Vatsal 11/17

Recall our original goal: $L(g,\chi,1) \neq 0$ for most anticyclotomic χ.

Gross's formula says, for any choice $P \in X_n$,

$$L(g,\chi,1)\cdot \mathrm{const}(g) = \Big| \sum_{\sigma \in \mathrm{Pic}\,\mathcal{O}_n} \chi(\sigma)\psi(P^\sigma) \Big|^2$$

Expanding, we get

$$= \frac{1}{p^n} \sum_{P \in X_n} \sum_{\sigma \in \mathrm{Pic}\,\mathcal{O}_n} \chi(\sigma)\psi(P^\sigma)\psi(P)$$

(Vatsal puts $p^{n-\delta}$, we assume $\delta = 0$ — we already discussed where it comes from)

Write $G_n = \mathrm{Pic}\,\mathcal{O}_n = \mathrm{Gal}(H_n/K) = G_1 \times \Delta_n$ $\quad \Delta_n \cong \mathbb{Z}/p^{n-1}\mathbb{Z}$

also $\chi = \chi_t \chi_w$ $\quad \chi_t$: char on G_1, χ_w: char on Δ_n

Vatsal says: χ faithful $\Leftrightarrow$ χ_w has order p^{n-1}

$\Leftrightarrow$ χ_w nontrivial on $S_n \subseteq \Delta_n$

sgp of elts order p.

(I'm actually confused by this) $\leftarrow$ now embarrassed that I was

According to a theorem of Shimura (which?), if $L(g,\chi_t\chi_w,1) \neq 0$ some faithful χ_w, same holds for all other faithful χ_w. So we'll prove

$$(*) \quad \frac{1}{p^{n-2}} \sum_{\chi_w \text{ faithful}} L(g,\chi_t\chi_w,1) \xrightarrow{n\to\infty} \text{nonzero value}$$

happens to be the correct normalization.

Hence we consider

$$S_n(\sigma) = \sum_{\chi_w \text{ faithful}} \chi_w(\sigma) \qquad \sigma \in \Delta_n.$$

Lemma 2.12 $\sum_{\sigma\in\Delta_n} S_n(\sigma)\cdot\sigma = p^{n-1}\cdot\sigma_0 - p^{n-2}\sum_{\sigma\in S_n}\sigma$

in $\bar{\mathbb{Q}}[\Delta_n]$.

pf must be an easy exercise.

Using this lemma, $(*)$ becomes

$$\frac{1}{p^{n-1}}\sum_{p\in X_n}\sum_{\tau\in G_1}\chi_t(\tau)\psi(p^\tau)\psi(p) - \frac{1}{p^{n-2}}\sum_{p\in X_n}\sum_{\tau\in G_1,\sigma\in S_n}\chi_t(\tau)\psi(p^{\sigma\tau})\psi(p)$$

We rewrite the latter part: recall

$$\sum_{\sigma\in S_n}\psi(p^{\sigma\tau}) = a_p\psi(Q^\tau)-\psi(\check{Q}^\tau)$$

where Q = predecessor of p, $\check{Q}$ = pre ~ of Q,

$S_n = Gal(H_n/H_{n-1})$ — remember Hecke action? :)

so it becomes

$$\frac{1}{p}\sum_{Q\in X_{n-1}}\sum_{\tau\in G_1}\chi_t(\tau)\left(a_p\psi(Q^\tau)-\psi(\check{Q}^\tau)\right)\left(a_p\psi(Q)-\psi(\check{Q})\right).$$

Prop 2.14 Let

$$E_D = \prod_{q|D}\left(1+\chi_t(Frq)\frac{a_q}{q+1}\right) \quad \left(\neq 0 \text{ b/c } \underbrace{\begin{matrix}\chi_t(Frq)=\pm1\\ |a_q|\le 2\sqrt{q}\end{matrix}}_{\text{Weil bound}}\right)$$

$$E_p = \left(1+a_p^2\frac{p-1}{p+1}\right)$$

and write $|G_n| = cp^{n-1}$. Then

(i) $$\lim_{n\to\infty}\frac{1}{cp^{n-1}}\sum_{p\in X_n}\sum_{\tau\in G_1}\chi_t(\tau)\psi(p^\tau)\psi(p) = E_D|\psi|^2$$

(ii) $$\lim_{n\to\infty}\frac{1}{cp^{n-2}}\sum_{Q\in X_{n-1}}\sum_{\tau\in G_1}\chi_t(\tau)\left(a_p\psi(Q^\tau)-\psi(\check{Q}^\tau)\right)\left(a_p\psi(Q)-\psi(\check{Q})\right)$$
$$= E_pE_D|\psi|^2.$$

(Here $|\cdot|$ is L^2-norm on $\Gamma\backslash T$)

We will work out one τ at a time. Then diagonal embedding

both can be understood as integrating over the image of an "elliptic" orbit via $G \xrightarrow{\Delta} \Gamma_1 \times \Gamma_2 \backslash G \times G$.

They're proved in nearly identical manner, so let me focus on (i) and be brief about (ii)

Let $\tilde{G} := PGL_2 \mathbb{Q}_p$ as in Vatsal,
Γ some discrete cocpt sgp
$\tilde{V}$ an open cpt sgp.

To prove (i), we set $\tilde{V} = R_p^\times / \mathbb{Z}_p$, stabilizer of P of cond 1
To prove (ii), $\tilde{V}$ = stab of a path $P \to P_1$, P_1 has cond p.

Let $\phi : \tilde{G} \to \mathbb{C}$ left inv't under Γ, right inv't under $\tilde{V}$.

To prove (i), $\phi(P) = \psi(P)$
(ii), $\phi(P) = a_p \psi(P) - \psi(\check{P})$.

Next write $\Gamma' = \tau_p^{-1} \Gamma \tau_p$ for some $\tau_p \in \tilde{G}$,
$$\xi(g_1, g_2) = \phi(g_1)\phi(\tau_p g_2), \quad \xi(g) = \xi(g, g)$$
consider

$$\tilde{G} \xrightarrow{\tilde{\Delta}} \Gamma \times \Gamma' \backslash \tilde{G} \times \tilde{G}$$
$$\cup|$$
$$\tilde{X} = \overline{\tilde{\delta}(G)} = \text{orbit of } \tilde{H} \subseteq \tilde{G} \times \tilde{G} \text{ closed sgp}$$
$\uparrow$
Ratner
$$= \Gamma_{\tilde{H}} \backslash \tilde{H} \qquad \Gamma_{\tilde{H}} = \Gamma \times \Gamma' \cap \tilde{H}$$
has "Haar measure" μ.

<u>Prop 5.2</u> If $\Gamma \backslash T$ not bipartite, then
$$\lim_{n \to \infty} \frac{1}{cp^{n-1}} \sum_{P \in X_n} \xi(P) = \int_{\Gamma_{\tilde{H}} \backslash \tilde{H}} \tilde{\xi}(h) \, d\mu(h).$$

Today's goal is to assume this, and prove 2.14.
Before that, one last piece of number theory!

<u>Prop 3.6</u> Let $\tau \in G_1$, $\tilde{\tau} \in \hat{K}$ corr. adele by CFT, $\tau_p = b_p f(\tilde{\tau}_p)$ (comes from some $b \in B^\times$)
Then $\tau_p \in \mathbb{Q}_p^\times B^\times$ (i.e. "rational" elt of $B_p^\times$) $\iff \tau \in G_0$.
Here $G_0 = \langle \mathrm{Frob}\, q : q \text{ ramifies over } K/\mathbb{Q} \rangle$

<u>pf</u> skip, for time reason.

* M, N, D, p are all coprimes to one another.

By this proposition, $\tau \in G_0 \Leftrightarrow \tau_p \in \mathbb{Q}_p^* B^*$
$\Leftrightarrow \Gamma$ + $\Gamma' = \tau_p^{-1} \Gamma \tau_p$ comm.
$\Leftrightarrow \tilde{H} =$ diagonal $\tilde{\Delta}$.

If $\tau \in G_1 \setminus G_0$, then $\tilde{H} = \tilde{\Delta} \cdot (PSL_2 \mathbb{Q}_p \times PSL_2 \mathbb{Q}_p)$
Sanghoon must've proved this last time. I'll explain again later.

Anyway, start w/ the case $\tau \in G_0$ so that Γ + Γ' comm. and $\tilde{H} = \tilde{\Delta}$. Write $\Gamma(\tau) = \Gamma \cap \Gamma'$. The strategy is to compute the "degree" of $S_\tau = \Gamma(\tau) \backslash \tilde{H} / \tilde{V}$ over $S_1 = \Gamma \backslash \tilde{H} / \tilde{V}$ (say, as a covering space) to reduce it to the case $\tau = 1$, which we've done already w/ Thm 1.5.
S_τ is just a finite set, so can write

$$\int_{T_H \backslash \tilde{H}} \xi(h) \, d\mu(h) = \frac{1}{|S_\tau|} \sum_{x \in S_\tau} \xi(x) =: A_d$$

where $d = q_1 q_2 \cdots q_r$
where $\tau = Fr\, q_1 \cdots Fr\, q_r$.

We claim
$$A_d = \frac{a_d}{t_d} A_1, \quad \text{where } a_d = a_d(\Psi), \ t_d = \deg(T_d)$$

To this end, adelize S_1 back:
$$\Gamma \backslash \tilde{G} / \tilde{V} = B^* \backslash B^* \hat{\mathbb{Q}}^* B_p^* V / V = B^* \backslash J / V$$
where $V = \prod V_\ell \subset \hat{R}^*$ w/ $V_\ell =$ elts of $R_\ell^* \equiv 1 \bmod M$ if $\ell \neq p$
$V_p =$ inverse img of $\tilde{V}$ via $R_p^* \to R_p^* / \mathbb{Z}_p^*$

Point is to essentially make $B^* \cap V = \Gamma$.
(Helpful exercise: Pick your favorite cong sgp Γ of $SL_2 \mathbb{R}$, and find $K = \prod K_p$ s.t. $\Gamma \backslash SL_2 \mathbb{R} = GL_2 \mathbb{Q} \backslash GL_2 \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Q}}^* K$.)

cise
- strong approx $\Rightarrow J \trianglelefteq \hat{B}^*$ finite index
- $S_1 = B^* \backslash J / V$ inherits Hecke actions $\{T_n\}_{n \geq 1}$ from $B^* \backslash \hat{B}^* / \hat{R}^*$.

Next we have to adelize S_τ. We'll take Vatsal's computation for granted.
Start w/ $\tau_p \in \tilde{G}$, $\tau_p = b_p f(\tilde{\tau}_p)$ b was chosen so that $b f(\tilde{\tau})$ at $\ell \neq p$ th place is in $\mathbb{Q}_\ell^* \hat{R}_\ell^*$

$*$ $\tau \xrightarrow{\text{artin}} \tilde{\tau} \longrightarrow f(\tilde{\tau})$ adjusted by $b \in B^*$ so as to do nothing on $\ell \neq p$ places.

$G \longrightarrow Cl(R)$

sic

(for the sake of the normal form stuff*)

Similarly as earlier,

$\Gamma_\tau \backslash \hat{G} / \hat{V}_p = B^* \backslash J / \hat{Q}^* V_\tau$,

V_τ def'd by $(V_\tau)_\ell = b_\ell^{-1} V_\ell b_\ell \quad \ell \neq p$

$(V_\tau)_p = \hat{V}_p$.

Because of how τ was defined ($\tau = \prod Fr_{q_i}$)

again, exercise!

$(V_\tau)_{q_i} = f(\tilde{\pi}_i) V_{q_i} f(\tilde{\pi}_i)^{-1}$ $\quad \pi_i$: uniformizer of q_i

and

$(V_\tau)_\ell = V_\ell$ all other $\ell \neq q_i$

well cuz b_ℓ ($\ell \neq p$) is chosen to make $b_\ell f(\tilde{\tau}_\ell)$ a unit in R_ℓ^* and at p, τ is trivial so $\tau_p = b_p$.

Again similarly,

$S_\tau = \Gamma \cap \Gamma_\tau \backslash \hat{G} / \tilde{V} = B^* \backslash J / \hat{Q}^* V(\tau) \qquad V(\tau) := V \cap V_\tau$.

Consider the projection to

$S_1 = B^* \backslash J / V$.

Fiber above $x \in S_1$ is ... one place at a time

$x V_{q_i} \qquad \coprod_j x \gamma_j (V_{q_i} \cap f(\tilde{\pi}_i) V_{q_i} f(\tilde{\pi}_i)^{-1})$

$f(\tilde{\pi}_i)$ is simply a matrix of det q_i.

Must compute the cosets γ_j ... should correspond to Hecke correspondence $T_{q_i}^{**}$ so that the fiber above x equals $T_d(x)$, and so

$**$ i.e. should be $\gamma_j = \begin{pmatrix} 1 & j \\ 0 & q_i \end{pmatrix}$ or $\begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix}$

$$A_d = \frac{1}{|S_\tau|} \sum_{x \in S_\tau} \xi(x) = \frac{1}{|S_1| t_d} \sum_{x \in S_\tau} \xi(x) = \frac{a_d}{|S_1| t_d} \sum_{x \in S_1} \xi(x) = \frac{a_d}{t_d} A_1$$

$\deg T_d$

as desired.

Now, the case $\tau \in G_1 \setminus G_0$.

In this case, $\tilde{H} = \tilde{\Delta} \cdot (PSL_2\mathbb{Q}_p \times PSL_2\mathbb{Q}_p)$.
Let $\Gamma_{\tilde{H}} = \Gamma \times \Gamma' \cap \tilde{H}$.
Must compute

$$\int_{\Gamma_{\tilde{H}} \setminus \tilde{H}} \xi(g_1, g_2)\, d\mu \quad , \text{ by prop 5.2.}$$

Note $\xi(g_1, g_2) = \Psi(g_1, \tau_p g_2)$ is left invt under $(\gamma, 1)$ where $\gamma \in \Gamma$ is any elt that interchanges even/odd vertices of T
and right invt under $\tilde{V}_* := \tilde{V} \times \tilde{V}$.

Claim $\tilde{G}_* = \tilde{H}\tilde{V}_* \cup (\gamma, 1) \cdot \tilde{H}\tilde{V}_*$.

Assuming the claim, by invariances of ξ suffices to consider

$$\int_{\Gamma_* \setminus \tilde{G}_*} \xi(g_1, g_2)\, d\mu = \sum_{x,y} \Psi(x)\Psi(y)$$

$\Gamma \times \Gamma'$

$$= 0 \text{ by Lemma 3.11}$$

pf of claim $\tilde{G} = PSL_2\mathbb{Q}_p \cdot \tilde{V} \cup \begin{pmatrix} 1 & \\ & p \end{pmatrix} PSL_2\mathbb{Q}_p \cdot \tilde{V}$ (exercise).
So $\tilde{G}_* = \tilde{H}\tilde{V}_* \cup \underbrace{\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix}, 1\right)}_{\text{modify appropriately.}} \tilde{H}\tilde{V}_*$.

In summary,

$$\frac{1}{cp^{n-1}} \sum_{p \in X_n} \sum_{\tau \in G_1} \chi_t(\tau)\Psi(p)\Psi(p^\tau)$$

$$\longrightarrow \sum_{\tau \in G_0} \chi_t(\tau) \frac{a_d}{t_d} A_1 = \prod_{q \mid D} \left(1 + \chi_t(\mathrm{Fr}\, q) \frac{a_q}{t_q}\right) A_1$$

by multiplicativity of the coefficient, as desired.