7/14 Statement of the main theorem.

First we need to understand what $L(g, \chi, s)$ means.

g: cuspidal newform of wt 2 level N
equivalently, holomorphic diff form on $X_0(N) = \Gamma_0(N) \backslash \mathbb{H}$
has a Fourier expansion
$$g(\tau) = \sum_{n=1}^{\infty} a_n(g) q^n, \qquad q := e^{2\pi i \tau}, \quad \tau \in \mathbb{H}.$$

χ: finite order Hecke character of $\overset{\#\text{fld}}{K}$, of conductor $\mathfrak{f}$.
To understand, need a crash course on algebraic number theory.
First, the notion of ring of integers $\mathcal{O}_K$ of K:

$$\begin{array}{ccl} K & - & \mathcal{O}_K : \text{integral closure of } \mathbb{Z} \text{ in } K. \\ | & & | \\ \mathbb{Q} & - & \mathbb{Z} \end{array}$$

Next, the primes of $\mathcal{O}_K$.
If $p \in \mathbb{Z}$ is a prime, can factor
$$p\mathcal{O}_K = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i} \qquad \mathfrak{p}_i : \text{prime ideals of } \mathcal{O}_K$$

We say $\mathfrak{p}_i$ lies over p.
e_i is called ramification degree of $\mathfrak{p}_i$. If $e_i > 1$, $\mathfrak{p}_i$ is ramified. If not, unramified.
$\mathcal{O}_K/\mathfrak{p}_i \cong \mathbb{Z}/p^{f_i}\mathbb{Z}$ some f_i: inertia degree.

Thm $\sum_{i=1}^{g} e_i f_i = [K : \mathbb{Q}]$.

If $K/\mathbb{Q}$ is Galois, $e_1 = \cdots = e_g$, $f_1 = \cdots = f_g$
b/c Gal permutes the $\mathfrak{p}_i$'s

Example $K = \mathbb{Q}(i)$, $p=5$.

$\mathcal{O}_K$ is hard to compute in general. In this case $\mathcal{O}_K = \mathbb{Z}[i]$.
$p\mathcal{O}_K$ is probably also hard, but we can look at

$$\mathbb{Z}[x] / (5, x^2+1) \cong \mathbb{F}_5[x] / \underset{\substack{\uparrow \\ \text{two solns in } \mathbb{F}_5 : \ 2 \ \& \ 3}}{(x^2+1)} \cong \mathbb{F}_5 \oplus \mathbb{F}_5$$

So we know $e=1$, $f=1$, $g=2$.
Indeed, $(5) = (1+2i)(1-2i)$ as an ideal of $\mathbb{Z}[i]$.

If $p=7$, then $\mathbb{F}_7[x] / \underset{\substack{\uparrow \\ \text{irred mod } 7}}{(x^2+1)} \cong \mathbb{F}_{7^2}[X]$

so (7) is a prime in $\mathbb{Z}[i]$. $e=1$, $f=2$, $g=1$.

Exercise Try $p=2$. one gets $e=2$, $f=1$, $g=1$.

we say 5 is split over K, 7 is inert, 2 ramifies.
Same terminology applies to any quadratic field.

Next we discuss (finite) adeles & ideles.
For details, consult Tom Weston's or W. Stein's notes on adeles.

Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z} \underset{\text{exercise}}{=} \prod_p \mathbb{Z}_p$.

Write $\hat{K} := K \otimes \hat{\mathbb{Z}}$ (finite adele of K).
has profinite topology.

Example $\hat{\mathbb{Q}} = \prod_p \mathbb{Q}_p$.

$$\hat{K} = \prod_p K \otimes \mathbb{Q}_p = \prod_p \Big(\underbrace{\prod_{v|p} K_v}_{K \otimes \varprojlim_n \mathcal{O}_K / p^n \mathcal{O}_K} \Big) = \prod_v K_v$$

Exercise Try working out w/ $K = \mathbb{Q}(i)$.
What are the K_v's, explicitly?

Finally, we can say, finite order Hecke char of K means
$\chi : K^* \backslash \hat{K}^* \to \mathbb{C}^*$ (continuous; won't discuss topology here)
Conductor $\mathfrak{f}$ ($\mathfrak{f}$ ideal of $\mathcal{O}_K$) means

χ_v (χ restricted to v-th place) is trivial on $1 + \mathfrak{f}\mathcal{O}_v$.

Example If $K = \mathbb{Q}$, this is precisely a Dirichlet character.
Let $\chi : \mathbb{Q}^* \backslash \hat{\mathbb{Q}}^* \to \mathbb{C}^*$, conductor (m) $(m \in \mathbb{Z})$.
m is a unit on $\mathcal{O}_p$ for all $p \nmid m \Rightarrow 1 + (m)\mathcal{O}_p = \mathcal{O}_p$
$\Rightarrow \chi_p$ trivial on $\mathcal{O}_p$.
By strong approximation, χ factors thru $(\mathbb{Z}/m\mathbb{Z})^*$.
Exercise look this $\int$ up, and prove this claim.

For general K, the description of χ is not as simple as Dirichlet characters.

Finally we can define $L(g, \chi, s)$.
Irrelevant for us, but I'll do it anyway.
Thm
aka $\mathrm{Pic}(\mathcal{O}_K) \to \underbrace{K^* \backslash \hat{K}^* / \hat{\mathcal{O}}_K^*}_{\text{same pf: algebraic geometry on } \mathrm{Spec}\,\mathcal{O}_K} \cong Cl(K) := (\text{fractional ideals of } \mathcal{O}_K) / (\text{fractional principal ideals of } \mathcal{O}_K)$

Exercise Compute both sides for $K = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-5})$

So it makes sense to define
$$\Theta_\chi := \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ \text{ideal}}} \chi(\mathfrak{a}) q^{N\mathfrak{a}} = \sum a_n(\chi) q^n \quad \text{where } q = e^{2\pi i z} \quad z \in \mathbb{H}, \quad N\mathfrak{a} = \prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \sigma\mathfrak{a}$$
modular form on $\Gamma_0(C_\chi)$ for some C_χ.
Then
$$L(g, \chi, s) = \zeta(2s-2) \sum_{(n, NC_\chi)=1} a_n(g) a_n(\chi) n^{-s}$$
$$= \text{"Rankin-Selberg convolution" of } g \,\&\, \Theta_\chi$$

Now restrict to case $K/\mathbb{Q}$ is im. quad.

<u>Def</u> χ is anticyclotomic if $1 \neq \sigma \in Gal(K/\mathbb{Q})$
then $\chi^\sigma = \chi^{-1}$.

How does σ act on χ? One place at a time: $\chi_v : K_v \to \mathbb{C}^*$
if v inert: $[K_v : \mathbb{Q}_p] = 2$ and σ acts by the nontriv Gal elt.
v split: if $\bar{v}$ is its conjugate, $\chi_v^\sigma = \chi_{\bar{v}}$
v ramify: exercise.

Any such χ factors through $Gal(K_f/K)$,
i.e. $\chi : Gal(K_f/K) \longrightarrow \mathbb{C}^*$.

$$\begin{array}{c} Gal(K_f/K) \\ \wr\Vert \\ K^* \backslash \hat{K}^* / \hat{\mathcal{O}}^*_{K,f}\, \hat{\mathbb{Q}}^* \end{array} \quad \text{by class field theory}$$

where $\mathcal{O}_{K,f} = \mathbb{Z} + f\mathcal{O}_K \; (\subseteq \mathcal{O}_K)$
(Sth like $\mathcal{O}_{K,f}$ – s/ring of $\mathcal{O}_K$ ctg $\mathbb{Z}$ – is called an <u>order</u> of K)
K_f is indirectly def'd by this reln, and is called
the <u>ring class field</u> w/ conductor f.

Furthermore any such χ is anticyclotomic.
<u>Exercise</u> show this.

Now $L(g, \chi, s)$ satisfies the functional equation

$$L(g, \chi, s) = e \cdot A^{s-1} \cdot L(g, \chi^{-1}, 2-s)$$

where $e = \pm 1$ is called ~~the~~ root number
and A is some constant.

In our context, (g level N, χ anti of K, conductor f)
if f and N are coprime, then $e = -\chi_D(N)$,
where $D =$ disc K and χ_D is the Dirichlet character

$$\begin{array}{ccc} (\mathbb{Z}/D\mathbb{Z})^* & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \Vert & & \Vert \\ Gal(\mathbb{Q}(\zeta_{|D|})/\mathbb{Q}) & & Gal(K/\mathbb{Q}) \end{array}$$

induced by $K \hookrightarrow \mathbb{Q}(\zeta_{|D|})$

We say we're in <u>definite case</u> if $\epsilon = +1$
<u>indefinite case</u> if $\epsilon = -1$.

Almost there!

Last <u>defn</u> $H_n :=$ ring cl. fld of K w/ conductor p^n.
$H_\infty := \cup H_n$.

(discuss Gal gp of H_∞)

<u>Theorem</u> (1.4 in "Uniform. ")

Suppose
- g newform of wt 2 on $\Gamma_0(N)$. N sq free
- $K = \mathbb{Q}(\sqrt{-D})$ im. quad. fld
- in definite case $\Leftrightarrow$ # prime factors of N inert over K is odd.
- $(p, ND) = 1$.

Then
- If p is ordinary for g ($p \nmid a_p(g)$) then $L(g, \chi, 1) \neq 0$ for all but finitely many (primitive) anti. char χ of p-power conductor.
- If p is supersingular for g ($p \nmid a_p(g)$) the same holds provided $p \nmid h_K := \# Cl(K)$.

<u>Next up</u> Gross' formula:

$$\frac{L(g, \chi, 1)}{\Omega_g} = \frac{1}{p^n} \sum_{P \in X_n} \sum_{\sigma \in G_n} \chi(\sigma) \Psi(P^\sigma) \Psi(P)$$

Ω_g: some constant

where X_n: Gross pts of cond p^n
$G_n = Pic(\mathcal{O}_n)$ ($\mathcal{O}_n$: ring cl. fld of cond p^n)
$= Gal(H_n/K)$
$\Psi : PGL_2 \mathbb{Q}_p / PGL_2 \mathbb{Z}_p \to \mathbb{R}$ constructed from g.

Chan-Ho will discuss Gross pts next time.