E : elliptic curve over $\mathbb{Q}$, of conductor N

K : $\mathbb{Q}(\sqrt{-d})$ $(d>0)$ $\quad \mathrm{disc}(K) = D$

K_∞ : anticyclotomic $\mathbb{Z}_p$ - extension of K

$(p, ND) = 1$ $\quad$ p odd prime unramified in K

H_n : ring class field of conductor p^n

$$G_\infty = \mathrm{Gal}(H_\infty / K) = \underbrace{G_1}_{\text{finite}} \times \underbrace{\Delta_\infty}_{\cong \mathbb{Z}_p}$$

$$\left(H_\infty = \bigcup_{n=1}^{\infty} H_n \right)$$

$G_n = G_1 \times \Delta_n$, $\quad \Delta_n$: Cyclic of order p^{n-1}

$\chi : G_n \to \mathbb{C}^*$ $\qquad \chi = \chi_t \cdot \chi_w$

χ_t : "tame" character of G_1

$$\psi(P^\sigma)\psi(P)$$

$$L^{al}(g, \chi, 1) := \frac{L(g, \chi, 1)}{\Omega} = \frac{1}{p^{n-\delta}} \sum_{P \in X_n} \sum_{\sigma \in G_n} \chi(\sigma)$$

$$L_n^{av}(g, \chi_t, 1) = \frac{1}{p^{n-2-\delta}} \sum_{\chi_w \in Y_n} L^{al}(g, \chi, 1)$$

(Y_n = set of faithful characters of Δ_n)

THM 2.11 ◦ $\lim_{n\to\infty} L_n^{av}(g, \chi_t, 1)$ exists and is nonzero

◦ For all $n \geq 0$, $\exists\ \chi_w \in Y_n$: $L(g, \chi_t\chi_w, 1) \neq 0$

◦ If the order of χ_t is prime to p, or if p is an ordinary prime for f, then $L(g, \chi_t\chi_w, 1) \neq 0$ for all $\chi_w \in Y_n$ as soon as n is sufficiently large.

—

g : cuspidal newform of weight 2 on the group $\Gamma_0(N)$

Write $g(z) = \sum_{n=1}^{\infty} a_n w^n$, $w = e^{\frac{2\pi i z}{M}}$

$q | D$ prime $\mathfrak{q}$: unique ramified prime of O_K above q

Frob

$$E_D = \prod_{q|D} \left(1 + \chi_t(\mathrm{Frob}(\mathfrak{q}))\frac{a_q}{q+1}\right)$$

Since $\chi_t(\mathrm{Frob}(\mathfrak{q})) = \pm 1$ and $|a_q| \leq 2\sqrt{q}$, it follows that $E_D \neq 0$

$E_p = \left(a_p^2\left(\frac{p-1}{p+1}\right) + 1\right)$ $e_n = |G_n| = c \cdot p^{n-\delta-1}$

Prop 2.14

- $$\lim_{n\to\infty} \frac{1}{cp^{n-\delta-1}} \sum_{P\in X_n} \sum_{\tau\in G_1} \chi_t(\tau)\psi(P^\tau)\psi(P) = E_D \cdot |\psi|^2$$

- $$\lim_{n\to\infty} \frac{1}{cp^{n-\delta-2}} \sum_{Q\in X_{n-1}} \sum_{\tau\in G_1} \chi_t(\tau)\left(a_p\psi(Q^\tau) - \psi(\check{Q}^\tau)\right)\left(a_p\psi(Q) - \psi(\check{Q})\right) = E_p E_D |\psi|^2$$

THM 4.1

C_1, C_2 : conjugacy classes of oriented Eichler orders of level N^+

$\tau \in G_1 \setminus G_0$

$\Rightarrow$ $\exists$ a Heegner point P of conductor p^n :

$P = (f_1, R_1)$ and $P^\tau = (f_2, R_2)$

(the order R_i represents the class C_i)

Furthermore, we can choose P so that $\mathrm{Cond}(P) = p^{n}$ has any desired parity

Action of the local Galois group

Prop 3.4 There are $\delta_0 \times p^{n-1-\delta}$ good vertices at distance $n-\delta$ from v_0, each corresponding to a distinct Heegner point $P \in X_n'$.

If a vertex v is good and corresponds to the Heegner point P', then v^{σ} corresponding to P'^{σ} is $f(\tilde{\sigma})v$.

Action of the ideal class group

Prop 3.6

$P = (f, R)$: a Heegner point of level 1

$\tau \in G_1$, fix an idele $\tilde{\tau} \in \hat{K}$ whose Artin symbol in $\mathrm{Gal}(H_\infty/K) = G_1 \times \Delta_\infty$ is equal to τ.

Let $P' = (f, R')$ be a good Heegner point in X_n' corresponding to the vertex v.

- P'^{τ} may be represented by (f', S'), S' normal form, S_p' corresponds to $v^{\tau} = \tau_p v$ for some fixed $\tau_p \in B_p$
- $\tau_p = b_p f(t_p)$

$G_\infty = \mathrm{Gal}(H_\infty/K) = G_1 \times \Delta_\infty$

$\tau \in G_1$ Choose $\tilde{\tau} \in \hat{K}$: Artin symbol of $\tilde{\tau}$ in $\mathrm{Gal}(H_\infty/K)$ is τ

$\tilde{\tau} = (t_\ell)_{\ell \neq p} \times t_p$

$P' = (f, R') \Rightarrow P'^\tau = (f, S)$, $\hat{S} = f(\tilde{\tau})\hat{R}' f(\tilde{\tau})^{-1}$

$\exists\, b \in B^\times$: $b_\ell S_\ell b_\ell^{-1} = R_\ell$ for all $\ell \neq p$

May assume $b_\ell t_\ell \in \mathbb{Q}_\ell^\times R_\ell^\times$ for $\ell \neq p$

$\Rightarrow P'^\tau = (f, S) = (f', S')$ $\quad f' = bfb^{-1}$

$S' = bSb^{-1}$

V: the vertex of the tree T corresponding to $P' = (f, R')$

$V^\tau = b_p f(t_p) V$

Let $\tau_p = b_p f(t_p)$. Then $\tau_p \in \mathbb{Q}_p^\times B^\times \subset B_p^\times$

$\Longleftrightarrow \tau \in G_0$

$G_0 \subset G_1$: the genus subgroup

generated by the elts ~~of~~ $\mathrm{Frob}(\mathfrak{q})$

($\mathfrak{q}$: unique prime of $\mathcal{O}_K$ above q, $q \mid D$)

$$\mathrm{Cl}(B) = B^\times \backslash \hat{B}^\times / \hat{R}^\times = R[\tfrac{1}{p}]^\times \backslash \underline{B_p^\times / \mathbb{Q}_p^\times R_p^\times}$$

$$\cong PGL(2,\mathbb{Q}_p)/PGL(2,\mathbb{Z}_p)$$

$\exists M > 0 :\ \Gamma(M) \subset R[\tfrac{1}{p}]^\times$

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R[\tfrac{1}{p}]^\times \;\middle|\; \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod M \right\}$$

$Pr : B_p \longrightarrow PGL(2,\mathbb{Q}_p)$

$\Gamma = \mathrm{Im}(Pr(\Gamma(M)))$, $\Gamma^+ = \Gamma \cap PSL(2,\mathbb{Q}_p)$

$\cdots (\ast)$

$\mathcal{G} = \Gamma(M) \backslash T$ is regular and not bi-partite.

<u>Goal</u> $\Gamma \tau_p \Gamma$ is dense in $B_p^\times$

<u>Prop 4.4</u> Let Γ be as in $(\ast)$. If $\alpha \in B_p^\times$ is such that $\Gamma_\alpha = \alpha \Gamma \alpha^{-1}$ is commensurable with Γ, then $\alpha \in \mathbb{Q}_p^\times B^\times$

Ratner's Theorems for p-adic gps

G: p-adic Lie group $\Gamma < G$ lattice

$A \subset \Gamma \backslash G$ homogeneous if $\exists x \in \Gamma \backslash G$ and $\exists$ a closed subgroup $H < G$:

- $xHx^{-1} \cap \Gamma$ is a lattice in xHx^{-1}, and
- $A = xH$

$u \in G$ is called Ad-unipotent if $\mathrm{Ad}(u) \in GL(\mathfrak{g})$ is unipotent.

1-parameter subgroup of G

= additive homomorphism $\mathbb{Q}_p \to G$

THM (Ratner 95)

$U \subset G$ be any subgroup gen'd by Ad-unip. 1-parameter subgroups.

Then for $\forall x \in \Gamma \backslash G$, $\overline{xU}$ is homogeneous.

($\exists H < G$: $\overline{xU} = xH$)

Coro 4.9 $G = SL(2, \mathbb{Q}_p)$ $\Gamma < G$ cocpt $\alpha \in G$. If Γ and $\alpha\Gamma\alpha^{-1}$ are not commensurable, then $\Gamma_\alpha \cdot \Gamma$ is dense in G. $(\Gamma_\alpha = \alpha\Gamma\alpha^{-1})$

[Pf] $G_* = G \times G$

$\Gamma_* = \Gamma_\alpha \times \Gamma$ $\quad x_* = id$

$U_* = \Delta = \{(g, g) \in G \times G\} \cong G$

is gen'd by Ad-unip. elements.

$\Rightarrow \overline{\Gamma_* U_*} = \Gamma_* H$ in $\Gamma_* \backslash G_*$ for some H

$\Gamma_* \cap H$ is a lattice in H

$\Gamma_* \cap U_* \cong \Gamma_\alpha \cap \Gamma$ is not a lattice in $G \cong U_*$

Thus, $\exists g \in G$: $(g, 1) \in H$.
$\pm Id \neq$

For any $g' \in G$, $(g' g g'^{-1}, 1) \in H$

$\Rightarrow (G, 1) \in H \Rightarrow H = G_*$ $(\Delta \subset H)$

Consider $G \times G \to G$, $(p, q) \mapsto pq^{-1}$. $\square$

Let $\widetilde{G} = B_p^\times / \mathbb{Q}_p^\times \cong PGL(2, \mathbb{Q}_p)$

$G \cong PSL(2, \mathbb{Q}_p) \subset \widetilde{G}$

$$\begin{array}{ccc} B_p & \longrightarrow & PGL(2, \mathbb{Q}_p) \\ \cup & & \\ \{ g \in B_p \mid N(g) = 1 \} & \to & G \\ R[\frac{1}{p}]^\times & \longrightarrow & \\ \Gamma(M) & \longrightarrow & \Gamma \end{array}$$

$\widetilde{\Delta} = \{ (g, g) \in PGL_2 \times PGL_2 \}$

$\Delta = \{ (g, g) \in PSL_2 \times PSL_2 \}$

$\Gamma_+ = \Gamma \cap G$

$\tau \in G_1 \setminus G_0$ $\quad \tau_p \in B_p^\times$ giving the action of τ

$\tau_p \notin \mathbb{Q}_p^\times B^\times \Rightarrow \Gamma$ and $\tau_p^{-1} \Gamma \tau_p$ are not commensurable

$\Rightarrow \Gamma_+$ and $\tau_p^{-1} \Gamma_+ \tau_p$ are not commensu- rable.

$\Rightarrow$ the closure of $\cancel{(\Gamma \times \tau_p^{-1} \Gamma \tau_p) \cdot PSL_2 \times PSL_2}$

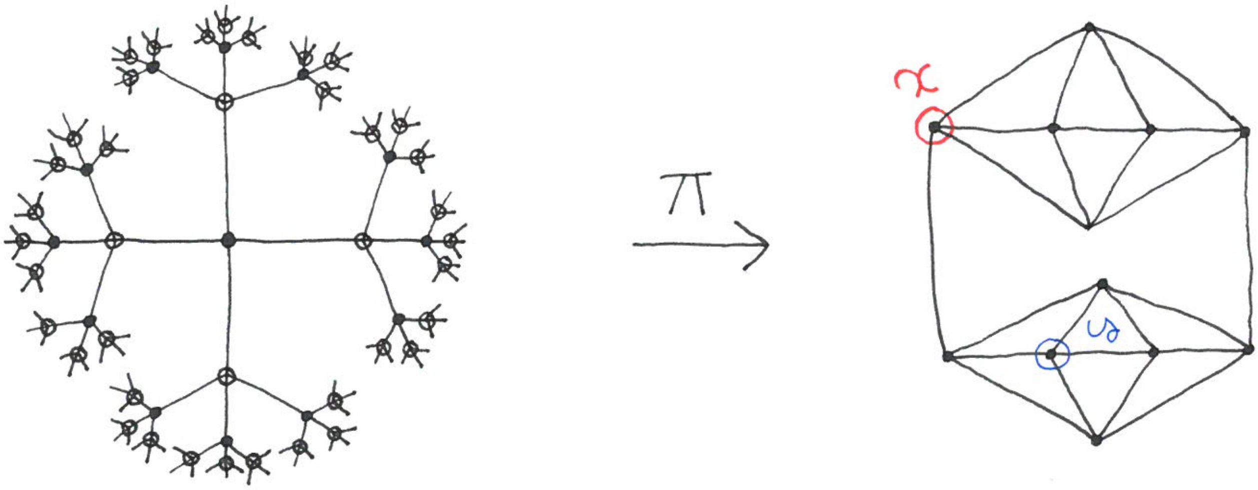
$(\Gamma \times \tau_p^{-1} \Gamma \tau_p) \cdot \Delta$ contains $G \times G$

in $\Gamma \times \tau_p^{-1} \Gamma \tau_p \backslash \widetilde{G} \times \widetilde{G}$

By Coro 4.9, $\Gamma_+ \tau_p^{-1} \Gamma_+ \tau_p$ is dense in G.
$(\tau_p^{-1} \Gamma_+ \tau_p \Gamma_+)$

V_0: ~~a~~ the vertex of the tree $\mathcal{T}$ corresponding to $P = (f, R)$ of level 1

$\mathcal{G} = \Gamma \backslash \mathcal{T}$ is not bipartite



Choose $(w, v) \in V\mathcal{T} \times V\mathcal{T}$ such that

$$\pi(w) = x \text{ and } \pi(v) = y$$

Since $\Gamma_+ \tau_p \Gamma_+$ is dense in G,

$\exists\, \gamma_1, \gamma_2 : \gamma_1 \tau_p \gamma_2 v = w$

$\Rightarrow$ ~~$(\gamma_2 \gamma_1 \tau$~~ $(\tau_p \gamma_2 v, \gamma_2 v) \xrightarrow{\pi} (x, y)$

$\gamma_i \in \Gamma_+$ do not disturb the parity conditions.

Ratner's uniform distribution thm

G : p-adic Lie gp $\quad U = u(t) : \mathbb{Q}_p \to G$

$\Gamma < G$ lattice $\quad x \in \Gamma \backslash G \Rightarrow \overline{xU} = xH$

$\Gamma \cap xHx^{-1}$ is a lattice in xHx^{-1}

$\exists$ H-inv. prob. msr $\overset{\mu}{\vee}$ on $xH \cong x^{-1}\Gamma x \cap H \backslash H$

$(\mathrm{Stab}_H(x) = x^{-1}\Gamma x \cap H) \qquad \cong \Gamma \cap xHx^{-1} \backslash xHx^{-1}$

$$T_h : \Gamma\backslash G \to \Gamma\backslash G$$
$$x \mapsto xh$$

Let λ be an additive Haar msr on $\mathbb{Q}_p$

$S \in \mathbb{R}$. $\quad F(S) = \{ x \in \mathbb{Q}_p \mid |x| \leq S \}$

<u>THM</u> ϕ : bdd. conti. ftn on $\begin{matrix} X = xH \cong \\ \Gamma \cap xHx^{-1} \backslash xHx^{-1} \end{matrix}$

Then $$\lim_{S \to \infty} \frac{1}{\lambda(F(S))} \int_{F(S)} \phi(xu(t))\, d\lambda(t)$$

$$= \int_X \phi(\underset{y}{\cancel{x}})\, d\mu(\underset{y}{\cancel{x}})$$