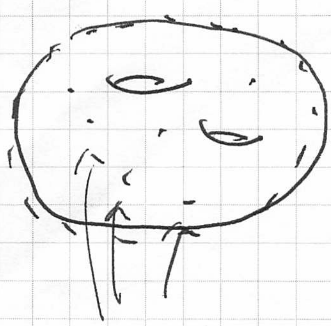
Gross pts. : very sketch.

0. What do we wanna do ?



arithmetically interesting space

- modular
- Shimura curves
- Hida var.

Heegner
Gross

some pts encode the information of the space.

$\Downarrow$ ← dynamics

equidistributed $\Rightarrow$ nonvanishing of values of the function on $\underbrace{\text{arithmetic of}}_{\text{L-values}}$ the space.

1. Space : "Shimura variety-like"

$\underline{GL_2/\mathbb{Q}}$.

$$SL_2(\mathbb{R}) \subset \mathfrak{h} \underset{\substack{\uparrow \\ \text{orbit-stab}}}{\simeq} SL_2(\mathbb{R}) / \underbrace{SO_2(\mathbb{R})}_{\text{stab}(i)}$$

$\Gamma \subseteq SL_2(\mathbb{Z})$ arithmetic subgp
(e.g. cong. subgp)

$$\rightsquigarrow \Gamma \backslash \mathfrak{h} \simeq \Gamma \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R}).$$

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<u>adelically</u> more.

$$GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q}) / K \cdot \underbrace{SO_2(\mathbb{R})}_{\text{max'l cpt.}} \cdot \underbrace{\mathbb{R}^\times}_{\text{center}} .$$

$$\left(\simeq \coprod_i \Gamma_i \backslash \mathfrak{h} \right)$$ via strong approx. <u>at ∞.</u>

where $K \leq GL_2(\hat{\mathbb{Z}})$. $\Gamma_i = GL_2(\mathbb{Q}) \cap x_i K_f x_i^{-1}$

$\updownarrow$ $x_i \in \underbrace{G(\mathbb{Q})\backslash G(\mathbb{A}_f) / K_f}_{\text{finite}}$.

$\Gamma \leq SL_2(\mathbb{Z})$

<u>Quaternion alg. /$\mathbb{Q}$.</u>

i, j, k

... why? very similar to GL_2 ... how?

B, a quat. alg. /$\mathbb{Q}$.

$$\rightsquigarrow \underbrace{B \otimes_\mathbb{Q} \mathbb{Q}_v}_{{}^{\text{"}}B_v} \simeq M_2(\mathbb{Q}_v), \quad \forall\forall\, v \leq \infty.$$

Indeed, $\{ v \leq \infty : B \otimes_\mathbb{Q} \mathbb{Q}_v \not\simeq M_2(\mathbb{Q}_v) \}$

determines B up to isom.

$(=: \underline{\mathrm{ram}}(B))$

$\uparrow$

<u>even</u> number of places of $\mathbb{Q}$.

: "Brauer gp reason".

$$N^- = \prod_{v \in \mathrm{ram}(B) \setminus \{\infty\}} v$$

The behavior ~~of~~ at ∞ place change the situation dramatically.

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong \begin{cases} M_2(\mathbb{R}) & \rightsquigarrow \text{indefinite} \\ \mathbb{H} \quad \text{Hamiltonian} & \rightsquigarrow \cancel{\text{indefinite}}\ \text{definite} \end{cases}$$

$$\left(B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \begin{cases} M_2(\mathbb{Q}_\ell) \\ \text{unique div. alg.} \end{cases} \right)$$

definite: $\underset{\text{nrd } 1}{\mathbb{H}^1} \subseteq \mathbb{H}^\times$ is compact. $\rightsquigarrow$ no geometry at ∞

$SL_2(\mathbb{R}) \subseteq GL_2(\mathbb{R})$ $\rightsquigarrow$ same hy. $\rightsquigarrow$ will ~~take~~ quotient by non-cong. subgp $\to$ Shimura curve.

... how about level str.? $\underset{\text{only}}{\underline{\Gamma_0(N)}}$ or $\Gamma_1(N)$, ...

$R \subseteq B$, $\mathbb{Z}$-order of B.

R is <u>an Eichler order of B of level N^+</u>

: $\underset{\parallel}{R_\ell}$ $R \otimes \mathbb{Z}_\ell \simeq M_2(\mathbb{Z}_\ell)$ $\quad \forall \ell \nmid N^+ N^-$.

: $(R \otimes \mathbb{Z}_\ell)^\times \simeq \left\{ \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \bmod \ell^n \right\}$ if $\ell^n \| N^+$. $\longleftrightarrow$ $\Gamma_0(\ell^n)$-level str.

$\to R \leftrightarrow \Gamma_0(N)$-level str.

$R \otimes \mathbb{Z}_\ell \subseteq B_\ell$ unique $\mathbb{Z}_\ell$-order of B_ℓ if $\ell \| N^-$.

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(left) ideal class sets. $I \subset R$. $I\alpha = J$ $I \sim J$. .

$$B^\times \backslash \hat{B}^\times / \hat{R}^\times$$

Prop. indef. $\Rightarrow$ $\# = 1$. $\leftarrow$ but $\exists$ ~~geometry~~ at ∞.

def $\Rightarrow$ $\# < \infty$.

$\uparrow$ finite graph.

(pf) strong approx. $\square$

Orientation of Eichler order: omitted.

Gross curves and Shimura curves.

① indef. $\rightsquigarrow$ Shimura curves

$$\mathbb{P} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \coprod \mathfrak{h}^{\pm}. \circlearrowleft B_\infty^\times \leftarrow B^\times.$$

: homogeneous space.

$$\mathbb{P} \overset{!}{=} \mathrm{Hom}(\mathbb{C}, B_\infty). \qquad p^+ \longleftrightarrow f.$$

How? $f \in \mathrm{Hom}(\mathbb{C}, B_\infty)$

$B^\times \circlearrowright \mathbb{P}$ conjugation.

$\rightarrow$ $\mathbb{C}^\times \rightarrow B_\infty^\times \circlearrowright \mathbb{P}$.

$\rightsquigarrow$ $\exists$ 2 fixed pts p^+, p^- in $\mathbb{P}$.

ordering: $\mathbb{C}^\times \circlearrowright \mathrm{Tot}_{p^+}(\mathbb{P})$

via $z \mapsto \bar{z}/z$.

$\mathbb{C}^\times \circlearrowright \cot_{p^-}(\mathbb{P})$

via $z \mapsto z/\bar{z}$.

$\underline{N^- > 1}$

$$\Rightarrow \quad \underline{X_0^{N^-}(N^+)} = {}_{B^\times}\backslash\left(\hat{B}^\times \times \overset{\substack{\mathrm{Hom}(\mathbb{C}, B_\infty) \\ \shortparallel}}{\mathbb{P}}\right)/\hat{R}^\times . \simeq \Gamma\backslash\mathbb{P}$$

$\underline{\text{cpt}}$. $\quad N^-$ even.

$N^- = 1 \rightsquigarrow$ get modular curve. $Y_0(N^+)$

$$R^\otimes \simeq {}_{\Gamma \cap SL_2(\mathbb{Z})}\backslash\mathbb{P}.$$

$$\Gamma = \left(\operatorname{Im} \underline{R} \to R_\infty \to M_2(\mathbb{Z})\right)^\times .$$

② def. ⇝ Gross curves.

$\mathbb{P}$ = the conic over $\mathbb{Q}$.

$\mathbb{P}(F) = \{ x \in B \otimes F : \mathrm{nrd}(x) = \cancel{\mathrm{tr}}\ \mathrm{trd}(x) = 0 \}$

for $\mathbb{Q}$-alg. F.

- $B^\times \circlearrowright \mathbb{P}$ by conjugation.

  and $\mathrm{Aut}(\mathbb{P}) = B^\times$.

- ~~For~~ K img. quad. field.

  $\Rightarrow \mathrm{Hom}(K, B) = \mathbb{P}(K)$

$$B^\times \backslash \hat{B} \times \mathbb{P} / \hat{R}^\times \simeq \coprod_j \Gamma_i \backslash \mathbb{P}$$

$$\Gamma_i = (g_i \hat{R}^\times g_i^{-1} \cap B^\times) / \{\pm 1\}$$

$\{g_i\}$ representatives of

$$B^\times \backslash \hat{B}^\times / \hat{R}^\times.$$

Hecke action? omitted.

2. Galois theory.

$$\begin{array}{c} K \\ | \quad \text{img. quad.} \\ \mathbb{Q} \end{array}$$

$$K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_K^{(p),\times} \cdot \hat{\mathbb{Q}}^\times .$$ ← our interest.

$$\begin{array}{c} \wr| \\ \underbrace{\mathrm{Gal}(K_\infty/K)}_{\simeq \mathbb{Z}_p} \times \underbrace{\Delta}_{\text{torsion.}} \end{array}$$

ring class ext'n

$$\begin{array}{c} \tilde{K}_\infty \\ | \\ K_\infty \\ | \\ K \\ | \\ \mathbb{Q} \end{array} \quad \begin{array}{l} \tilde{G}_\infty \\ \circlearrowleft \text{ conjugate action} \\ \quad\quad \downarrow \\ c \quad\quad \text{involution} \end{array}$$

How $\tilde{G}_\infty$ acts on Gross and Shimura curves?

Shimura curves : CM theory
(Shimura's reciprocity law)

Gross curves : purely combinatorial.

$\wr$

$\exists\, K \overset{\Psi}{\hookrightarrow} B$ if $\begin{cases} N^- & \text{inert} \\ N^+ & \text{split} \end{cases}$

with

$\mathcal{O}_K \hookrightarrow R.$ (optimal embedding)

$\Psi(\mathcal{O}_K)$
$= \Psi(K) \cap R$

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$$\mathcal{O}_K[\tfrac{1}{p}]^\times$$

$$K_p^\times / \mathcal{O}_p^\times \cdot \boxed{}$$

$$1 \to (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \mathcal{O}_p^\times \to K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_{K[\frac{1}{p}]}^{(p),\times} \hat{\mathcal{O}}^\times \to K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_{K[\frac{1}{p}]}^\times \to 1$$

$\circlearrowright$ "Bruhat–Tits tree"

$\circlearrowright$ $\mathrm{Hom}(K, B)$ simply and transitively.

$$B^\times \backslash \hat{B}^\times / \hat{R}^\times \underset{\text{strong approx.}}{\simeq} R[\tfrac{1}{p}]^\times \backslash B_p^\times / R_p^\times \cdot (\mathcal{O}_p^\times)$$

$$R[\tfrac{1}{p}]^\times \backslash GL_2(\mathcal{O}_p) / GL_2(\mathbb{Z}_p) \cdot \mathcal{O}_p^\times.$$

finite graph.

<u>adelically</u> $K \hookrightarrow B$

$$\rightsquigarrow \quad K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_K^{(p),\times} \cdot \hat{\mathcal{O}}^\times \longrightarrow B^\times \backslash \hat{B}^\times / \hat{R}^\times$$

$$\sigma \mapsto \Phi(\sigma).$$

$$\vdots \quad b$$

$$\Phi(\sigma) \cdot b \quad \text{left translation.}$$

Combinatorially

~~$\Psi \rightsquigarrow$~~

$K \xrightarrow{\Psi} B$, $B_p^\times / R_p^\times \cdot \mathcal{O}_p^\times$.

Ψ $\quad \ni v_0$

S $\quad \circlearrowleft$

$\sigma \quad \rightsquigarrow$ "$\Psi^{\sigma}(\sigma) \cdot v_0$"

$\uparrow$

Indeed, we will ~~do~~ "average".

... fixed part in the tree?

① p splits in K $p = \mathfrak{p} \cdot \bar{\mathfrak{p}}$

$| \, |_{\mathfrak{p}} : K_p^\times / \mathcal{O}_p^\times \to \mathbb{Z}$ $\qquad U_K[\frac{1}{p}]$

$\alpha \mapsto \mathrm{ord}_{\mathfrak{p}}\left(\frac{\alpha}{\bar{\alpha}}\right)$ $\qquad$ ~~index~~ $\tilde{\mathbb{Z}}$ ~ gen by fundamental p-units

$U_0 := \mathrm{Ker} \, | \, |_{\mathfrak{p}} = \mathbb{Z}_p^\times \subset K_p^\times / \mathcal{O}_p^\times$ max'l cpt.

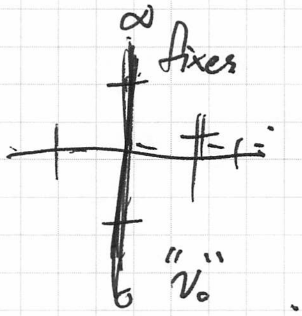
$\cup$ $\qquad \hookrightarrow U_0$ fixes a sequence

U_1

$\cup$

U_2 natural filtration.

$\vdots$



② p inert in K.

$K_p^\times / \mathcal{O}_p^\times$ already cpt $\rightsquigarrow$ fixes "v_0".

3. modular forms.

Gross ~~pt.~~ curves $\quad B^\times \backslash \hat{B}^\times \times \mathbb{P} / \hat{R}^\times$.

$\downarrow$

Jacobian of Gross curves

$$J = \mathbb{Z}[B^\times \backslash \hat{B}^\times / \hat{R}^\times].$$

modular forms = ~~fns~~ on J.

$$f : B^\times \backslash \hat{B}^\times / \hat{R}^\times \longrightarrow \mathbb{C} \rightsquigarrow S_2^{N^-}(N^+, \mathbb{C})$$

$\mathbb{C}$ $\int$ replace. $\quad$ nonconst.

$\mathcal{O}$, fin. ext'n of $\mathbb{Z}_p$.

... why?

<u>J-L</u> $\quad S_2^{N^-}(N^+, \mathbb{C}) \overset{\text{noncanonical}}{\simeq} S_2(\Gamma_0(N))^{N^-\text{-new}}$.

"same Hecke eigensystems in different Hecke modules"

4. Gross pts. definite analogue of Heegner pts

① ~~Gross curves.~~ CM pts in general.

B a quat. alg ($M_2(\mathbb{Q})$, indef, def.

$\rightsquigarrow$ $G(A) = (B \otimes A)^\times$ $\quad A = \mathbb{Q}$-alg

alg. gp.

$Z \subset G$ center.

$\Psi(K^\times) =: T$, max'l torus of $G/\mathbb{Q}$.

$\hat{R}^\times = H \subset G(\mathbb{A}_f)$ level str. $(\approx \hat{R}^\times)$ . . .

$$\rightsquigarrow CM_R = T(\mathbb{Q}) \backslash G(\mathbb{A}_f) / \hat{R}^\times \twoheadrightarrow B^\times \backslash \hat{B}^\times / \hat{R}^\times.$$

$$T(\mathbb{Q}) = K^\times \sim T(\mathbb{A}_f) = \mathbb{A}_f^\times \xrightarrow{\text{rec}} \mathrm{Gal}(K^{ab}/K)$$

<u>indef case</u>

$$CM_R \overset{1-1}{\longleftrightarrow} CM_R \times \mathfrak{h}$$

$$g \qquad\qquad (g, x)$$

where $x \in \mathfrak{h}$ fixed by $T(\mathbb{R}) \subset G(\mathbb{R}) \simeq GL_2(\mathbb{R})$

$\rightsquigarrow$ "classical" CM pts.

$/K^{ab}$ by Shimura's reciprocity law.

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On Gross curves.

$$B^\times \backslash \hat{B}^\times \times \mathrm{Hom}(K, B) / \hat{R}^\times = X^{N^-}(N^+)(K)$$

$$\ni (g, \Psi)$$

action: $b \cdot (g, \Psi) \cdot x = (bgx, b\Psi b^{-1})$

(g, Ψ) is a CM pt. of cond c

if $\Psi : \mathcal{O}_c \hookrightarrow gRg^{-1} \cap B$ optimal.

$$\begin{array}{ccc} \cap & & \cap \\ \hat{K} & \hookrightarrow & \hat{B}. \end{array}$$

1-1 correspondence.

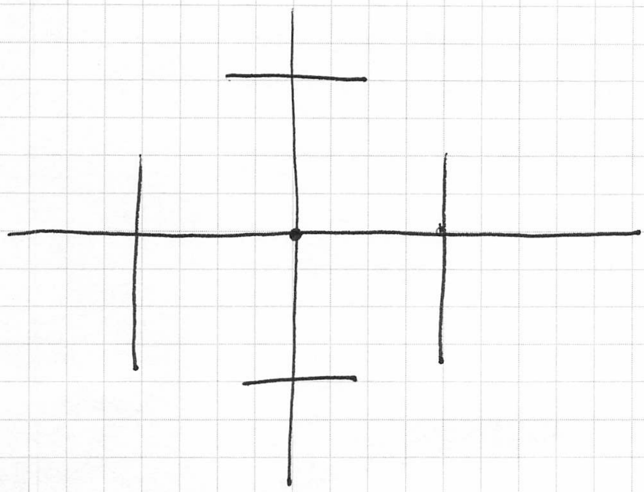
Fix Ψ_0.

$\Psi_0(K) \backslash \hat{B}^\times / \hat{R}^\times \ni g$ is of cond c

if $\Psi_0(K) \cap gRg^{-1} = \mathcal{O}_c$.

$g \longmapsto (g, \Psi_0)$ is 1-1 correspondence.

On the tree.



Each vertex = homothety class of $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$.

Fix "v_0" $\simeq \mathbb{Z}_p^2$. (or line)

↑

$PGL_2(\mathbb{Z}_p)$ fixes this.

$$U_0 \supset U_1 \supset \cdots \supset U_n \supset \cdots \supset \cdots$$

$$U_1 = 1+p\mathcal{O}_K / 1+p\mathbb{Z}_p, \qquad U_n = 1+p^n\mathcal{O}_K / 1+p^n\mathbb{Z}_p .$$

choose a seq. of vertices v_n

~~$v_n = SL$~~

$\mathrm{stab}(v_n) = U_n$.

↑

Gross pt of cond. p^n?