Next.
$$\begin{tikzcd} X_n \arrow[r] \arrow[dr, "\text{emb.}"'] & X \\ & T_p \arrow[u, "\text{quotient}"'] \end{tikzcd}$$

We want to construct these maps and play graph theory.

(copies of) $T_p = PGL_2\mathbb{Q}_p / PGL_2\mathbb{Z}_p$
= lattices (rank 2 $\mathbb{Z}_p$-module) in $\mathbb{Q}_p^2$ modulo "det"

Some useful facts

Thm (Strong approximation)
Let G be a linear alg gp $/k$ global field,
S a finite nonempty set of places of k,
$\mathbb{A}_k^S$ the adele ring minus the places in S.
Then $G(k)$ is dense in $G(\mathbb{A}_k^S)$ i.e. for any $g \in G(\mathbb{A}_k)$, can find $x \in G(k)$ s.t. $g_p - x_p$ is arbitrarily small $\forall p \notin S$.

A definite quaternion alg is not a linear alg gp $/\mathbb{Q}$, but if we impose S = nonempty set of finite places it holds true.

not needed

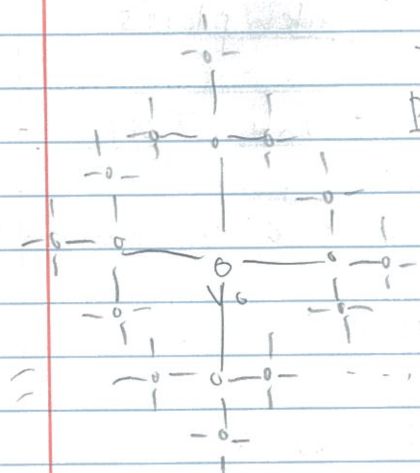
Thm (local Artin reciprocity)
Let k be a local field. There exists a unique map
$$\mathrm{Art}: k^* \longrightarrow \mathrm{Gal}(k^{ab}/k)$$
$$\text{any uniformizer} \longmapsto \mathrm{Frob}$$
and if l/k is any finite abelian extension,
$$k^*/\mathrm{Nm}(l^*) \xrightarrow[\sim]{\mathrm{Art}} \mathrm{Gal}(l/k).$$

In fact, Art induces
$$\hat{k}^* \xrightarrow{\sim} \mathrm{Gal}(k^{ab}/k)$$
where $\hat{k}^*$ is def'd by the exact sequence
$$0 \to U \to \hat{k}^* \to \hat{\mathbb{Z}} \to 0$$
cf. we have
$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d, phantom, "\shortparallel"] & k^* \arrow[r, "\mathrm{ord}"] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\ & \mathcal{O}_k^* & & \langle \mathrm{Frob} \rangle & \end{tikzcd}$$

Fix any Eichler order R, and $p \nmid N$.
If R' any other Eichler order, can find $b \in B^*$ s.t.
$b_\ell R'_\ell b_\ell^{-1} = R_\ell$ for all $\ell \neq p$, using strong approximation.
$S := bR'b^{-1}$ is called the normal form of R'
and this is used to embed Heegner points into T_p
as follows: if $(f, S) \in X_n$ and S is in normal form,
S corresponds to $v \in T_p$ where $S_p = \mathrm{stab}(v)$
e.g. if $v = PGL_2\mathbb{Z}_p$, $S_p = PGL_2\mathbb{Z}_p$
$v = a\,PGL_2\mathbb{Z}_p$, $S_p = a \cdot PGL_2\mathbb{Z}_p \cdot a^{-1}$

Now that we know how to embed $X_n \hookrightarrow T$,
we discuss the compatible actions of Gal on T.

Fix any $P_0 \in X_0$ i.e. maximal order of K.
There are $\#Cl$ many of them.
Say it corresponds to $v_0 \in T_p$.



Then $\mathrm{Gal}(H_1/H)$ takes v_0 to neighboring
$\frac{p \pm 1}{u}$ vertices ($-$ if p splits in K, $+$ if inert)
which corresponds to pts of cond p.
n.b. $|\mathrm{Gal}(H_1/H)| = \frac{p \pm 1}{u}$.

If v corresponds to pt of cond p^n, $n \geq 1$,
$\mathrm{Gal}(H_n/H)$ permutes pts of the same cond.,
and $\mathrm{Gal}(H_{n+1}/H_n)$ sends v to a neighboring pt
of cond p^{n+1}. i.e. Gal really acts like matrix $\begin{pmatrix} 1 & i \\ 0 & \ell \end{pmatrix}$.

How does $\mathrm{Gal}(H/K)$ act? Say $\tau \in \mathrm{Gal}(H/K)$.
If $P = (f, R)$, $P^\tau = (f, \underbrace{f(\tau)\hat{R}f(\tau)^{-1} \cap B}_{\text{need to put into normal form}})$,
so find $b \in B^*$ s.t. $b\,\boxed{\text{that}}\,b^{-1}$
is in normal form.
At the end we have $P^\tau = (bfb^{-1}, S)$ w/ S normal form. We
associate this to $v \in T_p$ s.t. $S_p = \mathrm{stab}(v)$.

It may be confusing that (f, S) and (bfb^{-1}, S) maps to the same pt on $\mathcal{T}$, so I prefer to imagine that there are distinct copies of $\mathcal{T}$.

As desired, we're finally set up for graph theory.

$$\begin{array}{ccc} \text{Heegner pts} & \subset & \mathcal{T}_p \\ & & \downarrow \\ & & X = R[\tfrac{1}{p}]^* \backslash \mathcal{T}_p \end{array}$$ a finite graph.

It'd be so much nicer if we could replace X by a regular graph, i.e. any vertex has the same # of edges.
Analogous situation in indefinite case: $PSL_2\mathbb{Z} \backslash PSL_2\mathbb{R}$ has two singularities due to the elliptic elements of $PSL_2\mathbb{Z}$.
The "irregularity" happens because when there is an element of finite order $w \in R[\frac{1}{p}]^*$, w acts as a stabilizer of a vertex say $v \in \mathcal{T}$, and w identifies some neighbors of v.

(possible action of w)

The solution is to increase level, as in indefinite case.
recall $\Gamma(N)$ for $N \geq 5$ has no elliptic element.
We define $\Gamma(N) \subseteq R[\frac{1}{p}]^*$ by the same prescription i.e. elts that are identity mod N. Then for $N \geq 5$ again $X' := \Gamma(N) \backslash \mathcal{T}_p$ is a regular graph. We'll work on X' and show that the desired result descends to X (there's an obvious map $X' \to X$)

Caveat In Vatsal's presentation he makes no clear distinction between oriented/non-oriented graphs. $\mathcal{T}$ is oriented but X is considered not.

Some (unoriented) black boxes regarding graphs:

- If G is a regular graph of degree d and A is its adjacency matrix then by Perron-Frobenius its largest e/value (in absolute value) is $\lambda = d$, and this has multiplicity 1.
- G is <u>bipartite</u> if $\lambda = -d$ is also an e/value of A. This means we can divide the vertices into two s/sets s.t. every edge goes from one subset to the other, e.g:

G is not bipartite $\iff$ there is a path of odd length starting and ending at any one vertex.

We want X' to be not bipartite. To do this we set N so that $N \equiv 3 \pmod 4$ and p is a square mod N, so that p has odd order mod N.
Hence $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ has odd order modulo $\Gamma(N)$. So, this induces a loop of odd length in X', so X' is not bipartite.

Now it's a good time to recall thm 1.5:

<u>Thm</u> Enumerate $Cl(B) = \{R_1 \text{ --- } R_n\}$.
$\mu_i(n) := \#$ of pts of X_n mapping to R_i via $X_n \to X$.
$e_n := \# X_n$, $w_i := \# R_i^*$.
Then
$$\lim_{n\to\infty} \frac{\mu_i(n)}{e_n} = \frac{1}{w_i h} = \frac{1/w_i}{\sum_j 1/w_j}.$$

This sort of means X_n is equidistributed in X.
We show instead that X_n is equi— in X'.
this suffices because

<u>Lemma</u> Recall the projection $X' \to X$, let $\varphi : X \to \mathbb{R}$.
If $\tilde{\varphi}$ is the pullback to X', then

$$(*) \quad \sum_{v \in X'} \tilde{\varphi}(v) = \sum_{v \in X} \varphi(v)/w(v) \qquad w(v) = \# \operatorname{stab}(v)$$
If $v = R_i$, $w(v) = w_i$.

If φ is the "cusp form" coming from Jacquet-Langlands
(*) equals zero.

<u>proof</u> (*) is rather obvious:

$$\begin{array}{c} X' \\ \big\downarrow \\ X \end{array} \left.\vphantom{\begin{array}{c} X' \\ \big\downarrow \\ X \end{array}}\right\} \begin{array}{l} \text{say index is } N, \text{ then } w(v) \mid N \\ \text{and } N/w(v) \text{ is the number of preimage of } v. \end{array}$$

(So to be correct, have to multiply RHS by N)

To show the next part, equip $L^2(X')$ w/ the inner product $(\phi_1, \phi_2) = \sum_{v \in X'} \phi_1(v)\overline{\phi_2}(v)$
and Laplacian $(\Delta\phi)(x) = \sum_{y \sim x} \phi(y)$, self-adjoint wrt $(,)$.

But recall T_p coincides with Δ, so $\tilde{\varphi}$ is a Δ-e/fn, w/ e/value $a_p = a_p(g)$. On the other hand $\mathbb{1}$ (const fn on X') is also an e/fn, w/ e/value $p+1$.
Since $a_p \neq p+1$ (Weil bound) $(\tilde{\Psi}, \mathbb{1}) = 0$.
$(*) = 0$ follows immediately $\square$

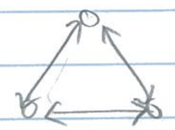
So if $\eta_i(n) :=$ # pts of X_n it suffices to show
$\lim_{n \to \infty} \frac{\eta_i(n)}{e_n} = \frac{1}{|X'|}$, i.e. X_n is equidistributed in X'.

But X_n is the set of points that are n steps away from v_0. So we have to show

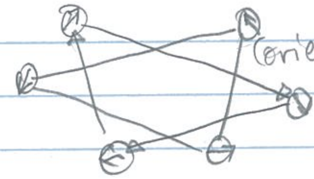
<u>Prop</u> Suppose G is a (unoriented) regular graph, on m vertices. Fix $v_0 \in G$, let W_n = set of all terminal vertices of walks from v_0 of length n, and counted w/ proper multiplicity <u>without backtracking</u> (i.e. $v_1 \to v_2 \to v_1$ is not permitted in a walk). Also assume G is not bipartite. Then for any $\phi : G \to \mathbb{C}$ we have

$$\lim_{n \to \infty} \frac{1}{d(d-1)^{n-1}} \sum_{y \in W_n} \phi(y) = \frac{1}{m} \sum_{x \in G} \phi(x).$$

pf (sketch) Draw the "line graph" $L(G)$ of G,
e.g.



G



$L(G)$

(oriented) lines become vertices, vertices become lines.

* we give orientations here!

There's a correspondence b/w walks w/o backtracking in G and plain walks on $L(G)$. So it suffices to show equidistribution in $L(G)$.

If G is regular, $L(G)$ is regular, if G is not bipartite, $L(G)$ is not bipartite. So now showing equi~~m~~ becomes very easy: simply consider

$$\left(\frac{1}{\deg L(G)} A(L(G)) \right)^n \cdot \phi .$$

Largest e/value is $\underset{\parallel}{\overset{d-1}{\deg L(G)}}$, all other e/values die out b/c there's no $\lambda = -\deg L(G)$ anymore. So it only picks up the average of ϕ. $\square$

In our case it's a little more subtle, as if p is split, there are $p-1$ paths rather than $p+1$. But fortunately it's not too big an obstacle (see Prop 3.16 of Vatsal). We skip the argument.