$\varphi: \mathbb{Q}_p \to G$, $\left.\dfrac{d\varphi(t)}{dt}\right|_{t=0} \neq 0$ continuous hom.

S = finite set of valuations of $\mathbb{Q}$

Ex $S = \{p\}$, $S = \{p, q\}$, $S = \{p, q, \infty\}$

$G_S = \prod_{s \in S} G_s$

(Erg msrs) $H \leq G_S$ closed $U \leq$ gen'd by one-parameter Ad-unip. subgps of G_S. Then for $\forall \Gamma < H$ discrete, every ergodic U-inv. Borel prob. msr on $\Gamma \backslash H$ is algebraic.

i.e., $\exists x = x(\mu) \in \Gamma\backslash \cancel{G}H$: $\mu(x\Lambda(\mu)) = 1$

where $\Lambda(\mu) = \{ \cancel{g} h \in \cancel{G} H \mid$ the action of h on $\Gamma\backslash H$ preserves μ

(Orbit closures) H, U as above, $\Gamma < H$ lattice

For any $x \in \Gamma\backslash H$, the closure $\overline{xU}$ of the orbit xU in $\Gamma\backslash H$ is homogeneous.

(Uniform distribution) $\mathcal{U} \leq H$ one-para Ad-unip. of G_S

Given any lattice Γ of H, every point $x \in \Gamma\backslash H$ is generic for $\mathcal{U}$. i.e., $\lim_{S\to\infty} \dfrac{1}{\lambda(F(S))} \displaystyle\int_{F(S)} f(xu(t))\, d\lambda(t) \to \int_{\Gamma\backslash H} f\, d\nu_L$

for some $\mathcal{U} \leq L \leq G$

(Dichotomy)

Goal If $\Gamma\backslash\mathcal{T} = \Gamma\backslash PGL(2,\mathbb{Q}_p)/PGL(2,\mathbb{Z}_p)$ is not bipartite, then

$$\lim_{n\to\infty}\frac{1}{\delta_0 p^{n-\delta-1}}\sum_{P\in X_n'} f(P) = \lim_{n\to\infty}\frac{1}{c\,p^{n-\delta-1}}\sum_{P\in X_n} f(P)$$

$$= \int_{(\Gamma\times\Gamma')\cap 目_L\backslash 目_L} f(h)\,d\mu(h)$$

$\phi : \cancel{\infty}\ \Gamma\backslash PGL(2,\mathbb{Q}_p)/V$ bdd, conti, real-valued

$\phi(P') = \phi(gV)$ $\quad f : \Gamma\times\tau_p^{-1}\Gamma\tau_p\backslash G\times G$

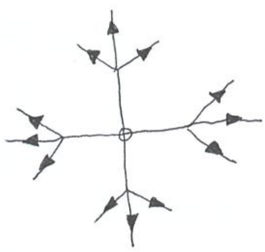
$f(g_1,g_2) = \phi(g_1)\phi(\tau_p g_2)$

$f(P') = f(g,g)$

$\# X_n' = \delta_0 p^{n-\delta-1}$ $\quad \# X_n = c\,p^{n-\delta-1}$

($X_n = \{$Heegner pts of level $p^{n-\delta}\}$)

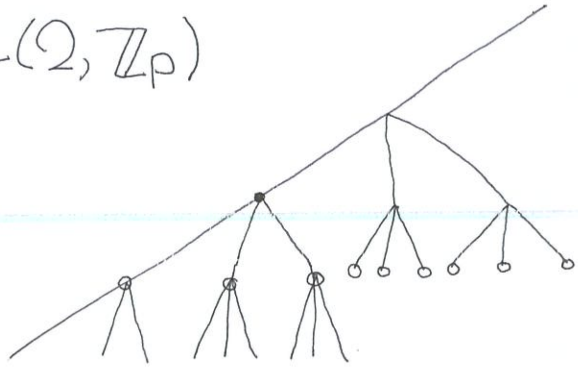
Want: $\displaystyle\lim_{n\to\infty}\frac{1}{\delta_0 p^{n-\delta-1}}\sum_{v\in X_n'} f(P(v)) = \int_{\Gamma\times\Gamma' 目_L\backslash 目_L} f(h)\,dh$



$$\boxed{\lim_{n\to\infty}\frac{1}{(p+1)p^{n-1}}\sum_{z\in Z_n} f(z) = \int_{\Gamma\times\Gamma'\cap 目_L\backslash 目_L} f(h)\,dh}$$

(Prop 3.16)

$$\mathcal{T} = G/K = PGL(2, \mathbb{Q}_p) / PGL(2, \mathbb{Z}_p)$$



$$U = \{ \cancel{t \in \mathbb{Q}_p} \mid t \in \mathbb{Q}_p \}$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{(p+1)p^{2n-1}} \sum_{z \in Z_n} f(z) = \int_{PGL(2,\mathbb{Z}_p)} f(ku(t))\, d\mu(k)$$

$$t = ap^{-n}, \quad a \in \mathbb{Z}_p^* \qquad \cancel{= \int_{PGL(2,\mathbb{Z}_p)} f(u(t)k)\, d\mu(k)}$$

λ : additive Haar measure on $\mathbb{Q}_p$, $\lambda(\mathbb{Z}_p) = 1$

$$F_n = \{ t \in \mathbb{Q}_p \mid \mathrm{ord}_p(t) \geq -n \} \qquad \lambda(F_n) = p^n$$

$$\lim_{n \to \infty} \frac{1}{\lambda(F_n)} \int_{F_n} f(ku(t))\, d\lambda(t) = \int_{X_k} f(x)\, d\mu_k(x)$$

$\cancel{\triangle} \Big\{$ ① kU is dense in kH a.e. k

② $\displaystyle\int_{X_k} f(x)\, d\mu_k(x) = \int_{\Gamma_L \backslash \mathbb{H}_L} f(x)\, d\mu(x)$ a.e. k

③ $$\lim_{n \to \infty} \frac{1}{\lambda(F_n')} \int_{F_n'} \int_{K} f(ku(t))\, dk\, d\lambda(t) = \int_{\Gamma_L \backslash \mathbb{H}_L} f(x)\, d\mu(x)$$

$\Delta = \{(g,g) \in G \times G \mid g \in G\}$ acts on

$\Gamma \times \tau_p^{-1}\Gamma\tau_p \backslash G \times G$ $\Rightarrow H = \{(g,g) : g \in PSL_2$

$\overline{(\Gamma \times \tau_p^{-1}\Gamma\tau_p)\cdot\Delta} = (\Gamma \times \tau_p^{-1}\Gamma\tau_p)\cdot\Delta$

or $(\Gamma \times \tau_p^{-1}\Gamma\tau_p)\cdot\Delta(PSL_2 \times PSL_2$

$\Rightarrow H = PSL_2 \times PSL_2$.

$\{(kU, kU)\} \subset kH \subset kL = L$

Since $(k,k) \in \Delta \subset L$

Need to prove

$(k,k)H$ is closed in $\Gamma \times \tau_p^{-1}\Gamma\tau_p \backslash G \times G$

Since $(\Gamma \times \tau_p^{-1}\Gamma\tau_p) \cap H$ is a lattice in H

① kU is dense in kH:

Γ and $\tau_p^{-1}\Gamma\tau_p$ are comm. $\Rightarrow$ every unipotent orbit is dense in $\Gamma \cap \tau_p^{-1}\Gamma\tau_p \backslash PSL_2(\mathbb{Q}_p)$

Not commensurable $\Rightarrow$ Use Thm 4.15:

$\Gamma \subset G$ cocpt lattice, one para. unip. group U satisfies $U \subset S \subset G$. Then μ is preserved by cSc^{-}

$(S \cong PSL_2(\mathbb{Q}_p))$ for some $c \in G$, c centralizes U

The closure of kUk^{-1} is invariant under the action of

$kU \subset kH$ dense

$\Leftrightarrow kUk^{-1} \subset kH$ dense

$S(U') = \{(g, ugu^{-1}) \mid g \in PSL_2\}$ for some $u \in U'$

$U' = \underline{kUk^{-1}}$

$G \times G$

$\cup$

$S \cong PSL_2(\mathbb{Q}_p)$

$\cup$

U'

$$\overline{(\Gamma \times \tau_p^{-1}\Gamma\tau_p)\cdot(U' \times U')}$$

$$= (\Gamma \times \tau_p^{-1}\Gamma\tau_p)\cdot H', \quad H' \leq H$$

$$cS(U')c^{-1} \subset H'$$

ⓐ $cS(U')c^{-1} \subsetneq H'$:

$$cS(U')c^{-1} = \{(cgc^{-1}, cugu^{-1}c^{-1}) \mid g \in PSL_2\}$$

Let $u_* = (c, cu)$. Then

$$\{(g,g) \mid g \in PSL_2\} = u_*^{-1} cS(U')c^{-1} u_*$$

$$\subsetneq u_*^{-1} H' u_* \leq PSL_2(\mathbb{Q}_p) \times PSL_2(\mathbb{Q}_p)$$

$$\Rightarrow H' = H, \quad kUk^{-1} \subset kH' \text{ dense}$$

ⓑ $cS(U')c^{-1} = H'$:

$\Gamma \times \tau_p^{-1}\Gamma\tau_p$ is a lattice in $(\Gamma \times \tau_p^{-1}\Gamma\tau_p) \cap cS(U')c^{-1}$

$\Rightarrow \Gamma$ and $u\tau_p^{-1}\Gamma\tau_p u^{-1}$ are commensurable

$\mathrm{Comm}_{B_p}(\Gamma) = \underbrace{\mathbb{Q}_p^\times \cdot B^\times}_{\text{countable}}$

$\left.\begin{array}{l} \det(u) = 1 \\ u\tau_p^{-1} \in \mathbb{Q}_p^\times B^\times \\ \tau_p \notin \mathbb{Q}_p^\times B^\times \end{array}\right\} \Rightarrow u \neq 1$

② $\cancel{X_k = (\Gamma \times \tau_p \Gamma \tau_p}$ $\qquad X_k = \overline{\Gamma k U}$

$$(\Gamma \times \tau_p \Gamma \tau_p^{-1})(k,k)H = Y_k \qquad X = Y_1$$

$$\overline{(\Gamma \times \tau_p \Gamma \tau_p^{-1})(k,k)(u,u \mid u \in U)} = X_k$$

$$Y_k \cong (\Gamma \times \tau_p \Gamma \tau_p^{-1}) \cap kHk^{-1} \backslash kHk^{-1}$$

$$(Y_k, \mu_k) \rightsquigarrow (kHk^{-1}, \mu) \rightsquigarrow (H, \mu)$$

$$\int_{X_k} f(x)\, d\mu_k(x) = \int_X f(x)\, d\mu(x)$$

$$= \int_{\Gamma_L \backslash L} f(x)\, d\mu(x) \quad (= A)$$

③ ② implies that

$$\int_{PGL(2,\mathbb{Z}_p)} \left(\lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} f(ku(t))\, d\lambda(t) \right) d\mu(k)$$

$$= \lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} \left(\underbrace{\int_{PGL(2,\mathbb{Z}_p)} f(ku(t))\, \cancel{d\lambda(t)}\, d\mu(k)}_{=A} \right) d\lambda(t)$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_{n-1}} \left(\underbrace{\int_{PGL(2,\mathbb{Z}_p)} f(ku(t))\, d\mu(k)}_{= \frac{1}{p} A} \right) d\lambda(t)$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{\lambda(F_n')} \int_{F_n'} \text{〃}\ d\lambda(t) = A \Rightarrow \lim_{n\to\infty} \frac{1}{(p+1)p^{2n-1}} \sum_{d(v,v_0)=2n} f(v_0) = A$$

Detail of ② and ③

$KHK^{-1} = H$

$$\mathcal{D}(g) = \{ x \in \Gamma \backslash G_S \mid xg^n \to \infty \}$$

$$(xg^n)^{-1} \gamma_n (xg^n) \to e_S$$

(Dichotomy)

$\Gamma < G_S$ discrete $\mu : \overset{u(t)}{\mathcal{U}}$-inv. Borel prob. msr on $\Gamma \backslash G_S$. Then either

(1) $\mu(\underline{\underline{\mathcal{D}(a(\tau))}}) = 1$ for $\forall a(\tau) \in A(\tau)$ with $|\tau| > 1$

or

(2) $c SL_2(\mathcal{U}, A) c^{-1} \subset \Lambda(\mu)$ for some $c \in C(\mathcal{U})$ and μ is algebraic.

Summary

$\overline{\Gamma} = \Gamma(M) \subset R[\frac{1}{p}]^\times$

$$\lim_{n \to \infty} \frac{1}{(p+1)p^{2n-1}} \sum_{d(v, v_0) = 2n} f(v) = A$$

to arbitrary vertex $v_0 \in T$ $\qquad \gamma \in R[\frac{1}{p}]^\times$

to vertices at odd distance

$$T_{2n+1}(v_0) = \sum_{i=1}^{p+1} T_{2n}(v_i) - pT_{2n-1}(v_0)$$

$$\lim_{n \to \infty} \frac{1}{(p+1)p^{2n}} \sum_{d(v, v_0) = 2n+1} f(v) = A$$

$(h_n)_{n=1}^{\infty} \in H$ $\quad \pi_H : \cancel{H \cap \Gamma}\ H \to \Gamma \cap H \backslash H$

$\pi_G : \quad G \to \Gamma \backslash G$

$\pi_H(h_n)$ does not converge. (in $\Gamma \cap H \backslash H$)

$H = \bigcup_{n=1}^{\infty} B_n$, B_n compact $\quad \varepsilon_n = \mu(\Gamma \cap H \backslash H - \pi_H(B_n)) \to 0$

$\pi_H(h_m) \notin \pi_H(B_n)$ for large m

$\cancel{U_n^{-1} U_n} B_n U_n U_n^{-1}$ $\qquad \underline{\underline{\mu(U_n) > \varepsilon_n}}$

$\Rightarrow \pi(\overset{h_m U_n}{\cancel{U_n h_m}}) \cap \pi(\overset{B_n U_n}{\cancel{U_n B_n}}) = \emptyset$

$\Rightarrow \mu(\pi_H(\cancel{U_n h}\, h_m U_n)) \leq \mu(\Gamma \cap H \backslash H - \pi_H(B_n U_n)) \leq \varepsilon_n$

$\Rightarrow \exists\, e \neq \gamma_m \in \Gamma$: $\gamma_m h_m u = h_m u'$ for some $u, u' \in U_n$

$\Rightarrow h_m^{-1} \gamma_m h_m \in U_n U_n^{-1}$

$h_m^{-1} \gamma_m h_m \to e$

$\lim \pi_H(h_m) = \pi_H(z)$. $\theta_m h_m \to z$

$h_m^{-1} \theta_m^{-1} \underbrace{\theta_m \gamma_m \theta_m^{-1}}_{\downarrow e} \theta_m h_m \to e$

$\Rightarrow \gamma_m = e$ for large e

$G = PGL(2, \mathbb{Q}_p)$

$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| \, t \in \mathbb{Q}_p \right.$

$$\lim_{n \to \infty} \frac{1}{\lambda(F_n)} \int_{F_n} f(ku(t))\, d\lambda(t) = \int_{X_k} f(x)\, d\mu_k(x) = \int_{\Gamma \times \Gamma' \cap L \backslash L} f(\ell)\, d\ell$$

$\forall$ a.e. $k \in K$

$\Gamma \backslash G$ U-orbit $\qquad \overline{\Gamma k U} = X_k$

$\Gamma \backslash G \times \Gamma' \backslash G'$ U-orbit

G-orbit

$L = \Delta$ or $\Delta \cdot (PSL_2 \times PSL_2)$

$(\Gamma \times \Gamma')(k,k)U$

$H = \Delta \cap (PSL_2 \times PSL_2)$

$\overline{(\Gamma \times \Gamma')(k,k)H} = Y_k$

or $PSL_2 \times PSL_2$

$\cong \Gamma_k \backslash kHk^{-1}$

$\tau^{-1} \tau g \tau^{-1}$

$\phi(\tau g)$

$= \phi(\Gamma \tau g)$

$= \phi(\Gamma \tau \underline{\tau^{-1} \Gamma \tau g})$