathcal{D}_x^{(k-1)} \nabla_0 F_{si}\|_{\mathcal{L}_s^{5/4,\infty} \mathcal{L}_x^2(0,1]} + \|\mathcal{D}_x^{(k-1)} \nabla_0 F_{si}\|_{\mathcal{L}_s^{5/4,2} \mathcal{L}_x^2(0,1]} \leq C_{k,\mathbf{E}} \cdot \sqrt{\mathbf{E}} \tag{5.6.9}$$

for $k \geq 1$. At this point, applying Corollary 5.4.6, we obtain (5.6.8). □

Combining Propositions 5.6.1 and 5.6.2, Theorem F follows.

5.7 Short time estimates for (HPYM) in the caloric-temporal gauge: Proof of Theorem G

The goal of this section is to prove Theorem G. As discussed in §5.1, this theorem follows from a local-in-time analysis of the wave equations of (HPYM). As such, its proof will follow closely that of Theorem B, which is essentially a 'H^1_x local well-posedness (in time)' statement for (HPYM) in the caloric-temporal gauge.

To begin with, let us recall the following definition from §4.4.1:

$$\mathcal{E}(t) := \sum_{m=1}^{3} \Big(\|\nabla_x^{(m-1)} F_{s0}(t)\|_{\mathcal{L}_s^{1,\infty} \mathcal{L}_x^2(0,1]} + \|\nabla_x^{(m)} F_{s0}(t)\|_{\mathcal{L}_s^{1,2} \mathcal{L}_x^2(0,1]} \Big).$$

Given a time interval $I \subset \mathbb{R}$, we define $\mathcal{E}(I)$ to be $\sup_{t \in I} \mathcal{E}(t)$.

The main reason why the analysis in Chapter 4 is insufficient to prove Theorem G is because of Proposition 4.4.3, which gives an estimate for $\mathcal{E}(t)$ *only under the hypothesis* that either the size of the initial data or the s-interval is small. The following proposition is a replacement of Proposition 4.4.3, which utilizes the smallness of the conserved energy $\mathbf{E}(t)$ instead. It is based on the covariant parabolic estimates derived in §5.3.

Proposition 5.7.1. *Let $I \subset \mathbb{R}$ be an open interval and $t \in I$. Consider a regular solution $A_{\mathbf{a}}$ to* (HPYM) *in the caloric-temporal gauge on $I \times \mathbb{R}^3 \times [0,1]$ such that*

$$\mathbf{E}(t) := \mathbf{E}[\overline{\mathbf{F}}(t)] < \delta_C, \quad {}^{(\underline{A})}\mathcal{I}(t) \leq D,$$

where $D > 0$ is an arbitrarily large number and $\delta_C > 0$ is the small constant in Proposition 5.3.2. Then the following estimate holds:

$$\mathcal{E}(t) \leq C_{D,\mathbf{E}(t)}. \tag{5.7.1}$$

In §5.7.1, we shall give a proof of Proposition 5.7.1. Assuming Proposition 5.7.1, the proof of Theorem G is a straightforward adaptation of that for Theorem B. We shall present a sketch in §5.7.2.

5.7.1 Improvement of estimates for F_{s0}

The goal of this subsection is to prove Proposition 5.7.1. Consider a regular solution $A_{\mathbf{a}}$ to (HPYM) on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ ($T_0 > 0$). Recall from Chapter 2 that F_{s0} satisfies the covariant parabolic equation

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell) F_{s0} = 2[F_0{}^\ell, F_{s\ell}]. \tag{5.7.2}$$

Recall furthermore that $[\mathbf{D}_i, (\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell)]B = \mathbb{O}(F, \mathbf{D}_x B)$. This implies that $\mathbf{D}_x^{(k)} F_{s0}$ for $k \geq 1$ satisfies the following schematic parabolic equation.

$$(\mathbf{D}_s - \mathbf{D}^\ell \mathbf{D}_\ell)(\mathbf{D}_x^{(k)} F_{s0}) = \sum_{j=0}^{k} \mathbb{O}(\mathbf{D}_x^{(j)} \mathbf{F}, \mathbf{D}_x^{(k-j)} \mathbf{F}_s). \tag{5.7.3}$$

Now recall that the hyperbolic Yang-Mills equation holds along $s = 0$. In particular, the constraint equation $\mathbf{D}^\ell F_{\ell 0}(s = 0) = 0$ holds, which is equivalent to $F_{s0}(s = 0) = 0$. Taking this extra ingredient into account, it follows that F_{s0} obeys an *improved bound* compared to the one proved in §5.3, as stated below.

Proposition 5.7.2 (Improved estimate for F_{s0}, with covariant derivatives)**.** *Let $T_0 > 0$ and $t \in (-T_0, T_0)$. Consider a regular solution $A_{\mathbf{a}}$ to* (HPYM) *on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ such that $\mathbf{E}(t) < \delta_C$, where $\delta_C > 0$ is the small constant in Proposition 5.3.2. Then the following estimate holds for each integer $k \geq 0$:*

$$\|\mathcal{D}_x^{(k-1)} F_{s0}(t)\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_x(0,1]} + \|\mathcal{D}_x^{(k)} F_{s0}(t)\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x(0,1]} \leq C_{k,\mathbf{E}(t)} \cdot \mathbf{E}(t), \tag{5.7.4}$$

When $k = 0$, we omit the first term on the left-hand side.

Proof. We shall fix $t \in (-T_0, T_0)$ and therefore omit writing t. Let us begin with the case $k = 0$. Applying Lemma 5.2.6 to the covariant parabolic equation for F_{s0}, along with the fact that $F_{s0} = 0$

at $s = 0$ thanks to (YM), it follows that

$$\|F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} \leq 2 \| \int_0^s e^{(s-\overline{s})\triangle} [F_0{}^{\ell}, F_{s\ell}](\overline{s}) \, \mathrm{d}\overline{s} \|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x}.$$

Using Lemma 5.2.6, Hölder, (5.3.4) (Proposition 5.3.2) and (5.3.7) (Corollary 5.3.3), we have

$$\begin{aligned} \| \int_0^s e^{(s-\overline{s})\triangle} [F_0{}^{\ell}, F_{s\ell}](\overline{s}) \, \mathrm{d}\overline{s} \|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} \leq & \| [F_0{}^{\ell}, F_{s\ell}] \|_{\mathcal{L}^{1+1,2}_s \mathcal{L}^1_x} \\ \leq & \|F_0{}^{\ell}\|_{\mathcal{L}^{3/4,\infty}_s \mathcal{L}^2_x} \|F_{s\ell}\|_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x} \leq C_{\mathbf{E}} \cdot \mathbf{E}. \end{aligned}$$

Therefore, we have proved $\|F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} \leq C_{\mathbf{E}} \cdot \mathbf{E}$.

For $k \geq 1$, we proceed by induction. Suppose, for the purpose of induction, that the cases $0, \cdots, k-1$ has already been established. Using the energy integral estimate (5.2.6) with $\ell = 1 + \frac{k-1}{2}$ to (5.7.3) for $\mathbf{D}^{(k-1)}_x F_{s0}$, we see that

$$\begin{aligned} & \|\mathcal{D}^{(k-1)}_x F_{s0}\|_{\mathcal{L}^{1,\infty}_s \mathcal{L}^2_x} + \|\mathcal{D}^{(k)}_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} \\ & \quad \leq C \|\mathcal{D}^{(k-1)}_x F_{s0}\|_{\mathcal{L}^{1,2}_s \mathcal{L}^2_x} + C \sum_{j=0}^{k-1} \|\mathbb{O}(\mathcal{D}^{(j)}_x \mathbf{F}, \mathcal{D}^{(k-1-j)}_x \mathbf{F}_s)\|_{\mathcal{L}^{1,1}_x \mathcal{L}^2_x}. \end{aligned}$$

The first term on the right-hand side is acceptable by the induction hypothesis; we therefore focus on the second term. Let us use Hölder to estimate $\mathcal{D}^{(j)}_x \mathbf{F}$ in $\mathcal{L}^{3/4,2}_s \mathcal{L}^6_x$ and $\mathcal{D}^{(k-1-j)}_x \mathbf{F}_s$ in $\mathcal{L}^{5/4,2}_s \mathcal{L}^3_x$. Next, we apply Corollary 5.2.2 to each. Then using Proposition 5.3.2 and Corollary 5.3.3, the sum is estimated by

$$\sum_{j=0}^{k-1} \|\mathcal{D}^{(j+1)}_x \mathbf{F}\|_{\mathcal{L}^{3/4,2}_s \mathcal{L}^2_x} \|\mathcal{D}^{(k-1-j)}_x \mathbf{F}_s\|^{1/2}_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x} \|\mathcal{D}^{(k-j)}_x \mathbf{F}_s\|^{1/2}_{\mathcal{L}^{5/4,2}_s \mathcal{L}^2_x} \leq C_{k,\mathbf{E}} \cdot \mathbf{E},$$

Therefore, (5.7.4) holds for the case k, which completes the induction. □

Suppose furthermore that $A_{\mathbf{a}}$ is in the caloric-temporal gauge, so that $A_s = 0$ in particular. Combining Proposition 5.7.2 and Corollary 5.4.6, the covariant derivative estimate (5.7.4) leads to the corresponding estimate for usual derivatives. This is the content of the following corollary, whose easy proof we omit.

Corollary 5.7.3 (Improved estimate for F_{s0}, with usual derivatives)**.** *Assume that the hypotheses of Proposition 5.7.1 hold. Furthermore, assume that $A_{\mathbf{a}}$ satisfies the caloric-temporal gauge condition.*

Then the following estimate holds for $0 \leq k \leq 29$:

$$\|\nabla_x^{(k)} F_{s0}(t)\|_{\mathcal{L}_s^{1,\infty}\mathcal{L}_x^2(0,1]} + \|\nabla_x^{(k)} F_{s0}(t)\|_{\mathcal{L}_s^{1,2}\mathcal{L}_x^2(0,1]} \leq C_{k,{}^{(\underline{A})}\mathcal{I}(t),\mathbf{E}(t)} \cdot \mathbf{E}(t). \tag{5.7.5}$$

The estimate (5.7.5) is more than sufficient to prove Proposition 5.7.1.

5.7.2 Proof of Theorem G

With Proposition 5.7.1, we are ready to give a proof of Theorem G. We shall basically follow the proof of Theorem B, replacing Proposition 4.4.3 by Proposition 5.7.1. We recommend the reader to take a look at §4.4 for the statements of Propositions 4.4.1 – 4.4.4, Theorems C, D, and the proof of Theorem B.

Proof of Theorem G. Let $A_{\mathbf{a}}$ be a regular solution to the hyperbolic-parabolic Yang-Mills equation in the caloric-temporal gauge on $(-T_0, T_0) \times \mathbb{R}^3 \times [0,1]$ such that (5.1.5) is satisfied. For simplicity, we shall consider the case in which I_0 is centered at $t = 0$, i.e. $I_0 = (-d/2, d/2)$ for $d > 0$ to be determined. As we shall see, the proof only utilizes the hypotheses (5.1.5) on I_0; therefore, the same proof applies to other $I_0 \subset (-T_0, T_0)$ as well.

We claim that

$$\mathcal{F}(I_0) + \underline{\mathcal{A}}(I_0) \leq BD, \tag{5.7.6}$$

for a large enough absolute constant B, to be determined later, provided that $|I_0| = d$ is small enough. Note that Theorem G then follows immediately from the claim, thanks to Propositions 4.4.1 and 4.4.2.

We shall use a bootstrap argument. The starting point is provided by Proposition 4.4.4, which implies

$$\mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0') \leq 2\mathcal{I}(0),$$

for some subinterval $I_0' \subset I$ containing 0 such that $|I_0'| > 0$ is sufficiently small (by upper semi-continuity of $\mathcal{F}, \underline{\mathcal{A}}$ at 0). Note that the right-hand side is estimated by BD, provided we choose $B \geq 2$.

Next, let us assume the following *bootstrap assumption*:

$$\mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0') \leq 2BD$$

for $I_0' := (-T', T') \subset I$. Applying Theorems C, D, and using Proposition 5.7.1 to control $\mathcal{E}(I_0')$, we obtain

$$
\begin{aligned}
\mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0') \leq & C\mathcal{I} + (T')^{1/2} C_{C_{D,\mathbf{E}[\overline{\mathbf{F}}]}, \mathcal{F}(I_0'), \underline{\mathcal{A}}(I_0')} (C_{D,\mathbf{E}[\overline{\mathbf{F}}]} + \mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0'))^2 \\
& + (T')\Big(C_{\mathcal{F}(I_0'), \underline{\mathcal{A}}(I_0')} C_{D,\mathbf{E}[\overline{\mathbf{F}}]} + C_{C_{D,\mathbf{E}[\overline{\mathbf{F}}]}, \mathcal{F}(I_0'), \underline{\mathcal{A}}(I_0')} (C_{D,\mathbf{E}[\overline{\mathbf{F}}]} + \mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0')^2 \Big).
\end{aligned}
$$

Here, we used the hypotheses (5.1.5) on $I_0' \subset I_0$. Using the bootstrap assumption and choosing d small enough depending on $D, \mathbf{E}[\overline{\mathbf{F}}]$ and B (note that $T' \leq d$), we can make the second and third terms on the right-hand $\leq \frac{B}{2} D$. Then choosing $B > 2C$, we see that

$$
\mathcal{F}(I_0') + \underline{\mathcal{A}}(I_0') \leq BD,
$$

which 'beats' the bootstrap assumption. By a standard continuity argument, (5.7.6) then follows. $\square$

Appendix A

Estimates for gauge transforms

The goal of this appendix is to establish some estimates for gauge transforms (namely Lemma 4.3.6 and Propositions 3.5.1, 3.5.2) that had been deferred in the main body of the thesis. These estimates will be derived from a lemma concerning an abstract ODE (Lemma A.2.1), which models the ODEs satisfied by various gauge transforms occurring in this thesis.

A brief outline of this appendix is in order. After a brief review of Littlewood-Paley theory in §A.1, we shall formulate and prove the abstract ODE lemma (Lemma A.2.1) in §A.2 that we referred to. Then in §A.3, we shall discuss how the estimates given by Lemma A.2.1 can be used to estimate gauge transformations of connection 1-forms and covariant tensors. Then the rest of the appendix will be concerned with application of Lemma A.2.1: In §A.4, estimates for gauge transforms for (YM) to the temporal gauge will be proved in a quite general setting (Proposition A.4.2). As a special case, we shall obtain Lemma 4.3.6. Finally, in §A.5, we shall give proofs of Propositions, concerning gauge transforms for (cYMHF) to the caloric gauge.

The results in this section generalize the materials in [25, Appendix B], which were confined to $d = 3$, $\gamma = 1$.

A.1 Review of Littlewood-Paley theory

To prove the abstract ODE lemma (Lemma A.2.1), we shall need a more direct characterization of homogeneous fractional Sobolev spaces. For this purpose, we shall use the *Littlewood-Paley theory*, whose brief recap we shall give below.

For $d \geq 1$, let χ be a smooth radial function on $\mathbb{R}^d$ such that $\chi = 1$ on $\{|x| \leq 1\}$ and $\chi = 0$ in $\{|x| \geq 2\}$. Define χ_0 by $\chi_0(x) = \chi(x) - \chi(2x)$. Then $\chi_0(x)$ is a smooth compactly supported radial function such that $\text{supp}\chi_0 \subset \{\frac{1}{2} \le |x| \le 2\}$ and $\sum_{k\in\mathbb{Z}} \chi_0(x/2^k) = 1$ for every $x \ne 0$. For $k \in \mathbb{Z}$ and $\phi \in \mathcal{S}_x$, we define the *Littlewood-Paley projections* $P_k\phi$ and $P_{\le k}\phi$ by the formulae

$$\widehat{P_k\phi}(\xi) = \chi_0(\xi/2^k)\widehat{\phi}(\xi), \quad \widehat{P_{\le k}\phi}(\xi) = \chi(\xi/2^k)\widehat{\phi}(\xi).$$

Note that $\sum_k P_k\phi = \phi$ and $\sum_{k'\le k} P_{k'}\phi = P_{\le k}\phi$ in an appropriate sense (the only ambiguity being at $\xi = 0$). For general $\phi \in \mathcal{S}'_x$, $P_k\phi$ is defined by duality. The following definition of $\dot{W}^{\gamma,p}_x$-norm using the Littlewood-Paley projections is standard.

Definition A.1.1 (Littlewood-Paley characterization of the $\dot{W}^{\gamma,p}_x$-norm)**.** Let $d \ge 1$, $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $\phi \in \mathcal{S}'_x(\mathbb{R}^d)$ a tempered distribution on $\mathbb{R}^d$. The $\dot{W}^{\gamma,p}_x$*-norm*[1] of ϕ is defined to be

$$\|\phi\|_{\dot{W}^{\gamma,p}_x} := \|\Big(\sum_k 2^{2\gamma k}|P_k\phi|^2\Big)^{1/2}\|_{L^p_x},$$

and we shall define $\dot{W}^{\gamma,p}_x$ to be the space of all tempered distributions ϕ with $\|\phi\|_{\dot{W}^{\gamma,p}_x} < \infty$.

As usual, we shall use the notation $\dot{H}^\gamma_x$ in the case $p = 2$, i.e. $\dot{H}^\gamma_x = \dot{W}^{\gamma,2}_x$.

Remark A.1.2. Recall that in the main body of the thesis, the $\dot{H}^\gamma_x$-norm for a *function* ϕ, when $\gamma \ge 0$ is an integer, had been defined using ordinary weak derivatives as

$$\|\phi\|_{\dot{H}^\gamma_x} = \|\partial^{(\gamma)}_x\phi\|_{L^2_x},$$

whereas for non-integral values of γ, it had been defined by using the operator $|\partial_x|^{\gamma'}$. It is a consequence of the standard Littlewood-Paley theory that this definition of $\dot{H}^\gamma_x$ is equivalent to the one given in Definition A.1.1.

In many instances, instead of dealing with a general element of $\dot{W}^{\gamma,p}_x$, we shall work with Schwartz or H^∞_x functions and use an approximation argument to pass to the general case. The following density lemma, which is an easy consequence of the Littlewood-Paley theory and tempered distributions, can be used to neatly characterize the closure of these spaces with respect to certain $\dot{W}^{\gamma,p}_x$-norms.

Lemma A.1.3 (Density of $\mathcal{S}_x$)**.** *Let* $d \ge 1$, $1 < p < \infty$ *and* $0 < \gamma \le \frac{d}{p}$.

1. *Consider the non-endpoint case* $0 < \gamma < \frac{d}{p}$ *and define* $q \ge p$ *by* $\frac{d}{q} = \frac{d}{p} - \gamma$. *Then the space* $\mathcal{S}_x(\mathbb{R}^d)$ *of Schwartz functions is a dense subspace of* $\dot{W}^{\gamma,p}_x \cap L^q_x(\mathbb{R}^d)$.

[1] To be pedantic, the $\dot{W}^{\gamma,p}_x$-'norm' defined above is only a *semi-norm*, which is equal to zero for any tempered distribution whose Fourier support lies in $\{0\}$ (i.e. a polynomial). As such, some authors mod out these tempered distributions to define the space $\dot{W}^{\gamma,p}_x$; see [10, Chapter 6].

2. *Corresponding to the endpoint case $\gamma = \frac{d}{p}$, the space $\mathcal{S}_x$ is dense in $\dot{W}_x^{d/p,p} \cap \mathring{C}_x(\mathbb{R}^d)$, where*

$$\mathring{C}_x(\mathbb{R}^d) := \{\phi \in C_x(\mathbb{R}^d) : \phi(x) \to 0 \text{ as } |x| \to \infty\}.$$

Remark A.1.4. When $0 < \gamma < \frac{d}{p}$, $\| \cdot \|_{\dot{W}_x^{\gamma,p}}$ is in fact a *norm* on $\mathcal{S}_x(\mathbb{R}^d)$ by Sobolev; more precisely, we have $\|\phi\|_{L_x^p} \leq C_{d,\gamma,p} \|\phi\|_{\dot{W}_x^{\gamma,p}}$ for $\phi \in \mathcal{S}_x(\mathbb{R}^d)$. By the first statement of Lemma A.1.3, we see that the closure of $\mathcal{S}_x(\mathbb{R}^d)$ with respect to the $\dot{W}_x^{\gamma,p}$-norm is exactly $\dot{W}_x^{\gamma,p} \cap L_x^p$. Similarly, the second statement implies that the closure of $\mathcal{S}_x(\mathbb{R}^d)$ with respect to the norm $\| \cdot \|_{\dot{W}_x^{d/p,p} \cap L_x^\infty}$ is exactly $\dot{W}_x^{d/p,p} \cap \mathring{C}_x$.

Remark A.1.5. The above statements remain true with $\mathcal{S}_x$ replaced by H_x^∞.

In the next section, we shall analyze the product of two functions ϕ_1, ϕ_2 by the *Littlewood-Paley trichotomy*, which simply refers to the following decomposition:

$$P_{k_0}(\phi_1 \phi_2) = (\text{HL}) + (\text{LH}) + (\text{HH}),$$

where

$$\begin{aligned}
(\text{HL}) &= \sum_{k_1 : k_1 = k_0 + O(1)} P_{k_1} \phi_1 P_{\leq k_0 + O(1)} \phi_2, \\
(\text{LH}) &= \sum_{k_2 : k_2 = k_0 + O(1)} P_{\leq k_0 + O(1)} \phi_1 P_{k_2} \phi_2, \\
(\text{HH}) &= \sum_{k_1, k_2 : k_1 - k_2 = O(1)} P_{k_0}(P_{k_1} \phi_1 P_{k_2} \phi_2),
\end{aligned}$$

for appropriate constants $O(1)$, up to a negligible overlap.

We refer to [36] for more on the Littlewood-Paley theory.

A.2 An ODE estimate

In this section, we shall prove an abstract lemma concerning an ODE which models those satisfied by gauge transforms to the temporal or caloric gauge. The principal tool would be the Littlewood-Paley theory, as briefly reviewed in the previous section.

Lemma A.2.1 (ODE estimates)**.** *Let $d \geq 1$, X a finite-dimensional normed space, $J \subset \mathbb{R}$ an interval, and $\omega_0 \in J$. Consider an X-valued functions $\Xi_0 \in C_x(\mathbb{R}^d)$, $F \in C_{\omega,x}(J \times \mathbb{R}^d)$ and an*

$L(X)$-valued[2] *function $A \in C_{\omega,x}(J \times \mathbb{R}^d)$. Let $\Xi : J \times \mathbb{R}^d \to X$ be the unique solution to the ODE*

$$\begin{cases} \partial_\omega \Xi = A(\Xi) + F \quad (\omega \in J), \\ \Xi(\omega_0) = \Xi_0, \end{cases} \tag{A.2.1}$$

Then the following statements hold.

1. *Suppose that for some $1 \leq p \leq \infty$, we have*

$$\Xi_0 \in L^p_x, \quad \int_{\omega_0}^{\omega} F(\omega') \,\mathrm{d}\omega' \in C_\omega(J, L^p_x), \quad \|A\|_{L^1_\omega L^\infty_x(J)} < \infty. \tag{A.2.2}$$

Let $K_0 := \|A\|_{L^1_\omega L^\infty_x(J)}$. Then $\Xi \in C_\omega(J, L^p_x)$ and the following estimate holds.

$$\|\Xi\|_{L^\infty_\omega L^p_x(J)} \leq e^{K_0} \Big(\|\Xi_0\|_{L^p_x} + \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} F(\omega') \,\mathrm{d}\omega' \|_{L^p_x} \Big). \tag{A.2.3}$$

2. *Suppose, in addition to* (A.2.2) *for $p = \infty$, that for some $\gamma \geq \frac{d}{2}$, we have*

$$\Xi_0 \in \dot{H}^\gamma_x, \quad \int_{\omega_0}^{\omega} F(\omega') \,\mathrm{d}\omega' \in C_\omega(J, \dot{H}^\gamma_x), \quad \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} A(\omega') \,\mathrm{d}\omega' \|_{\dot{H}^\gamma_x} < \infty. \tag{A.2.4}$$

Let $K_1 := \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} A(\omega') \,\mathrm{d}\omega' \|_{\dot{H}^\gamma_x}$. Then $\Xi \in C_\omega(J, \dot{H}^\gamma_x)$ and the following estimate holds.

$$\begin{aligned} \|\Xi\|_{L^\infty_\omega \dot{H}^\gamma_x(J)} \leq & C e^{C K_0} \Big(\|\Xi_0\|_{\dot{H}^\gamma_x} + \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} F(\omega') \,\mathrm{d}\omega' \|_{\dot{H}^\gamma_x} \Big) \\ & + C e^{C K_0} K_1 \Big(\|\Xi_0\|_{L^\infty_x} + \|F\|_{L^1_\omega L^\infty_x} \Big). \end{aligned} \tag{A.2.5}$$

Remark A.2.2 (Remarks concerning regularity). If A and F possess further regularity in ω, by a standard argument, so does Ξ. For example, if $A, F \in C^k_\omega(J, L^p_x)$ for some $k \geq 0$, then $\Xi \in C^{k+1}_\omega(J, L^p_x)$.

Proof. By dividing J into two pieces and changing the orientation if necessary, we may assume that ω_0 is the left endpoint of J. For the arguments below, it will be useful to reformulate the ODE (A.2.1) in the integral form as follows:

$$\Xi(\omega) = \Xi_0 + \int_{\omega_0}^{\omega} A(\Xi)(\omega') \,\mathrm{d}\omega' + \int_{\omega_0}^{\omega} F(\omega') \,\mathrm{d}\omega'. \tag{A.2.6}$$

[2] $L(X)$ is the space of linear maps $X \to X$ equipped with the operator norm.

Proof of Statement 1. This is an immediate consequence of the inequality

$$\|\Xi(\omega) - \Xi_0\|_{L^p_x} \le \int_{\omega_0}^{\omega} \|A(\omega')\|_{L^\infty_x} \|\Xi(\omega')\|_{L^p_x} \, \mathrm{d}\omega' + \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} F(\omega') \, \mathrm{d}\omega' \|_{L^p_x},$$

and Gronwall. Letting $\omega \to \omega_0$, the continuity of Ξ with respect to the L^p_x-norm at ω_0 also follows; similar argument then applies to all $\omega \in J$.

Proof of Statement 2.

Beginning from the integral formulation (A.2.6), we shall analyze the expression

$$\| \int_{\omega_0}^{\omega} A(\Xi)(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^\gamma_x}$$

by using Definition A.1.1 and decomposing A and Ξ into Littlewood-Paley pieces according to the Littlewood-Paley trichotomy[3]. For simplicity, we shall abbreviate $A_k := P_k A$ and $\Xi_k := P_k \Xi$. Note that the Littlewood-Paley projections commute with ω-integrals.

- *Case 1. (HL) interaction.* This is when $k_0 = k_1 + O(1)$ and $k_2 \le k_1 + O(1)$. We claim

$$\begin{aligned} \| \sum_{k_0, k_1; k_0 = k_1 + O(1)} P_{k_0} \int_{\omega_0}^{\omega} A_{k_1}(\Xi_{\le k_1 + O(1)})(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^\gamma_x} & \\ \le & C K_1 (1 + K_0) \|\Xi\|_{L^\infty_\omega L^\infty_x} + C K_1 \|F\|_{L^1_\omega L^\infty_x} \\ \le & C e^{C K_0} K_1 \Big(\|\Xi_0\|_{L^\infty_x} + \|F\|_{L^1_\omega L^\infty_x} \Big). \end{aligned}$$

The second inequality follows from (1) and the fact that

$$\sup_{\omega \in J} \| \int_{\omega_0}^{\omega} F(\omega') \, \mathrm{d}\omega' \|_{L^\infty_x} \le \|F\|_{L^1_\omega L^\infty_x},$$

so it suffices to prove the first one. Using the orthogonality of Littlewood-Paley projections and a simple convolution estimate, we can remove k_0 from the left-hand side and arrive at

$$\le C \Big(\sum_{k_1} 2^{2\gamma k_1} \| \int_{\omega_0}^{\omega} A_{k_1}(\Xi_{\le k_1 + O(1)})(\omega') \, \mathrm{d}\omega' \|_{L^2_x}^2 \Big)^{1/2}.$$

To utilize the hypothesis (A.2.4), we shall use a frequency-localized variant of a trick, which seems to be due to Klainerman-Machedon [15]. The idea is to plug in (A.2.6) for $\Xi_{k_1 + O(1)}$. Then

[3] As X is a finite-dimensional normed space, Ξ and A may be viewed as a collection of scalar functions and $A(\Xi)(\omega')$ simply a linear combination thereof. For such objects, the standard Littlewood-Paley theory is easily applicable.

the summand of the square sum is estimated by

$$\begin{aligned}\leq & 2^{\gamma k_1} \| \int_{\omega_0}^{\omega} A_{k_1}(\omega') \, \mathrm{d}\omega' (P_{\leq k_1 + O(1)} \Xi_0) \|_{L^2_x} \\ & + 2^{\gamma k_1} \| \int_{\omega_0}^{\omega} A_{k_1}(\omega') (P_{\leq k_1 + O(1)} \int_{\omega_0}^{\omega'} A(\Xi)(\omega'') + F(\omega'') \, \mathrm{d}\omega'') \, \mathrm{d}\omega' \|_{L^2_x}.\end{aligned}$$

The first term is further estimated by

$$\leq C \| P_{k_1} \int_{\omega_0}^{\omega} A(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^{\gamma}_x} \| \Xi_0 \|_{L^\infty_x}$$

using Hölder, whose square sum is $\leq C K_1 \| \Xi_0 \|_{L^\infty_x} \leq C K_1 \| \Xi \|_{L^\infty_\omega L^\infty_x}$ as desired. On the other hand, using Fubini, the second term may be rewritten as follows:

$$2^{\gamma k_1} \| \int_{\omega_0}^{\omega} \Big(\int_{\omega''}^{\omega} A_{k_1}(\omega') \, \mathrm{d}\omega' \Big) \Big(P_{\leq k_1 + O(1)} (A(\Xi)(\omega'') + F(\omega'')) \Big) \, \mathrm{d}\omega'' \|_{L^2_x}.$$

This, in turn, is easily estimated by

$$\leq C \int_{\omega_0}^{\omega} \| P_{k_1} \int_{\omega''}^{\omega} A(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^{\gamma}_x} \Big(\| A(\omega'') \|_{L^\infty_x} \| \Xi(\omega'') \|_{L^\infty_x} + \| F(\omega'') \|_{L^\infty_x} \Big) \, \mathrm{d}\omega''$$

using Minkowski and Hölder. Its square sum may then be estimated by

$$\leq C K_1 (\| A \|_{L^1_\omega L^\infty_x} \| \Xi \|_{L^\infty_\omega L^\infty_x} + \| F \|_{L^1_\omega L^\infty_x})$$

where $\| A \|_{L^1_\omega L^\infty_x} \leq K_0$ by hypothesis. This proves our claim.

- *Case 2. (LH) interaction.* This is when $k_0 = k_2 + O(1)$ and $k_1 \leq k_2 + O(1)$. We claim

$$\| \sum_{k_0, k_1; k_0 = k_2 + O(1)} P_{k_0} \int_{\omega_0}^{\omega} A_{\leq k_2 + O(1)} (\Xi_{k_2})(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^{\gamma}_x} \leq C \int_{\omega_0}^{\omega} \| A(\omega') \|_{L^\infty_x} \| \Xi(\omega') \|_{\dot{H}^{\gamma}_x} \, \mathrm{d}\omega'.$$

Proceeding as in Case 1, the left-hand side is bounded by

$$\leq C \Big(\sum_{k_2} 2^{2\gamma k_2} \| \int_{\omega_0}^{\omega} A_{\leq k_2 + O(1)} (\Xi_{k_2})(\omega') \, \mathrm{d}\omega' \|_{L^2_x}^2 \Big)^{1/2}$$

which, by Minkowski and Hölder, is estimated by

$$\leq C \int_{\omega_0}^{\omega} \Big(\sum_{k_2} \| A_{\leq k_2 + O(1)}(\omega') \|_{L^\infty_x}^2 \| \Xi_{k_2}(\omega') \|_{\dot{H}^\gamma_x}^2 \Big)^{1/2} \, \mathrm{d}\omega'.$$

Note that $\| A_{\leq k_2 + O(1)}(\omega') \|_{L^\infty_x} \leq C \| A(\omega') \|_{L^\infty_x}$ uniformly in k_2; then the remaining square sum is equal to $\| \Xi(\omega') \|_{\dot{H}^\gamma_x}$. Thus, the claim follows.

- *Case 3. (HH) interaction.* This is when $k_1 = k_2 + O(1)$ and $k_0 \leq k_2 + O(1)$. We claim that

$$\| \sum_{k_1, k_2; k_1 = k_2 + O(1)} P_{\leq k_2 + O(1)} \int_{\omega_0}^{\omega} A_{k_1}(\Xi_{k_2})(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^\gamma_x} \leq C \int_{\omega_0}^{\omega} \| A(\omega') \|_{L^\infty_x} \| \Xi(\omega') \|_{\dot{H}^\gamma_x} \, \mathrm{d}\omega'.$$

By orthogonality, the left-hand side is bounded by

$$\leq C \Big(\sum_{k_0} 2^{2\gamma k_0} \| \sum_{k_1, k_2; k_1 = k_2 + O(1), \, k_0 \leq k_2 + O(1)} P_{k_0} \int_{\omega_0}^{\omega} A_{k_1}(\Xi_{k_2})(\omega') \, \mathrm{d}\omega' \|_{L^2_x}^2 \Big)^{1/2}.$$

Using Minkowski and Hölder, and furthermore the fact that

$$\sum_{k_1; k_1 = k_2 + O(1)} \| A_{k_1}(\omega') \|_{L^\infty_x} \leq C \| A(\omega') \|_{L^\infty_x} \quad \text{uniformly in } k_2,$$

the preceding expression is estimated by

$$\leq C \int_{\omega_0}^{\omega} \| A(\omega') \|_{L^\infty_x} \Big(\sum_{k_0} \Big(\sum_{k_2; k_0 \leq k_2 + O(1)} 2^{\gamma(k_0 - k_2)} \| \Xi_{k_2}(\omega') \|_{\dot{H}^\gamma_x} \Big)^2 \Big)^{1/2} \, \mathrm{d}\omega'.$$

By Cauchy-Schwarz (or Schur's test), the claim then follows.

- *Conclusion.* As a result of the above analysis, we have the inequality

$$\begin{aligned} \| \Xi(\omega) - \Xi_0 \|_{\dot{H}^\gamma_x} \leq & \sup_{\omega \in J} \| \int_{\omega_0}^{\omega} F(\omega') \, \mathrm{d}\omega' \|_{\dot{H}^\gamma_x} \\ & + C K_1 e^{C K_0} \Big(\| \Xi_0 \|_{L^\infty_x} + \| F \|_{L^1_t L^\infty_x} \Big) + \int_{\omega_0}^{\omega} \| A(\omega') \|_{L^\infty_x} \| \Xi(\omega') \|_{\dot{H}^\gamma_x} \, \mathrm{d}\omega'. \end{aligned}$$

The desired conclusion (A.2.5) follows by applying the triangle and Gronwall inequalities. The continuity of $\Xi(\omega) \in \dot{H}^\gamma_x$ in ω is also an easy consequence of the arguments so far, as in Statement 1. □

A.3 General lemma concerning gauge transformation

In application, Lemma A.2.1 will be used to derive estimates for a $\mathfrak{G}$-valued function U, which is a *gauge transform* of some connection 1-form A_i to certain gauge (caloric or temporal, in our context). In this short section, we shall formulate a simple lemma which relates the estimates we have for U to that for the corresponding gauge transformation of connection 1-forms and covariant tensors. The proof is an obvious application of the Leibniz rule and Lemma 3.1.4, and thus shall be omitted.

Lemma A.3.1. *Let U be a $\mathfrak{G}$-valued function in $H^\infty_x(\mathbb{R}^d)$, B a $\mathfrak{g}$-valued function in H^∞_x and $-\frac{d}{2} < \gamma < \frac{d}{2}$. Then the following statements hold.*

1. *For every integer $m \geq 0$, we have*

$$\begin{aligned} &\|\partial_x^{(m)}\big(UBU^{-1}\big)\|_{\dot{H}^\gamma_x} \\ &\quad \leq C_{d,\gamma,m} \sum_{k_1+k_2+k_3=m; k_i \geq 0} \|\partial_x^{(k_1)} U\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \|\partial_x^{(k_2)} B\|_{\dot{H}^\gamma_x} \|\partial_x^{(k_3)} U^{-1}\|_{\dot{H}^{d/2}_x \cap L^\infty_x}, \end{aligned} \tag{A.3.1}$$

and

$$\|\partial_x^{(m)}\big(\partial_i U U^{-1}\big)\|_{\dot{H}^\gamma_x} \leq C_{d,\gamma,m} \sum_{k=0}^{m} \|\partial_x^{(k)} \partial_i U\|_{\dot{H}^\gamma_x} \|\partial_x^{(m-k)} U^{-1}\|_{\dot{H}^{d/2}_x \cap L^\infty_x}. \tag{A.3.2}$$

2. *Let U' be another $\mathfrak{G}$-valued function in H^∞_x and B' a $\mathfrak{g}$-valued function in H^∞_x. In addition to the usual notations $\delta U = U - U'$ and $\delta B = B - B'$, we shall also use*

$$\delta(UBU^{-1}) := UBU^{-1} - U'B'(U')^{-1}, \quad \delta(\partial_i U U^{-1}) := \partial_i U U^{-1} - \partial_i U'(U')^{-1}.$$

Then for every integer $m \geq 0$, we have

$$\begin{aligned} &\|\partial_x^{(m)} \delta\big(UBU^{-1}\big)\|_{\dot{H}^\gamma_x} \\ &\quad \leq C_{d,\gamma,m} \sum_{k_1+k_2+k_3=m; k_i \geq 0} \|\partial_x^{(k_1)} (\delta U)\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \|\partial_x^{(k_2)} B\|_{\dot{H}^\gamma_x} \|\partial_x^{(k_3)} U^{-1}\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \\ &\qquad + C_{d,\gamma,m} \sum_{k_1+k_2+k_3=m; k_i \geq 0} \|\partial_x^{(k_1)} U'\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \|\partial_x^{(k_2)} \delta B\|_{\dot{H}^\gamma_x} \|\partial_x^{(k_3)} U^{-1}\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \\ &\qquad + C_{d,\gamma,m} \sum_{k_1+k_2+k_3=m; k_i \geq 0} \|\partial_x^{(k_1)} U'\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \|\partial_x^{(k_2)} B'\|_{\dot{H}^\gamma_x} \|\partial_x^{(k_3)} (\delta U^{-1})\|_{\dot{H}^{d/2}_x \cap L^\infty_x}, \end{aligned} \tag{A.3.3}$$

and

$$
\begin{aligned}
\|\partial_x^{(m)}\delta\big(\partial_i U U^{-1}\big)\|_{\dot{H}^\gamma_x} \leq & C_{d,\gamma,m}\sum_{k=0}^{m}\|\partial_x^{(k)}\partial_i\delta U\|_{\dot{H}^\gamma_x}\|\partial_x^{(m-k)}U^{-1}\|_{\dot{H}^{d/2}_x\cap L^\infty_x} \\
& + C_{d,\gamma,m}\sum_{k=0}^{m}\|\partial_x^{(k)}\partial_i U'\|_{\dot{H}^\gamma_x}\|\partial_x^{(m-k)}(\delta U^{-1})\|_{\dot{H}^{d/2}_x\cap L^\infty_x}.
\end{aligned}
\tag{A.3.4}
$$

Remark A.3.2. Let A_i be a connection 1-form on $\mathbb{R}^d$. Recall the following (partial list of) gauge transformation formulae:

$$
\begin{aligned}
A_i \mapsto & \widetilde{A}_i = UA_iU^{-1} - \partial_i U U^{-1}, \\
F_{ij} \mapsto & \widetilde{F}_{ij} = UF_{ij}U^{-1}, \\
\mathbf{D}_i F_{jk} \mapsto & \widetilde{D}_i\widetilde{F}_{ij} = U(\mathbf{D}_iF_{jk})U^{-1}.
\end{aligned}
$$

By (A.3.1), gauge transformation of covariant tensors such as $\widetilde{F}_{ij}$, $\mathbf{D}_i\widetilde{F}_{jk}$ etc. may be estimated in terms of the original object F_{ij}, $\mathbf{D}_iF_{jk}$, respectively, and appropriate bounds for $U - \mathrm{Id}$ and $U^{-1} - \mathrm{Id}$. On the other hand, to estimate $\widetilde{A}_i$, we need to use both (A.3.1) and (A.3.2).

Remark A.3.3. For U, U' $\mathfrak{G}$-valued functions on $C_t(I, H^\infty_x)$, we also have

$$
\|\partial_x^{(m)}\big(\partial_0 U U^{-1}\big)\|_{\dot{H}^\gamma_x} \leq C_{d,\gamma,m}\sum_{k=0}^{m}\|\partial_x^{(k)}\partial_0 U\|_{\dot{H}^\gamma_x}\|\partial_x^{(m-k)}U^{-1}\|_{\dot{H}^{d/2}_x\cap L^\infty_x}. \tag{A.3.2$'$}
$$

and

$$
\begin{aligned}
\|\partial_x^{(m)}\delta\big(\partial_0 U U^{-1}\big)\|_{\dot{H}^\gamma_x} \leq & C_{d,\gamma,m}\sum_{k=0}^{m}\|\partial_x^{(k)}\partial_0(\delta U)\|_{\dot{H}^\gamma_x}\|\partial_x^{(m-k)}U^{-1}\|_{\dot{H}^{d/2}_x\cap L^\infty_x} \\
& + C_{d,\gamma,m}\sum_{k=0}^{m}\|\partial_x^{(k)}\partial_0 U'\|_{\dot{H}^\gamma_x}\|\partial_x^{(m-k)}(\delta U^{-1})\|_{\dot{H}^{d/2}_x\cap L^\infty_x}.
\end{aligned}
\tag{A.3.4$'$}
$$

for every integer $m \geq 0$.

A.4 Estimates for gauge transform to the temporal gauge

In this section, we shall formulate and prove a general proposition concerning gauge transforms to the temporal gauge. As a consequence of our general result, Lemma 4.3.6 would follow.

Let $I \subset \mathbb{R}$ be an interval, $d \geq 2$ and $\gamma \geq \frac{d-2}{2}$. For a $\mathfrak{g}$-valued function $A_0 \in C_{t,x}(I \times \mathbb{R}^d)$, we define the norm $\mathcal{A}_0^\gamma(I)[A_0]$ to be

$$\mathcal{A}_0^\gamma(I)[A_0] := \|A_0\|_{L^\infty_t \dot{H}^{(d-2)/2}_x} + \|A_0\|_{L^\infty_t \dot{H}^\gamma_x} + \|A_0\|_{L^1_t(\dot{H}^{d/2}_x \cap L^\infty_x)} + \|A_0\|_{L^1_t \dot{H}^{\gamma+1}_x}.$$

For $\mathfrak{g}$-valued functions $A_0, A_0' \in C_{t,x}(I \times \mathbb{R}^d)$, we shall use the abbreviations

$$\mathcal{A}_0^\gamma(I) := \mathcal{A}_0^\gamma(I)[A_0], \quad \delta\mathcal{A}_0^\gamma(I) := \mathcal{A}_0^\gamma(I)[A_0 - A_0'].$$

Remark A.4.1. In the main body of the thesis, $d = 3$, $\gamma = 1$ and we had considered a solution $A_{\mathbf{a}}$ to (HPYM), which depends additionally on $s \in [0, s_0]$. In this case, the goal had been to apply a gauge transform to set $\widetilde{A}_0(s = 0) \equiv 0$. Therefore, we used the norms $\overline{\mathcal{A}}_0(I)$ and $\delta\overline{\mathcal{A}}_0$ defined with respect to $\overline{A}_0 := A_0(s = 0)$ and $\delta\overline{A}_0 := \delta A_0(s = 0)$, respectively.

We are now ready to state the main proposition concerning estimates for gauge transforms to the temporal gauge.

Proposition A.4.2 (Gauge transform to temporal gauge). *Let $d \geq 2$ and $\frac{d-2}{2} \leq \gamma \leq \frac{d}{2}$. Consider the ODE*

$$\begin{cases} \partial_t V = V A_0 \\ V(t = 0) = \mathring{V}, \end{cases} \tag{A.4.1}$$

on $I \times \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an interval containing 0 and A_0 is a $\mathfrak{g}$-valued function in $C_{t,x}(I \times \mathbb{R}^d)$ such that $\mathcal{A}_0^\gamma(I)[A_0] < \infty$.

1. *Suppose that $\mathring{V}$ is a $\mathfrak{G}$-valued function on $\{t = 0\} \times \mathbb{R}^d$ such that $\mathring{V} \in \dot{H}^{\gamma+1}_x \cap \dot{H}^{d/2}_x \cap C_x$. Then there exists a unique solution V to the ODE* (A.4.1)*, which is a $\mathfrak{G}$-valued function in $C_t(I, \dot{H}^{\gamma+1}_x \cap \dot{H}^{d/2}_x \cap C_x)$. Furthermore, the solution obeys the estimate*

$$\begin{aligned} &\|V - \mathrm{Id}\|_{L^\infty_t \dot{H}^{\gamma+1}_x(I)} + \|V - \mathrm{Id}\|_{L^\infty_t(\dot{H}^{d/2}_x \cap L^\infty_x)(I)} \\ &\quad \leq C_{\mathcal{A}_0(I)}(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{\gamma+1}_x} + \|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{d/2}_x \cap L^\infty_x} + \mathcal{A}_0(I)). \end{aligned} \tag{A.4.2}$$

The following estimate for $\partial_t(V - \mathrm{Id})$ also holds:

$$\begin{aligned} &\|\partial_t(V - \mathrm{Id})\|_{L^\infty_t \dot{H}^\gamma_x(I)} + \|\partial_t(V - \mathrm{Id})\|_{L^\infty_t \dot{H}^{(d-2)/2}_x(I)} \\ &\quad \leq C_{\mathcal{A}_0(I)} \cdot \mathcal{A}_0(I)(\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{d/2}_x \cap L^\infty_x} + 1). \end{aligned} \tag{A.4.3}$$

2. *Let A_0' be a $\mathfrak{g}$-valued function in $C_{t,x}(I \times \mathbb{R}^d)$ such that $\mathcal{A}_0^\gamma(I)[A_0'] < \infty$, and $\mathring{V}'$ a $\mathfrak{G}$-valued*

function on $\{t=0\}\times\mathbb{R}^3$ *such that* $\mathring{V}' \in \dot{H}^{\gamma+1}_x \cap \dot{H}^{d/2}_x \cap C_x$. *Without loss of generality, assume furthermore that*

$$\mathcal{A}^{\gamma}_0(I)[A'_0] \leq \mathcal{A}^{\gamma}_0(I)[A_0] =: \mathcal{A}^{\gamma}_0(I).$$

Denote by V' *the solution to the ODE* (A.4.1) *with* A_0 *and* $\mathring{V}$ *replaced by* A'_0, $\mathring{V}'$, *respectively. Then the difference* $\delta V := V - V'$ *obeys the following estimate.*

$$\begin{aligned} &\|\delta V\|_{L^\infty_t \dot{H}^{\gamma+1}_x(I)} + \|\delta V\|_{L^\infty_t \dot{H}^{d/2}_x \cap L^\infty_x(I)} \\ &\quad \leq C_{\mathcal{A}_0(I)}(\|\delta\mathring{V}\|_{\dot{H}^{\gamma+1}_x} + \|\delta\mathring{V}\|_{\dot{H}^{d/2}_x \cap L^\infty_x}) \\ &\quad\quad + C_{\mathcal{A}_0(I)}\delta\mathcal{A}_0(I)(\|\mathring{V}-\mathrm{Id}\|_{\dot{H}^{\gamma+1}_x} + \|\mathring{V}-\mathrm{Id}\|_{\dot{H}^{d/2}_x\cap L^\infty_x} + 1). \end{aligned} \tag{A.4.4}$$

Moreover, the following estimate for $\partial_t(\delta V)$ *also holds.*

$$\begin{aligned} &\|\partial_t(\delta V)\|_{L^\infty_t \dot{H}^{\gamma}_x(I)} + \|\partial_t(\delta V)\|_{L^\infty_t \dot{H}^{(d-2)/2}_x(I)} \\ &\quad \leq C_{\mathcal{A}_0(I)}\cdot \mathcal{A}_0(I)\|\delta\mathring{V}\|_{\dot{H}^{d/2}_x\cap L^\infty_x} + C_{\mathcal{A}_0(I)}\cdot \delta\mathcal{A}_0(I)(\|\mathring{V}-\mathrm{Id}\|_{\dot{H}^{d/2}_x\cap L^\infty_x}+1). \end{aligned} \tag{A.4.5}$$

3. *We have* smooth dependence on parameters*; in particular, the following statement holds: If* $A_0 \in C^\infty_t(I, H^\infty_x)$ *and* $\mathring{V} - \mathrm{Id} \in H^\infty_x$, *then the solution* V *satisfies* $V - \mathrm{Id} \in C^\infty_t(I, H^\infty_x)$.

4. *Finally, Statements 1, 2 and 3 remain true with* V, δV, $\mathring{V}$, $\delta\mathring{V}$ *replaced by* V^{-1}, δV^{-1}, $\mathring{V}^{-1}$ *and* $\delta\mathring{V}^{-1}$, *respectively.*

Proof. By the standard theory of ODE, the existence of a unique solution V to (A.4.1) follows. To derive estimates, we shall rewrite the ODE (A.4.1) as follows:

$$\begin{cases} \partial_t(V-\mathrm{Id}) = (V-\mathrm{Id})A_0 + A_0 \\ (V-\mathrm{Id})(t=0) = \mathring{V} - \mathrm{Id}, \end{cases}$$

Note that the unique solution V solves the preceding ODE on $I\times\mathbb{R}^d$. Then since

$$\|A_0\|_{L^1_t L^\infty_x(I)} + \sup_{t\in I}\|\int_0^t A_0(t')\,\mathrm{d}t'\|_{\dot{H}^{d/2}_x} + \sup_{t\in I}\|\int_0^t A_0(t')\,\mathrm{d}t'\|_{\dot{H}^{\gamma+1}_x} \leq \mathcal{A}^{\gamma}_0(I),$$

the estimate (A.4.2) is an immediate consequence of Lemma A.2.1. For the other estimate, namely (A.4.3), we begin with

$$\|\partial_t(V-\mathrm{Id})(t)\|_{\dot{H}^{\gamma'}_x} \leq C\|A_0(t)\|_{\dot{H}^{\gamma'}_x}(\|(V-\mathrm{Id})(t)\|_{\dot{H}^{d/2}_x\cap L^\infty_x}+1)$$

for $\gamma' = \frac{d-2}{2}, \gamma$, which follows from Lemma 3.1.4. Using (A.4.2), along with the observation that $\|\mathring{V} - \mathrm{Id}\|_{\dot{H}^{\gamma+1}_x}$ is not needed to estimate $\|(V - \mathrm{Id})(t)\|_{\dot{H}^{d/2}_x \cap L^\infty_x}$, we obtain (A.4.3).

For estimates concerning δV, namely (A.4.4) – (A.4.5), we consider the ODE

$$\partial_t(\delta V) = (\delta V)A'_0 + V\delta A_0$$

satisfied by δV, and proceed as before. Similarly, observe that V^{-1}, δV^{-1} satisfy

$$\partial_t(V^{-1} - \mathrm{Id}) = -A_0(V^{-1} - \mathrm{Id}) - A_0 \quad \text{and} \quad \partial_t(\delta V^{-1}) = -\delta A_0 V^{-1} - A'_0 \delta V^{-1},$$

respectively. By the same argument as before, the corresponding estimates for V^{-1} and δV^{-1} also follow. □

A.5 Estimates for gauge transform to the caloric gauge

In this section, we shall establish Propositions 3.5.1 and 3.5.2, whose proofs had been deferred in §3.5.

Proof of Proposition 3.5.1. Let us begin by rewriting (3.5.1) in terms of $U - \mathrm{Id}$ as follows:

$$\begin{cases} \partial_s(U - \mathrm{Id}) = (U - \mathrm{Id})A_s + A_s, \\ (U - \mathrm{Id})(s = s_1) = 0. \end{cases} \tag{3.5.1'}$$

Then Statements 1 and 2 are easy consequence of Lemma A.2.1 applied to (3.5.1′), along with the estimates (3.2.13), (3.2.14) for A_s. In order to prove (3.5.3) of Statement 3, we shall proceed by induction. That is, let $m \geq 1$ and assume

$$\sum_{k=0}^{m-1} \|s^{k/2}\partial_x^{(k+1)}(U - \mathrm{Id})\|_{L^\infty_s \dot{H}^\gamma_x(0,s_1]} \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}, \tag{A.5.1}$$

which holds for $m = 1$ by (3.5.2). We shall then prove

$$\|s^{m/2}\partial_x^{(m+1)}(U - \mathrm{Id})\|_{L^\infty_s \dot{H}^\gamma_x(0,s_1]} \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}. \tag{A.5.2}$$

For the simplicity of notation, let us use the shorthand $u_k := s^{k/2}\partial_x^{(k+1)}(U - \mathrm{Id})$. Let $s \in (0, s_1]$.

Integrating (3.5.1′) and applying $s^{m/2}\partial_x^{(m+1)}$, we obtain

$$\begin{aligned}u_m(s) = &-\int_s^{s_1} (s/s')^{m/2} u_m(s') \Big(s' A_s(s') \Big) \frac{\mathrm{d}s'}{s'} \\ &- \sum_{k=1}^{m} \int_s^{s_1} (s/s')^{m/2} u_{m-k}(s') \Big((s')^{(k+2)/2} \partial_x^{(k)} A_s(s') \Big) \frac{\mathrm{d}s'}{s'} \\ &- \int_s^{s_1} (s/s')^{m/2} U(s') \Big((s')^{(m+2)/2} \partial_x^{(m+1)} A_s(s') \Big) \frac{\mathrm{d}s'}{s'}.\end{aligned}$$

Taking the $\dot{H}^\gamma_x$-norm and applying Lemma 3.1.4, we obtain

$$\begin{aligned}\|u_m(s)\|_{\dot{H}^\gamma_x} \leq &\int_s^{s_1} (s/s')^{m/2} \|u_m(s')\|_{\dot{H}^\gamma_x} \Big(s' \|A_s(s')\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \Big) \frac{\mathrm{d}s'}{s'} \\ &+ \sum_{k=1}^{m} \int_s^{s_1} (s/s')^{m/2} \|u_{m-k}(s')\|_{\dot{H}^\gamma_x} \Big((s')^{(k+2)/2} \|\partial_x^{(k)} A_s(s')\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \Big) \frac{\mathrm{d}s'}{s'} \\ &+ \int_s^{s_1} (s/s')^{m/2} \|U(s')\|_{\dot{H}^{d/2}_x \cap L^\infty_x} \Big((s')^{(m+2)/2} \|\partial_x^{(m+1)} A_s(s')\|_{\dot{H}^\gamma_x} \Big) \frac{\mathrm{d}s'}{s'}.\end{aligned}$$

By the induction hypothesis (A.5.1) and (3.5.2), for $1 \leq k \leq m$, we have

$$\|u_{m-k}\|_{L^\infty_s \dot{H}^\gamma_x (0,s_1]} \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}, \quad \|U - \mathrm{Id}\|_{L^\infty_s \dot{H}^{d/2}_x \cap L^\infty_x (0,1]} \leq C_{d,\gamma,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}.$$

From Proposition 3.2.6, we also have

$$\sum_{k=0}^{m} \|s^{(k+1)/2+\ell_\gamma} \partial_x^{(k)} A_s\|_{L^\infty_s (\dot{H}^{d/2}_x \cap L^\infty_x)} + \|s^{(m+2)/2} \partial_x^{(m+1)} A_s\|_{L^\infty_s \dot{H}^\gamma_x} \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x}.$$

As $\ell_\gamma < 1/2$, observe that

$$\|s^{(k+2)/2} \partial_x^{(k)} A_s\|_{L^\infty_s (\dot{H}^{d/2}_x \cap L^\infty_x)} \leq \|s^{(k+1)/2+\ell_\gamma} \partial_x^{(k)} A_s\|_{L^\infty_s (\dot{H}^{d/2}_x \cap L^\infty_x)},$$

which allows us to use the first term on the left-hand side of the preceding estimate. As a consequence, we arrive at

$$\|u_m(s)\|_{\dot{H}^\gamma_x} \leq C_{d,\gamma,m,\|\overline{A}\|_{\dot{H}^\gamma_x}} \|\overline{A}\|_{\dot{H}^\gamma_x} \Big(\int_s^{s_1} (s/s')^{m/2} \|u_m(s')\| \frac{\mathrm{d}s'}{s'} + \int_s^{s_1} (s/s')^{m/2} \frac{\mathrm{d}s'}{s'} \Big).$$

Using the obvious bound

$$\int_s^{s_1} (s/s')^{m/2} \frac{\mathrm{d}s'}{s'} \leq C_m,$$

along with Gronwall's inequality, the desired estimate (A.5.2) follows. The remaining estimate

(3.5.4) of Statement 3 can be proved similarly.

Finally, as U^{-1} satisfies the ODE

$$\begin{cases} \partial_s U^{-1} = - A_s U^{-1}, \\ U^{-1}(s = s_1) = \mathrm{Id}, \end{cases}$$

repeating the above arguments easily establishes Statements 1 – 3 for U^{-1}. □

The proof of the difference analogue (Proposition 3.5.2) proceeds similarly; we omit the details.

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