<Vatsal talks>.

6/30

Goal. Understand
V. Vatsal, "Uniform dist. of Heegner pts"
Inv. math 2002.
Assuming very little knowledge on # theory & ergodic theory.

References: Vatsal, ——
Bertolini & Darmon, Inv. 1996
Gross, Heights & special values of L-series
Vigneras, Arithmétique des algèbres de quaternions

— A motivation.

genus 1 — complex dim 1 — def'd by poly in $\mathbb{Q}$

Let E be an elliptic curve over $\mathbb{Q}$
e.g. soln set of $y^2 = x^3 - x$.
Want: solns in $\mathbb{Q}$ aka $\mathbb{Q}$-points of E or $E(\mathbb{Q})$
or solns in a # fld (finite extn of $\mathbb{Q}$) $K/\mathbb{Q}$
i.e. $E(K)$

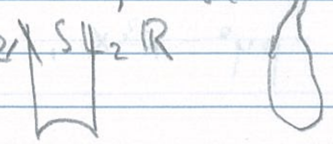
How do we find any of that?

Modularity Thm For every E an ell curve $/\mathbb{Q}$, there exists a surjection of "Riemann surfaces"

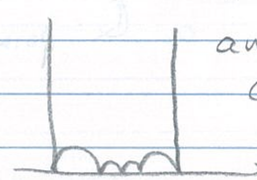
$$X_0(N) \longrightarrow E. \quad (N \text{ integer depending on } E)$$

Here
$$X_0(N) = \Gamma_0(N) \backslash \mathbb{H}, \quad \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in SL_2\mathbb{Z} \right\}.$$

e.g. $X_0(1) = SL_2\mathbb{Z} \backslash SL_2\mathbb{R}$



$X_0(11)$:



an ell curve

The map $X_0(N) \longrightarrow E$ can be re-interpreted as a map b/w "varieties" / $\mathbb{Q}$, the map also def'd / $\mathbb{Q}$

In particular, $X_0(N)$ can be written as a locus of some poly eqn in $\mathbb{Q}$. __Exercise__: find it!

__Hint__.

$$\begin{array}{ccc} X_0(N) & & \mathbb{Q}(j(\tau), j(N\tau)) \\ \Big\downarrow \text{"covering"} & \underset{\text{fields}}{\overset{\text{function}}{\rightsquigarrow}} & \Big| \\ X_0(1) & & \mathbb{Q}(j(\tau)) \end{array}$$

where $j(\tau) : X_0(1) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$.

So it suffices to find a min poly of $j(N\tau)$:

$$\prod_{\sigma \in \mathrm{Gal}} (j(\tau) - \sigma \cdot j(N\tau))$$

Q: what's the relevant Galois group?

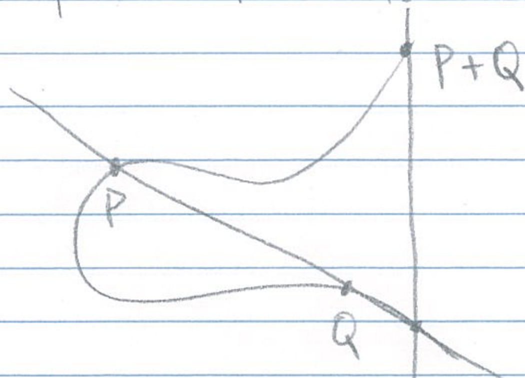
Thanks to modularity, we can exploit the rich structure of $X_0(N)$ to study E.
For example, using class field theory, we can construct on $X_0(N)$ what are called Heegner points.
It's simple to describe: eq. class on $\mathbb{H}$ of form

$$\frac{-b \pm \sqrt{-d}}{2a}, \quad d = 4ac - b^2, \quad a, b, c \in \mathbb{Z}.$$

This is a H_K-point, where H_K = Hilbert cl. fld of $K = \mathbb{Q}(\sqrt{-d})$
Modularity gives an H_K-pt on E, an abelian group
and adding up the point gives a K-point also of E!

__Aside__ Heegner used sth like this to find a $\mathbb{Q}$-point of $py^2 = x^3 - x$, where $p \equiv 5$ or 7 (8) prime

The points of E form an abelian group.



identity is pt at ∞.

In particular, if $K/\mathbb{Q}$ any # fld,

<u>Thm</u> (Mordell-Weil)

$$E(K) \cong \underbrace{\mathbb{Z}^r}_{\text{called rank}} \times \underbrace{(\text{fin. ab. gp})}_{\text{torsion}}$$

We may wonder whether the pts in $E(K)$ found via modularity + Heegner idea has finite or infinite order, or whether they span all of $E(K)$, etc

The philosophy is that this kind of properties are connected to certain values of what are called L-functions. To give a rough idea,

$L(s, E/\mathbb{Q}) := \sum_{n=1}^{\infty} a_n/n^s$

where $a_1 = 1$, $a_p = p+1 - |E(\mathbb{F}_p)|$ others are def'd in some way. Similarly, one can define $L(s, E/K)$.

Then

<u>Thm</u> (Gross-Zagier formula)

$$L'(1, E/K) = \boxed{\text{sth}} \cdot \text{height of the } K\text{-pt of } E \text{ we constructed}$$

In paticular, $L'(1, E/K) = 0 \iff$ that pt is torsion.

<u>Conjecture</u> (Birch & Swinnerton-Dyer)

won't state.

- What's done in Vatsal's paper (roughly).

Conjecture (Mazur) Under such such conditions ...
some family of L-value = 0 almost all the time
and this implies $E(H_\infty)$ is finitely generated
$\quad$ ↖ some infinite extn / $\mathbb{Q}$
(so Mordell-Weil need not apply).

On the other hand, by a formula of Gross,
L = something pretty

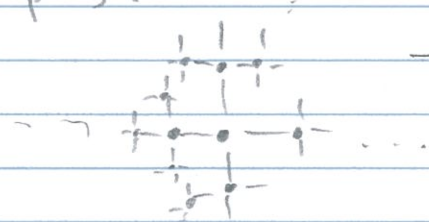
$$= \frac{1}{p^n} \sum_{p \in X_n} \sum_{\sigma \in G_n} \chi(\sigma)\psi(P^\sigma)\psi(P)$$

where
X_n: set of "Heegner pts" on Gross curve
of "disc n"

an analogue of a modular curve.

One can think $X_n \subseteq PGL_2 \mathbb{Q}_p / PGL_2 \mathbb{Z}_p =: T$
(Bruhat-Tits tree).

e.g. when $p = 3$:



— an analogue of $\mathbb{H}$.

G_n: just some group that permutes X_n.
X_n, G_n finite.
χ: some character of G_n
ψ: a map $\Gamma \backslash PGL_2 \mathbb{Q}_p / PGL_2 \mathbb{Z}_p =: G \longrightarrow \mathbb{R}$
an analogue of $\Gamma_0(N)$
coming from E somehow. (Actually modularity + Jacquet-Langlands)

Vatsal shows $L \neq 0$ as $n \to \infty$ for almost all χ.

How the proof goes, roughly:
fix a $\tau \in G_n$ (actually we consider $\tau \in$ proper s/set of G_n)
and consider

$$\sum_{p \in X_n} \Psi(p)\Psi(p^\tau)$$

A finite graph! interesting dynamics happens here.

$$T \longrightarrow G = \Gamma \backslash T$$

Heegner pts $\qquad$ classes

P_1, $P_2 \longrightarrow C_1$

$\vdots \longrightarrow C_2$

$\vdots$

Q1 How many P's go to each C_i?
Say answer is $\approx \rho_i \cdot |X_n|$.
So if $\tau = 1$,

$$\sum_{p \in X_n} \Psi(p)\Psi(p) = \int_G \rho_i \Psi^2(C_i).$$

Done by random walk theory on graphs.

Q2 τ doesn't preserve C_i's unfortunately. But by Ratner one can show

$$|C_i^\sigma \cap C_j| \approx \rho_i |C_i|.$$

So

$$\sum_{p \in X_n} \Psi(p)\Psi(p^\tau) \approx \left(\int_G \rho_i \Psi(C_i)\right)^2$$

$= 0$ easily from ppty of Ψ.