Vatsal 9/8

Last time

First recall: $u = \frac{1}{2}\#\mathcal{O}_K^*$.

$$\begin{array}{c|l}
\vdots & \\
H_3 & \\
p & \text{wild} \\
H_2 & \\
p & \text{wild} \\
H_1 & \\
\frac{p\pm 1}{u} & \text{tame at } p \\
H & \\
 & \\
K &
\end{array}$$

Vatsal writes $\mathrm{Gal}(H_\infty/K) = G_1 \times \Delta_\infty$
Δ_∞ has no p-power.

If $p \nmid \mathcal{Cl}(K)$, $G_1 \cong \mathrm{Gal}(H_1/K)$
(assume this throughout)

Now (remember $p \nmid DN$) D: disc K, $N = N^+N^-$

$$\begin{array}{ccc}
\text{Heegner pts of conductor } p^n & \longrightarrow & \text{Gross curve } (N^+, N^-) \\
\parallel & & \parallel \\
X_n := K^*\backslash\hat{K}^*/\hat{\mathcal{O}}_n^* & & X := B^*\backslash \hat{B}^* \times \mathrm{Hom}(K, B(\mathbb{Q}))/\hat{R}^* \\
(\mathcal{O}_n = \mathbb{Z} + p^n\mathcal{O}_K) & & \parallel \\
 & & \bigcup R_i^*\backslash \mathrm{Hom}(K,B)
\end{array}$$

(R_i: some other Eichler order of the same level)

why is it a genus 0 curve? (f: $\mathrm{Hom}(\mathbb{C}, B(\mathbb{R})) \cong \mathbb{P}^1(\mathbb{C})$)

more on X:

$B^*\backslash\hat{B}^*/\hat{R}^* = \mathcal{Cl}(R)$ by defn
$\cong \left\{\begin{matrix}\text{Eichler orders} \\ \text{of level } N^+\end{matrix}\right\}$ by considering the action of $\hat{B}^*$ on this set, by $\hat{b}\cdot R = \hat{b}\hat{R}\hat{b}^{-1} \cap B$.

We can also write
$B^*\backslash\hat{B}^*/\hat{R}^* = B^*\backslash\hat{B}^*/\hat{R}^*\hat{\mathbb{Q}}^*$ because $\mathbb{Q}^* \subseteq B^*$, $\hat{\mathbb{Z}}^* \subseteq \hat{R}^*$, and $\hat{\mathbb{Q}}^* = \mathbb{Q}^*\hat{\mathbb{Z}}^*$

$= B^*\backslash {\prod_\ell}'' B_\ell^*/R_\ell^*\mathbb{Q}_\ell^*$

$\overset{\text{if } \ell = p}{=} R[\frac{1}{p}]^*\backslash T_p$ (strong approximation)

Using this, we understand the Hecke action:
If $\ell \nmid N$, $B_\ell^*/R_\ell^*\mathbb{Q}_\ell^* \cong PGL_2\mathbb{Q}_\ell/PGL_2\mathbb{Z}_\ell$.
Pick an adele $\eta_i = \begin{cases} \begin{pmatrix}\ell & i \\ 0 & 1\end{pmatrix} \text{ or } \begin{pmatrix}1 & 0 \\ 0 & \ell\end{pmatrix} & \text{at place } \ell \\ 1 & \text{at other places}\end{cases}$

Then $T_\ell x = \sum_{i=0}^{\ell} \eta_i x$

Galois action on Heegner points.

But before that, still more on X:

clearly (f, R) is a point of $X = B^* \backslash \hat{B}^* \times \mathrm{Hom}(K, B) / \hat{R}^*$

where $f \in \mathrm{Hom}(K, B)$, R an Eichler order, and we have

$(f, R) \sim (b f b^{-1}, b \hat{r} \hat{R} \hat{r}^{-1} b^{-1} \cap B)$ for $b \in B^*$, $\hat{r} \in \hat{R}^*$.

$\sigma \in \mathrm{Gal}(H_n/K)$ acts by

$\mathrm{Gal}(H_n/K) \overset{\text{Artin rec.}}{\cong} X_n \longrightarrow X$ $\quad P = (f, R) \mapsto (f, f(\sigma) \hat{R} f(\sigma^{-1}) \cap B) = P^\sigma$

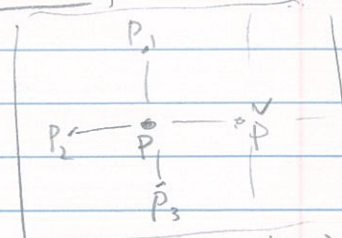
acts by translating

$\boxed{\text{n.b. } (f,R) \in \text{image of } X_n \iff f(K) \cap R = f(\mathcal{O}_n)}$. This action is free & transitive.

If $n \geq 1$ and $P \in X_n$, $T_p(P) = P_1 + \cdots + P_{n+1}$

one of them is in X_{n-1} — call it $\check{P}$

all others are in X_{n+1}.



(on Bruhat-Tits tree)

In fact, $T_p(P) = \check{P} + \sum_{\sigma \in \mathrm{Gal}(H_{n+1}/H_n)} P_*^\sigma$

where P_*: any nbd of P w/ conductor p^{n+1}.

<u>POINT</u> nothing said here is rocket science,

most can be proved by common sense + patient computation.

$$\begin{tikzcd} X_n \arrow[r] \arrow[dr, "\text{emb.}"'] & X \\ & \text{(copies of) } T_p = PGL_2\mathbb{Q}_p/PGL_2\mathbb{Z}_p \arrow[u, "\text{quotient}"'] \end{tikzcd}$$

<u>Next</u>. $X_n \longrightarrow X$

$= \text{lattices (rank 2 } \mathbb{Z}_p\text{-module) in } \mathbb{Q}_p^2 \text{ modulo "det"}$

We want to construct these maps and play graph theory.

Some useful facts

<u>Thm</u> (Strong approximation)

Let G be a linear alg gp $/k$ global field, S a finite nonempty set of places of k, $\mathbb{A}_k^S$ the adele ring minus the places in S. Then $G(k)$ is dense in $G(\mathbb{A}_k^S)$ i.e. for any $g \in G(\mathbb{A}_k)$, can find $x \in G(k)$ s.t. $g_p - x_p$ is arbitrarily small $\forall p \notin S$.

A definite quaternion alg is not a linear alg gp $/\mathbb{Q}$, but if we impose S = nonempty set of <u>finite</u> places it holds true.

not needed

<u>Thm</u> (local Artin reciprocity)

Let k be a local field. There exists a unique map

$$\mathrm{Art}: k^* \to \mathrm{Gal}(k^{ab}/k)$$

$$\text{any uniformizer} \mapsto \mathrm{Frob}$$

and if l/k is any finite abelian extension,

$$k^*/Nm(l^*) \xrightarrow[\sim]{\mathrm{Art}} \mathrm{Gal}(l/k).$$

In fact, Art induces

$$\widehat{k^*} \xrightarrow{\sim} \mathrm{Gal}(k^{ab}/k)$$

where $\widehat{k^*}$ is def'd by the exact sequence

$$0 \to U \to \widehat{k^*} \to \hat{\mathbb{Z}} \to 0$$

cf. we have

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d, phantom, "\shortparallel"] & k^* \arrow[r, "\mathrm{ord}"] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ & O_k^* & & \langle \mathrm{Frob} \rangle & \end{tikzcd}$$