MODULI SPACES OF MCKAY QUIVER REPRESENTATIONS: G-IRAFFES

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Abstract. This article introduces a (generalized) G-graph which is a generalized version of Nakamura G-graphs in [18]. As Nakamura G-graphs are associated with torus invariant G-clusters, our G-graphs are associated with torus invariant G-constellations. If a G-graph Γ satisfies a certain condition, then we call the G-graph a *G-iraffe*. For each G-iraffe Γ, we define a toric affine open set $U(\Gamma)$ and a family over the open set $U(\Gamma)$. Using G-iraffes, we describe local charts of the birational component Y_θ.

Introduction

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. A G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a *G-constellation* if its global sections $\mathrm{H}^0(\mathcal{F})$ are isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module. In particular, the structure sheaf of a G-invariant subscheme $Z \subset \mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$ as a G-module, which is called a *G-cluster*, is a G-constellation. It is known that G-clusters are θ-stable G-constellations for a particular choice of GIT stability parameter θ [10].

For a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$, Ito and Nakamura [11] introduced G-Hilb $\mathbb{C}^2$ which is the fine moduli space parametrising G-clusters and proved that G-Hilb $\mathbb{C}^2$ is the minimal resolution of $\mathbb{C}^2/G$. Nakamura showed that for a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$, G-Hilb $\mathbb{C}^3$ is a crepant resolution of the quotient variety $\mathbb{C}^3/G$. In his paper, he introduced (Nakamura) G-graphs to describe a local chart of G-Hilb for an abelian group G. He also claimed that every G-cluster is over the birational component, which is turned out to be false.

On the other hand, for a finite abelian group $G \subset \mathrm{GL}_n(\mathbb{C})$ and a generic GIT parameter $\theta \in \Theta$, Craw, Maclagan and Thomas [4] showed that the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. So the irreducible component is birational to the quotient variety $\mathbb{C}^n/G$. The component Y_θ is called the *birational component*[1] of $\mathcal{M}_\theta$. In their consecutive paper [5], they introduced a new technique to describe a local chart of the birational component of G-Hilb using Gröbner basis.

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[1]This component is also called the coherent component.

Moreover, they presented a counterexample of Nakamura's claim: there exists a G-cluster which does not lie over the birational component.

The motivation of this paper is from the question on why Nakamura's claim is wrong in general. Nakamura [18] defined an open set $U(\Gamma)$ associated to each Nakamura G-graph Γ. He assumed that $U(\Gamma)$ has a torus fixed point. We find out that if $U(\Gamma)$ has a torus fixed point, then $U(\Gamma)$ is an open set in the birational component of G-Hilb. In other words, there should be G-graphs such that $U(\Gamma)$ has no torus fixed point by the existence of G-clusters outside the birational component [5].

Main results. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3})$ where ϵ is a primitive rth root of unity. The group G acts naturally on $S := \mathbb{C}[x, y, z]$.

Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$$

which is an overlattice of $\overline{L} = \mathbb{Z}^3$ of finite index. Set $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The embedding of G into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt} \colon \overline{M} \longrightarrow G^\vee$$

where $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ is the character group of G.

We define a (generalized) G-graph Γ and an affine toric variety $U(\Gamma)$:

Definition 0.1. A *(generalized) G-graph* Γ is a subset of Laurent monomials in $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ satisfying:

(i) $\mathbf{1} \in \Gamma$.
(ii) $\mathrm{wt} \colon \Gamma \to G^\vee$ is bijective, i.e. for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ whose weight is ρ.
(iii) if $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.
(iv) Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x, y, z in Γ.

As is defined in [18], for a G-graph $\Gamma = \{\mathbf{m}_\rho\}$, define $S(\Gamma)$ to be the subsemigroup of M generated by $\dfrac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for all $\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. We prove the semigroup $S(\Gamma)$ is finitely generated. We define

$$U(\Gamma) = \mathrm{Spec}\, \mathbb{C}[S(\Gamma)],$$

which is an affine toric variety whose torus is $\mathrm{Spec}\, \mathbb{C}[M]$ and define a G-constellation $C(\Gamma)$ associated with Γ.

Definition 0.2. A generalized G-graph Γ is called a *G-iraffe* if the open set $U(\Gamma)$ has a torus fixed point.

We prove that for a finite group $G \subset \mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$ and a generic GIT parameter θ, there is a 1-to-1 correspondence between the set of torus fixed points in the birational component Y_θ and the set of θ-stable G-iraffes (see Proposition 6.7). Furthermore, we have the following theorem.

Theorem 0.3. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G-constellations. Assume that $\mathfrak{G}$ is the set of all θ-stable G-iraffes.*

(i) *The birational component Y_θ of $\mathcal{M}_\theta$ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$.*
(ii) *The normalization of Y_θ is isomorphic to the normal toric variety whose toric fan consists of the full dimensional cones $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{G}$ and their faces.*

In general, finding all θ-stable G-iraffes is a very difficult job. Nakamura [18] introduces G-igsaw transforms which finds all Nakamura G-graphs lying over the birational components. We expect that there is a method to find all θ-stable G-iraffes which is analogous to G-igsaw transforms in [18].

Remark 0.4 (Link to [4])**.** Craw, Maclagan, and Thomas [4] described Y_θ using a certain polyhedron P_θ. The vertices $\mathbf{v}_\alpha$ of the polyhedron P_θ correspond to fixed points p_α of the torus action. For each vertex $\mathbf{v}_\alpha$, they define a semigroup A_α such that $\operatorname{Spec} \mathbb{C}[A_\alpha]$ gives an affine open set through p_α.

In our description, since each torus fixed point p_α represents the isomorphism class of a θ-stable torus invariant G-constellation lying over Y_θ, we have a unique G-iraffe Γ_α and the semigroup $S(\Gamma_\alpha)$. We expect that our semigroup $S(\Gamma_\alpha)$ is equal to the semigroup A_α. ♦

Warning 0.5. In this paper, we restrict ourselves to the case where a group G is a finite cyclic group in $\mathrm{GL}_3(\mathbb{C})$. It is possible to generalize part of the argument to include general small abelian groups in $\mathrm{GL}_n(\mathbb{C})$ for any dimension n. However, we prefer to focus on this case where we can avoid the difficulty of notation. ♦

Layout of this paper.

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1. Moduli of quiver representations

In this section, we briefly review the construction of moduli spaces of quiver representations introduced in [12].

1.1. **Quivers and their representations.**

A *quiver* Q is a directed graph with a set of vertices $I = Q_0$ and a set of arrows Q_1. For an arrow $a \in Q_1$, let $\mathrm{h}(a)$ (resp. $\mathrm{t}(a)$) denote the head (resp. tail) of the arrow a:

$$\mathrm{t}(a) \xrightarrow{a} \mathrm{h}(a).$$

One can define the *path algebra* of a quiver Q to be the $\mathbb{C}$-algebra whose basis is nontrivial paths in Q and trivial paths corresponding to the vertices of Q and whose multiplication is given by the concatenation of two paths.

A *representation* of a quiver Q is a collection of $\mathbb{C}$-vector spaces V_i for each vertex $i \in I$ and linear maps $V_i \to V_j$ for each arrow from i to j. For a representation V, the I-tuple $(\dim_{\mathbb{C}} V_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$ is called the *dimension vector* of V denoted by $\underline{\dim}\,(V)$. A representation (U, ξ') of a quiver Q is called a *subrepresentation* of a representation (V, ξ) if U is an I-graded subspace of V such that $\xi_a(U_{\mathrm{t}(a)}) \subset U_{\mathrm{h}(a)}$ for all $a \in Q_1$ and ξ' is the restriction of ξ to U.

It is well known that the abelian category of representations of a quiver Q is equivalent to the category of finitely generated left modules of the path algebra of Q.

Let us fix a dimension vector $\mathbf{v} = (v_i)_{i \in I}$. Let $\mathrm{Rep}(Q, \mathbf{v})$ denote the representation space of Q with dimension vector $\mathbf{v}$:

$$\mathrm{Rep}(Q, \mathbf{v}) = \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{\mathrm{t}(a)}, V_{\mathrm{h}(a)}) = \bigoplus_{a: i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}),$$

which is an affine space. Note that the reductive group $\mathrm{GL}(\mathbf{v}) := \prod_{i \in I} \mathrm{GL}_{v_i}$ acts on $\mathrm{Rep}(Q, \mathbf{v})$ as basis change.

One can see that

$$\mathrm{Rep}(Q, \mathbf{v}) \longrightarrow \mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v}) := \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{\mathrm{GL}(\mathbf{v})}$$

is a categorical quotient and that $\mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v})$ is an affine variety.

Remark 1.1. Geometric points of $\mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v})$ correspond to $\mathrm{GL}(\mathbf{v})$-orbits of semisimple representations of Q whose dimension is $\mathbf{v}$ ◆

1.2. **Background: Geometric Invariant Theory.**

In this section, we present results from standard Geometric Invariant Theory (GIT), cf. [16].

Definition 1.2. Let G be a reductive group acting on an affine variety X. A surjective morphism $\psi\colon X \to Y$ is a *good quotient* if:

(i) ψ is constant on G-orbits.

(ii) for any open set $U \subset Y$, the natural map $\mathcal{O}_Y(U) \to \psi_*\mathcal{O}_X(U)$ induces $\mathcal{O}_Y(U) = (\psi_*\mathcal{O}_X)^G(U)$.
(iii) $\psi(W)$ is closed in Y for any G-invariant closed set $W \subset X$.
(iv) $\psi(W_1) \cap \psi(W_2) = \emptyset$ for two disjoint G-invariant closed sets W_1, W_2 of X.

Moreover, if Y is an orbit space, then $\psi\colon X \to Y$ is called a *geometric quotient*.

Consider an affine algebraic variety X with a reductive group G acting on it. Given a character $\chi\colon G \to \mathbb{C}^\times$, $f \in \mathbb{C}[X]$ is called a χ semi-invariant function if

$$f(g \cdot x) = \chi(g)f(x) \quad x \in X,\ \forall g \in G.$$

Let $\mathbb{C}[X]_{\chi^n}$ denote the $\mathbb{C}$-vector space of all χ^n semi-invariant functions. One defines the *semistable locus* as

$$X^{ss}(\chi) := \big\{x \in X \,\big|\, \exists n \geq 1, f \in \mathbb{C}[X]_{\chi^n} \text{ such that } f(x) \neq 0\big\}$$

and the *stable locus* as

$$X^{s}(\chi) := \left\{x \in X^{ss}(\chi) \,\Big|\, \begin{array}{c} G \cdot x \text{ is closed in } X^{ss}(\chi), \\ \text{the stabiliser } G_x \text{ is finite} \end{array}\right\}.$$

The quasiprojective variety

$$X /\!\!/_{\chi}\, G := \operatorname{Proj}\Big(\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^n}\Big)$$

is called a *GIT quotient* corresponding to χ. In particular, if the character $\chi = 0$, i.e. θ is trivial, then $\mathbb{C}[X]_{\chi^n} = \mathbb{C}[X]^G$ for all n so we have

$$X /\!\!/_0\, G = \operatorname{Spec} \mathbb{C}[X]^G$$

which is an affine variety. Thus we have a canonical projective morphism

$$X /\!\!/_{\chi}\, G \to \operatorname{Spec} \mathbb{C}[X]^G.$$

Remark 1.3. Let G be a reductive group acting on an affine variety X. Fix a character χ of G. For each positive integer d, define the dth *Veronese subalgebra* of $\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^n}$ to be

$$\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^{dn}}.$$

One can show that the inclusion of the subalgebra induces an isomorphism of algebraic varieties

$$X /\!\!/_{\chi}\, G \xrightarrow{\sim} X /\!\!/_{\chi^d}\, G.$$

Thus any positive multiple of a character χ gives the same GIT quotient as χ. ◆

As is well known by GIT [16], the quasiprojective variety $X /\!\!/_{\chi}\, G$ is a categorical quotient $X^{ss}(\chi)$ by G.

Theorem 1.4 (Geometric Invariant Theory [16]). *Let G be a reductive group acting on an affine variety X and χ a character of G. Then:*

(i) $\pi\colon X^{ss}(\chi) \to X /\!\!/_\chi G$ *is a good quotient of* $X^{ss}(\chi)$ *by* G.

(ii) *there exists an open subset* Y *of* $X /\!\!/_\chi G$ *such that* Y *is a geometric quotient of* $X^s(\chi)$ *by* G*, i.e. an orbit space.*

(iii) *the GIT quotient* $X /\!\!/_\chi G$ *is projective over the affine variety* $\operatorname{Spec} \mathbb{C}[X]^G$.

Remark 1.5. Let $\pi\colon X /\!\!/_\chi G \to X^s(\chi)/G$ be the GIT quotient with $X^s(\chi) = X^{ss}(\chi)$. Then π is a geometric quotient. Let U be a G-invariant affine open set in $X^{ss}(\chi)$. Then

$$\pi|_U\colon U \to \pi(U)$$

is a good quotient and $\pi(U) = \operatorname{Spec} \mathbb{C}[U]^G$ is an open set of $X^s(\chi)/G$. ♦

The following theorem is helpful to understand the local behaviour of the GIT quotients.

Theorem 1.6 (Luna's Étale Slice Theorem [9,15]). *Let G be a reductive group acting on an affine variety X. Assume that $\pi\colon X \to X /\!\!/ G$ is a good quotient. Let $x \in X$ be a point with closed G-orbit $G{\cdot}x$. Then there exists a G_x-invariant locally closed affine subset V of X containing x such that the G-action on X induces an étale G-equivariant morphism $\psi\colon G \times_{G_x} V \to X$. Moreover, ψ induces an étale morphism $V /\!\!/ G_x \to X /\!\!/ G$, and the following diagram*

$$\begin{array}{ccc} G \times_{G_x} V & \to & X \\ \downarrow & & \downarrow \\ V /\!\!/ G_x & \to & X /\!\!/ G \end{array}$$

is Cartesian.

1.3. **Moduli spaces of quiver representations.** This section explains a notion of stability on quiver representations introduced by King [12]. His main result is that the notion of stability on quiver representations and the notion of GIT stability are equivalent and that we can construct a fine moduli space of quiver representations in a certain case.

An element $\theta \in \mathbb{Q}^I$ can be thought as a group homomorphism from the Grothendieck group of representations of Q to $\mathbb{Q}$ defined by

$$\theta(V) := \sum_{i\in I} \theta_i \dim_C V_i = \theta \cdot \mathbf{v}$$

where V is a representation of Q with dimension vector $\mathbf{v}$.

Definition 1.7. Let V be a $\mathbf{v}$-dimensional representation of a quiver Q. For a parameter $\theta \in \mathbb{Q}^I$ satisfying $\theta \cdot \mathbf{v} = 0$, we say that:

(i) V is *θ-semistable* if $\theta(W) \geq 0$ for any subrepresentation W of V.
(ii) V is *θ-stable* if $\theta(W) > 0$ for any nonzero proper subrepresentation W of V.
(iii) θ is *generic* if every θ-semistable representation is θ-stable.

The parameter $\theta \in \mathbb{Q}^I$ plays the same role as χ does in Section 1.2. The character χ_θ defined by

$$\chi_\theta(g) := \prod_{i \in I} \det(g_i)^{\theta_i}$$

for $g = (g_i) \in \mathrm{GL}(\mathbf{v})$ vanishes on the diagonal matrices $\mathbb{C}^\times \in \mathrm{GL}(\mathbf{v})$ if and only if $\theta \cdot \mathbf{v} = 0$.

King [12] shows that a representation $V \in \mathrm{Rep}(Q, \mathbf{v})$ is θ-semistable (resp. θ-stable) if and only if the corresponding point $V \in \mathrm{Rep}(Q, \mathbf{v})$ is χ_θ-semistable (resp. χ_θ-stable). Moreover:

Theorem 1.8 (King [12])**.** *Let $\mathbf{v}$ be a dimension vector. Assume a parameter $\theta \in \mathbb{Q}^I$ satisfies $\theta \cdot \mathbf{v} = 0$.*

(i) *The quasiprojective variety*

$$\mathcal{M}_\theta(Q, \mathbf{v}) := \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]_{\chi_\theta^n}\right)$$

is a coarse moduli space of θ-semistable $\mathbf{v}$-dimensional representations of Q up to S-equivalence.
(ii) *If θ is generic, $\mathcal{M}_\theta(Q, \mathbf{v})$ is a fine moduli space of θ-stable $\mathbf{v}$-dimensional representations of Q.*
(iii) *The variety $\mathcal{M}_\theta(Q, \mathbf{v})$ is projective over* $\mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{\mathrm{GL}(\mathbf{v})}$.

Remark 1.9. By Luna's Étale Slice Theorem, if θ is generic, then the quotient map

$$\pi\colon \mathrm{Rep}^s(Q, \mathbf{v}) \to \mathcal{M}_\theta(Q, \mathbf{v})$$

is a principal $\mathrm{GL}(\mathbf{v})/\mathbb{C}^\times$-bundle. ♦

2. McKay quiver and G-constellations

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. Let ρ_i be the irreducible representation of G whose weight is i. Since G is abelian, every irreducible representation is one-dimensional and the number of irreducible representation is equal to the order of G. We can identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$. Note that the inclusion $G \subset \mathrm{GL}_3(\mathbb{C})$ induces a natural representation of G on $\mathbb{C}^3$, which can be decomposed as

$$\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}.$$

2.1. **McKay quiver representations.**

Definition 2.1. (McKay quiver) The *McKay quiver* of G is the quiver whose vertex set is the set I of irreducible representations of G and the number of arrows from ρ_i to ρ_j is the dimension of $\mathrm{Hom}_G(\rho_j, (\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}) \otimes \rho_i)$.

Since G has r irreducible representations, the McKay quiver of G has r vertices $\rho_0, \dots, \rho_{r-1}$. For two irreducible G-representations ρ_i and ρ_j,

$$\begin{aligned}\mathrm{Hom}_G\left(\rho_j, (\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}) \otimes \rho_i)\right) &= \mathrm{Hom}_G(\rho_j, \bigoplus\nolimits_{k=1}^{3} \rho_{\alpha_k} \otimes \rho_i) \\ &= \bigoplus_{k=1}^{3} \mathrm{Hom}_G(\rho_j, \rho_{i+\alpha_k}),\end{aligned}$$

and by Schur's lemma

$$\dim \mathrm{Hom}_G(\rho_j, \rho_{i+\alpha_k}) = \begin{cases} 1 & \text{if } j = i + \alpha_k \mod r, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the McKay quiver has $3r$ arrows. Let x_i, y_i, z_i denote the arrow from ρ_i to $\rho_{i+\alpha_1}$, $\rho_{i+\alpha_2}$, $\rho_{i+\alpha_3}$, respectively. We are interested in the McKay quiver with the following commutation relations:

$$\begin{cases} x_i y_{i+\alpha_1} - y_i x_{i+\alpha_2}, \\ x_i z_{i+\alpha_1} - z_i x_{i+\alpha_3}, \\ y_i z_{i+\alpha_2} - z_i y_{i+\alpha_3}. \end{cases} \tag{2.2}$$

Definition 2.3. A *McKay quiver representation* is a representation of the McKay quiver of dimension $(1, \dots, 1)$ with the relations (2.2), i.e. it is a collection of one-dimensional $\mathbb{C}$-vector spaces V_i for each $\rho_i \in G^\vee$, and a collection of linear maps from V_i to V_j assigned to each arrow from ρ_i to ρ_j which satisfy the commutation relations (2.2).

Example 2.4. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{12}(1, 5, 7)$, i.e. $r = 12$ and $a = 5$. The set of irreducible representations of G is $\{\rho_i \,|\, 0 \leq i \leq 11\}$ and the induced representation is isomorphic to $\rho_1 \oplus \rho_5 \oplus \rho_7$. The McKay quiver of G has 12 vertices and 36 arrows.

After fixing basis on vector spaces attached to vertices, the McKay quiver representations are in 1-to-1 correspondence with points of the closed subscheme of the affine space

$$\mathbb{C}^{3r} = \mathrm{Spec}\, \mathbb{C}[x_0, \dots, x_{r-1}, y_0, \dots, y_{r-1}, z_0, \dots, z_{r-1}]$$

defined by the commutation relations (2.2). ♦

Let $\mathrm{Rep}\, G$ denote the McKay quiver representation space of G. Note that its coordinate ring is

$$\mathbb{C}[\mathrm{Rep}\, G] = \mathbb{C}\left[x_i, y_i, z_i \,\middle|\, i \in I\right] / I_G$$

where I_G is the ideal generated by the quadrics in (2.2).

Let $\delta = (1, \dots, 1) \in \mathbb{Z}^I_{\geq 0}$. The reductive group $\mathrm{GL}(\delta) := \prod_{i\in I} \mathbb{C}^\times = (\mathbb{C}^\times)^r$ acts on $\operatorname{Rep} G$ by basis change. Note that $\mathrm{GL}(\delta)$-orbits are in 1-to-1 correspondence with isomorphism classes of the McKay quiver representations.

Consider the algebraic torus $\mathbf{T} = (\mathbb{C}^\times)^3$ acting on $\operatorname{Rep} G$ by

$$(t_1, t_2, t_3) \cdot (x_i, y_i, z_i) = (t_1x_i, t_2y_i, t_3z_i).$$

One can see that $\mathbf{T}$-action commutes with $\mathrm{GL}(\delta)$-action. This action naturally comes from the notion of *G-constellations*, which are a certain kind of coherent sheaves on $\mathbb{C}^3$ (see Remark 2.15).

We define the GIT parameter space Θ to be

$$\Theta := \left\{\theta \in \mathbb{Q}^I \,\middle|\, \theta \cdot \delta = 0\right\}.$$

By Theorem 1.8, we know that:

(i) the quasiprojective scheme

$$\mathcal{M}_\theta := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\operatorname{Rep} G]_{\chi_\theta^n}\right)$$

is a coarse moduli space of θ-semistable McKay quiver representations up to S-equivalence.

(ii) if θ is generic, $\mathcal{M}_\theta$ is a fine moduli space of θ-stable McKay quiver representations of Q.

(iii) $\mathcal{M}_\theta$ is projective over $\operatorname{Spec} \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$.

Remark 2.5. The affine scheme $\operatorname{Spec} \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ contains the quotient variety $\mathbb{C}^3/G$ as a closed subvariety. ♦

2.2. G-constellations.

Definition 2.6. A *G-constellation* on $\mathbb{C}^3$ is a G-equivariant $\mathbb{C}[x, y, z]$-module $\mathcal{F}$ on $\mathbb{C}^3$, which is isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module.

Remark 2.7. Any G-constellation $\mathcal{F}$ is isomorphic to $\bigoplus_i \mathbb{C}\rho_i$ as a vector space. ♦

The representation ring $R(G)$ of G is $\bigoplus_{\rho\in G^\vee} \mathbb{Z}\rho$. Define the GIT stability parameter space

$$\begin{aligned}\Theta &= \left\{\theta \in \operatorname{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right) = 0\right\}\\ &= \left\{\theta = (\theta^i) \in \mathbb{Q}^r \,\middle|\, \Sigma_{i\in I}\theta^i = 0\right\}.\end{aligned}$$

Definition 2.8. For a stability parameter $\theta \in \Theta$, we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \geq 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.

(ii) a G-constellation $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.

(iii) θ is *generic* if every θ-semistable object is θ-stable.

Remark 2.9. It is known that the language of G-constellations is the same as the language of the McKay quiver representations. Thus we can construct the moduli spaces of G-constellations by Geometric Invariant Theory as in Section 1. ♦

Let $\mathcal{M}_\theta$ denote the moduli space of θ-stable G-constellations. Ito and Nakajima [10] showed that G-Hilb $\mathbb{C}^3$ is isomorphic to $\mathcal{M}_\theta$ if θ is in the following set:

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for nontrivial } \rho \neq \rho_0\right\}. \tag{2.10}$$

Let Z be a G-orbit in the algebraic torus $\mathbf{T} := (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$. Then $\mathrm{H}^0(\mathcal{O}_Z)$ is isomorphic to $\mathbb{C}[G]$, thus it is a G-constellation. Moreover, since Z is a free G-orbit, $\mathcal{O}_Z$ has no nonzero proper submodules. Hence it follows that $\mathcal{O}_Z$ is θ-stable for any parameter θ. Thus for any parameter θ, there exists a natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$.

Remark 2.11. The existence of the natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$ can be proved by Luna's Étale Slice Theorem as is standard in the theory of moduli spaces of sheaves (e.g. see [9]). ♦

Lemma 2.12. *Let Z be a free G-orbit in $\mathbb{C}^3$. Then $\mathcal{O}_Z$ is a G-constellation supported on the free G-orbit Z. Conversely, if a G-constellation $\mathcal{F}$ is supported on a free G-orbit $Z \subset \mathbb{C}^3$, then $\mathcal{F}$ is isomorphic to $\mathcal{O}_Z$ as a G-constellation.*

Proof. For the first statement, one can refer to [17].

To prove the second statement, let $\mathcal{F}$ be a G-constellation whose support is a free G-orbit Z.

Then $\mathcal{F}$ has no nonzero proper submodules. Indeed, for a nonzero submodule $\mathcal{G}$ of $\mathcal{F}$, the support of $\mathcal{G}$ is a G-invariant nonempty subset of the free G-orbit Z. As Z is a free G-orbit, the support of $\mathcal{G}$ is Z. Since $\mathcal{F}_x$ is 1-dimensional for any $x \in Z$, it follows that $\mathcal{G}_x = \mathcal{F}_x$ and hence $\mathcal{G} = \mathcal{F}$.

Consider $\psi\colon \mathbb{C}[x, y, z] \to \mathcal{F}$ defined by $f \mapsto f * e_0$ where e_0 is a basis of $\mathbb{C}\rho_0$. As $\mathcal{F}$ has no nonzero proper submodules, ψ is surjective. Since the support of $\mathcal{F}$ is Z, it follows that I_Z is in the kernel of ψ. Thus we have

$$\mathcal{O}_Z = \mathbb{C}[x, y, z]/I_Z \geq \mathbb{C}[x, y, z]/\ker(\psi) \cong \mathcal{F}.$$

From the fact that both $\mathcal{O}_Z$ and $\mathcal{F}$ are G-constellations, it follows that $\mathcal{O}_Z \cong \mathcal{F}$ as $\dim_{\mathbb{C}} \mathcal{O}_Z = \dim_{\mathbb{C}} \mathcal{F}$. □

Craw, Maclagan and Thomas [4] proved the following theorem.

Theorem 2.13 (Craw, Maclagan and Thomas [4])**.** *Let $\theta \in \Theta$ be generic. Then $\mathcal{M}_\theta$ has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover Y_θ satisfies the following properties:*

(i) *Y_θ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^3/G$.*
(ii) *Y_θ is projective over the quotient variety $\mathbb{C}^3/G$.*

$$\begin{array}{ccc} Y_\theta & \xhookrightarrow[\text{irr.}]{} & \mathcal{M}_\theta \\ \downarrow & & \downarrow \\ \mathbb{C}^3/G & \xhookrightarrow[\text{closed}]{} & \mathcal{M}_0 \end{array}$$

Remark 2.14. We call the unique irreducible component Y_θ of $\mathcal{M}_\theta$ the *birational component.* For generic $\theta \in \Theta$, Craw, Maclagan and Thomas [4] constructed the birational component Y_θ as GIT quotient of a reduced irreducible affine scheme by an algebraic torus. From this, it follows that Y_θ is irreducible and reduced. ♦

Remark 2.15. Since the algebraic torus $\mathbf{T}$ acts on $\mathbb{C}^3$, $\mathbf{T}$ acts on the moduli space $\mathcal{M}_\theta$ naturally. Fixed points of the $\mathbf{T}$-action play a crucial role in the study of the moduli space $\mathcal{M}_\theta$. Note that this $\mathbf{T}$-action is the same as the $\mathbf{T}$-action in Section 2.1. ♦

3. Abelian group actions and toric geometry

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3})$ where ϵ is a primitive rth root of unity. The group G acts naturally on $S := \mathbb{C}[x, y, z]$. Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$$

which is an overlattice of $\overline{L} = \mathbb{Z}^3$ of finite index. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$. Set $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The dual lattices $\overline{M}$ and M can be identified with Laurent monomials and G-invariant Laurent monomials, respectively. The embedding of G into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee$$

where $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ is the character group of G. Note that M is the kernel of the map wt.

Remark 3.1. There are two isomorphisms of abelian groups $L/\mathbb{Z}^3 \to G$ and $\overline{M}/M \to G^\vee$. ♦

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.

$$\overline{M}_{\geq 0} = \left\{ x^{m_1} y^{m_2} z^{m_3} \in \overline{M} \,\middle|\, m_1, m_2, m_3 \geq 0 \right\}.$$

For a set $A \subset \mathbb{C}[x^\pm, y^\pm, z^\pm]$, let $\langle A \rangle$ denote the $\mathbb{C}[x, y, z]$-submodule of $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ generated by A.

Let σ_+ be the cone in $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ generated by e_1, e_2, e_3, i.e.

$$\sigma_+ := \mathrm{Cone}(e_1, e_2, e_3).$$

For the cone σ_+ and the lattice L, we define a corresponding affine toric variety

$$U_{\sigma_+} := \operatorname{Spec} \mathbb{C}[\sigma_+^\vee \cap M].$$

Note that U_{σ_+} is the quotient variety $X = \mathbb{C}^3/G = \operatorname{Spec} \mathbb{C}[x, y, z]^G$ as M is the G-invariant Laurent monomials.

4. Generalized G-graphs

Definition 4.1. A *(generalized) G-graph* Γ is a subset of Laurent monomials in $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ satisfying:

(i) $\mathbf{1} \in \Gamma$.
(ii) $\mathrm{wt}\colon \Gamma \to G^\vee$ is bijective, i.e. for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ whose weight is ρ.
(iii) if $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.
(iv) Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x, y, z in Γ.

For any Laurent monomial $\mathbf{m} \in \overline{M}$, let $\mathrm{wt}_\Gamma(\mathbf{m})$ denote the unique element $\mathbf{m}_\rho$ in Γ whose weight is $\mathrm{wt}(\mathbf{m})$.

Remark 4.2. Nakamura G-graphs Γ in [18] are G-graphs in this sense because if a monomial $\mathbf{m} \cdot \mathbf{n}$ is in Γ for two monomials $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{m}$ is in Γ. The main difference between Nakamura's definition and ours is that we allow elements to be Laurent monomials, not just genuine monomials. ♦

Example 4.3. Let G be the group of type $\frac{1}{7}(1, 3, 4)$. Then

$$\Gamma_1 = \left\{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\},$$

$$\Gamma_2 = \left\{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z}, \tfrac{y^3}{z^2}\right\}$$

are G-graphs. In Γ_1, $\mathrm{wt}_{\Gamma_1}(x) = \frac{z}{y}$ and $\mathrm{wt}_{\Gamma_1}(y^3) = \frac{z^2}{y^2}$. ♦

As is defined in [18], for a generalized G-graph $\Gamma = \{\mathbf{m}_\rho\}$, define $S(\Gamma)$ to be the subsemigroup of M generated by $\dfrac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for all $\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. Define a cone $\sigma(\Gamma)$ in $L_{\mathbb{R}} = \mathbb{R}^3$ as follows:

$$\begin{aligned}\sigma(\Gamma) &= S(\Gamma)^\vee \\ &= \left\{ \mathbf{u} \in L_{\mathbb{R}} \;\middle|\; \left\langle \mathbf{u}, \frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} \right\rangle \geq 0 \quad \forall \mathbf{m}_\rho \in \Gamma,\ \mathbf{m} \in \overline{M}_{\geq 0} \right\}.\end{aligned}$$

Observe that:

(i) $\sigma(\Gamma) \subset \sigma_+$,
(ii) $\left(\overline{M}_{\geq 0} \cap M\right) \subset S(\Gamma)$,
(iii) $S(\Gamma) \subset \left(\sigma(\Gamma)^\vee \cap M\right)$.

Define two affine toric open sets:

$$U(\Gamma) := \operatorname{Spec} \mathbb{C}[S(\Gamma)],$$
$$U^\nu(\Gamma) := \operatorname{Spec} \mathbb{C}[\sigma^\vee(\Gamma) \cap M].$$

One can see that $U^\nu(\Gamma)$ is the normalization of $U(\Gamma)$ and that the torus $\operatorname{Spec} \mathbb{C}[M]$ of $U(\Gamma)$ is isomorphic to $(\mathbb{C}^\times)^3/G$.

Craw, Maclagan and Thomas [5] showed that there exists a torus invariant G-cluster which does not lie over the birational component Y_θ. The following definition is implicit in [5].

Definition 4.4. A generalized G-graph Γ is called a *G-iraffe* if the open set $U(\Gamma)$ has a torus fixed point.

Remark 4.5. As is standard in toric geometry, note that $U(\Gamma)$ has a torus fixed point if and only if $S(\Gamma) \cap (S(\Gamma))^{-1} = \{\mathbf{1}\}$. The open set $U(\Gamma)$ does not need to have a torus fixed point. In other words, the cone $\sigma(\Gamma)$ is not necessarily a 3-dimensional cone. For counterexamples, see Appendix A. ♦

Example 4.6. For the G-graphs in Example 4.3,

$$\sigma(\Gamma_1) = \left\{\mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m}\rangle \geq 0, \text{ for all } \mathbf{m} \in \{\tfrac{y^5}{z^2}, \tfrac{z^3}{y^4}, \tfrac{xy}{z}\}\right\},$$
$$= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(3,2,5), \tfrac{1}{7}(1,3,4)\right), \text{ and}$$
$$\sigma(\Gamma_2) = \left\{\mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m}\rangle \geq 0, \text{ for all } \mathbf{m} \in \{\tfrac{y^4}{z^3}, \tfrac{z^4}{y^3}, \tfrac{xz^2}{y^3}\}\right\},$$
$$= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(6,4,3)\right).$$

In both cases, they are G-iraffes. One can see that $S(\Gamma_1) = \sigma(\Gamma_1)^\vee \cap M$ and $S(\Gamma_2) = \sigma(\Gamma_2)^\vee \cap M$. ♦

Lemma 4.7. *Let Γ be a G-graph. Define*

$$B(\Gamma) := \left\{\mathbf{f} \cdot \mathbf{m}_\rho \,\middle|\, \mathbf{m}_\rho \in \Gamma,\ \mathbf{f} \in \{x, y, z\}\right\} \setminus \Gamma.$$

Then the semigroup $S(\Gamma)$ is generated as a semigroup by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$. In particular, $S(\Gamma)$ is finitely generated as a semigroup.

Proof. Let S be the subsemigroup of M generated by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$ as a semigroup. Clearly, $S \subset S(\Gamma)$. For the inverse inclusion, it is enough to show that the generators of $S(\Gamma)$ are in S.

An arbitrary generator of $S(\Gamma)$ is of the form $\frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)}$ for some $\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. We may assume that $\mathbf{m}\cdot\mathbf{m}_\rho \notin \Gamma$. In particular, $\mathbf{m} \neq \mathbf{1}$. Since $\mathbf{m}$ has positive degree, there exists $\mathbf{f} \in \{x, y, z\}$ such that $\mathbf{f}$ divides $\mathbf{m}$, i.e. $\frac{\mathbf{m}}{\mathbf{f}} \in \overline{M}_{\geq 0}$ and $\deg(\frac{\mathbf{m}}{\mathbf{f}}) < \deg(\mathbf{m})$. Let $\mathbf{m}_{\rho'}$ denote $\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho)$. Note that

$$\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'}) = \mathrm{wt}_\Gamma(\mathbf{f}\cdot\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho) = \mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho).$$

Thus

$$\begin{aligned}\frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)} &= \frac{\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho)}\cdot\frac{\mathbf{f}\cdot\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho)}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)}\\ &= \frac{\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}}\cdot\mathbf{m}_\rho)}\cdot\frac{\mathbf{f}\cdot\mathbf{m}_{\rho'}}{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_{\rho'})}.\end{aligned}$$

By induction on the degree of monomial $\mathbf{m}$, the assertion is proved. □

5. G-graphs and local charts

Let Γ be a G-graph. Define

$$C(\Gamma) := \langle\Gamma\rangle/\langle B(\Gamma)\rangle,$$

then it can be seen that $C(\Gamma)$ is a torus invariant G-constellation. Note that $C(\Gamma)$ can be realised as follows: $C(\Gamma)$ is the $\mathbb{C}$-vector space with a basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x, y, z]$ and whose $\mathbb{C}[x, y, z]$-action is given by

$$\mathbf{m} * \mathbf{m}_\rho = \begin{cases}\mathbf{m}\cdot\mathbf{m}_\rho & \text{if } \mathbf{m}\cdot\mathbf{m}_\rho \in \Gamma,\\ 0 & \text{if } \mathbf{m}\cdot\mathbf{m}_\rho \notin \Gamma,\end{cases}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$.

Any submodule $\mathcal{G}$ of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a $\mathbb{C}$-basis of $\mathcal{G}$. We give a combinatorial description of submodules of $C(\Gamma)$.

Lemma 5.1. *Let A be a subset of Γ. The following are equivalent.*

(i) *The set A forms a $\mathbb{C}$-basis of a submodule of $C(\Gamma)$.*
(ii) *If $\mathbf{m}_\rho \in A$ and $\mathbf{f} \in \{x, y, z\}$, then $\mathbf{f}\cdot\mathbf{m}_\rho \in \Gamma$ implies $\mathbf{f}\cdot\mathbf{m}_\rho \in A$.*

Example 5.2. From Example 4.3, recall the G-graph

$$\Gamma = \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\},$$

where G is of type $\frac{1}{7}(1, 3, 4)$. For the element $y + y^2 + \frac{z}{y}$ in $C(\Gamma)$,

$$y * (y + y^2 + \tfrac{z}{y}) = y^2 + 0 + z = y^2 + z \in C(\Gamma).$$

Let $\mathcal{G}$ be the submodule of $C(\Gamma)$ generated by a basis e_1 of $\mathbb{C}\rho_1$. Then one can see that the set $A = \{z, \frac{z}{y}, \frac{z^2}{y}\}$ satisfies the condition (ii) in the lemma above. Indeed, A is a $\mathbb{C}$-basis of $\mathcal{G}$. ♦

Let p be a point in $U(\Gamma)$. Then there exists the evaluation map

$$\mathrm{ev}_p\colon S(\Gamma) \to (\mathbb{C}, \times),$$

which is a semigroup homomorphism.

To assign a G-constellation $C(\Gamma)_p$ to the point p of $U(\Gamma)$, firstly consider the $\mathbb{C}$-vector space with basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x, y, z]$. Endow it with the following $\mathbb{C}[x, y, z]$-action,

$$\mathbf{m} * \mathbf{m}_\rho := \mathrm{ev}_p\left(\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}\right)\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho), \tag{5.3}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and an element $\mathbf{m}_\rho$ in Γ.

Lemma 5.4. *With the notation as above, we have the following:*

(i) *$C(\Gamma)_p$ is a G-constellation for any $p \in U(\Gamma)$.*
(ii) *For any p, Γ is a $\mathbb{C}$-basis of $C(\Gamma)_p$.*
(iii) *$C(\Gamma)_p \ncong C(\Gamma)_q$, if p and q are different points in $U(\Gamma)$.*
(iv) *Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free G-orbit and p the corresponding point in the torus $\operatorname{Spec}\mathbb{C}[M]$ of $U(\Gamma)$. Then $C(\Gamma)_p \cong \mathcal{O}_Z$ as G-constellations.*
(v) *If Γ is a G-iraffe and p is the torus fixed point of $U(\Gamma)$, then $C(\Gamma)_p \cong C(\Gamma)$.*

Proof. From the definition of $C(\Gamma)_p$, The assertions (i), (ii) and (v) follow immediately. The assertion (iii) follows from the fact [3] that points on the affine toric variety $U(\Gamma)$ are in 1-to-1 correspondence with semigroup homomorphisms from $S(\Gamma)$ to $\mathbb{C}$.

It remains to show (iv). Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free G-orbit and p the corresponding point in $\operatorname{Spec}\mathbb{C}[M] \subset U(\Gamma)$. Define a G-equivariant $\mathbb{C}[x, y, z]$-module homomorphism

$$\mathbb{C}[x, y, z] \to C(\Gamma)_p, \qquad \text{given by } f \mapsto f * \mathbf{1}.$$

One can check the morphism is surjective and whose kernel is equal to the ideal of Z. This proves (iv). □

This is a family of McKay quiver representations in the following sense of [12].

Definition 5.5. *A family of representations of a quiver Q with relations over a scheme B* is a representation of Q with relations in the category of locally free sheaves over B.

Definition 5.6. A G-graph is said to be *θ-stable* if the G-constellation $C(\Gamma)$ is θ-stable.

Proposition 5.7. *Let Γ be a G-iraffe, that is, $U(\Gamma)$ has a torus fixed point. Let Y_θ be the birational component in $\mathcal{M}_\theta$. For a generic θ, assume that $C(\Gamma)$ is θ-stable. Then $C(\Gamma)_p$ is θ-stable for any $p \in U(\Gamma)$. Thus there exists an open immersion*

$$U(\Gamma) = \operatorname{Spec}\mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta.$$

Proof. Let us assume that the G-constellation $C(\Gamma)$ is θ-stable. Let p be an arbitrary point in $U(\Gamma)$ and $\mathcal{G}$ a submodule of $C(\Gamma)_p$. By the definition of $C(\Gamma)_p$, it is clear that $\mathcal{G}$ is a submodule of $C(\Gamma)$. Since $C(\Gamma)$ is θ-stable, $\theta(\mathcal{G}) > 0$, and thus $C(\Gamma)_p$ is θ-stable.

Now we introduce deformation theory of the G-constellation in $\mathcal{M}_\theta$. Deforming $C(\Gamma)$ involves $3r$ parameters $\left\{x_\rho, y_\rho, z_\rho \,\middle|\, \rho \in G^\vee\right\}$

$$\begin{cases} x * \mathbf{m}_\rho = x_\rho \operatorname{wt}_\Gamma(x \cdot \mathbf{m}_\rho), \\ y * \mathbf{m}_\rho = y_\rho \operatorname{wt}_\Gamma(y \cdot \mathbf{m}_\rho), \\ z * \mathbf{m}_\rho = z_\rho \operatorname{wt}_\Gamma(z \cdot \mathbf{m}_\rho), \end{cases}$$

such that the following quadrics vanish:

$$(5.8) \qquad \begin{cases} x_\rho y_{\operatorname{wt}(x\cdot\mathbf{m}_\rho)} - y_\rho x_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)}, \\ x_\rho z_{\operatorname{wt}(x\cdot\mathbf{m}_\rho)} - z_\rho x_{\operatorname{wt}(z\cdot\mathbf{m}_\rho)}, \\ y_\rho z_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)} - z_\rho y_{\operatorname{wt}(y\cdot\mathbf{m}_\rho)}. \end{cases}$$

Since Γ is a $\mathbb{C}$-basis, for $\mathbf{f} \in \{x, y, z\}$, $\mathbf{f}_\rho = 1$ if $\operatorname{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_\rho) = \mathbf{f} \cdot \mathbf{m}_\rho$. Define a subset of the $3r$ parameters

$$\Lambda(\Gamma) := \left\{\mathbf{f}_\rho \,\middle|\, \operatorname{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_\rho) = \mathbf{f} \cdot \mathbf{m}_\rho,\ \mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}\right\}.$$

Define an affine scheme $D(\Gamma)$ whose coordinate ring is

$$\mathbb{C}\left[x_\rho, y_\rho, z_\rho \,\middle|\, \rho \in G^\vee\right] / I_\Gamma$$

where $I_\Gamma = \left\langle \text{ the quadrics in (5.8)}, \mathbf{f} - 1 \,\middle|\, \mathbf{f} \in \Lambda(\Gamma)\right\rangle$.

By King's GIT [12], the affine scheme $D(\Gamma)$ is an open set of $\mathcal{M}_\theta$ which contains the point corresponding to $C(\Gamma)$. More precisely, for a θ-stable G-graph Γ, we have an affine open set $\widetilde{U_\Gamma}$ in the McKay quiver representation space $\operatorname{Rep} G$, which is defined by $\mathbf{f}_\rho$ to be nonzero for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$. Note that $\widetilde{U_\Gamma}$ is $\mathrm{GL}(\delta)$-invariant and that any point in $\widetilde{U_\Gamma}$ is θ-stable. Since the quotient map $\operatorname{Rep}^s G \to \mathcal{M}_\theta$ is a geometric quotient, by GIT (see Remark 1.5), it follows that

$$\widetilde{U_\Gamma} /\!\!/ \mathrm{GL}(\delta) = \operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$$

is an open set in $\mathcal{M}_\theta$. On the other hand, after changing basis, we can set $\mathbf{f}_\rho \in \Lambda(\Gamma)$ to be 1 for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$. One can see that this gives a slice[2] so that $D(\Gamma)$ is isomorphic to $\operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$.

[2]First, see that $\mathbb{C}[\widetilde{U_\Gamma}] = \operatorname{Rep} G[\Lambda(\Gamma)^{-1}]$. Note that $\mathrm{GL}(\delta)$-invariants in $\mathbb{C}[\widetilde{U_\Gamma}]$ are generated by cycles with inverting the arrows in $\Lambda(\Gamma)$. Assume that a is the linear map corresponding to an arrow from ρ to ρ'. For ρ, ρ', there exists an undirected path $\mathbf{p}_a$ in $\Lambda(\Gamma) \cap \Lambda(\Gamma)^{-1}$ from ρ to ρ', that is unique up to the commutation relations. This means that $a\mathbf{p}_a^{-1}$ is $\mathrm{GL}(\delta)$-invariants. From this, one can show that there exists an algebra isomorphism between $\mathbb{C}[D(\Gamma)]$ to $\mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$ defined by $a \mapsto a\mathbf{p}_a^{-1}$.

Note that there is a $\mathbb{C}$-algebra epimorphism from $\mathbb{C}[D(\Gamma)]$ to $\mathbb{C}[S(\Gamma)]$ defined by

$$\mathbf{f}_\rho \mapsto \frac{\mathbf{f} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_\rho)},$$

for $\mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}$. It follows that $U(\Gamma)$ is a closed subscheme of $D(\Gamma)$.

As Craw, Maclagan, and Thomas [4] proved that the birational component Y_θ is a unique irreducible component of $\mathcal{M}_\theta$ containing torus T which is isomorphic to $(\mathbb{C}^\times)^3/G$ as an algebraic group, $Y_\theta \cap D(\Gamma)$ is a unique irreducible component of $D(\Gamma)$ which contains the torus T. Note that $Y_\theta \cap D(\Gamma)$ is reduced by Remark 2.14.

We now prove that the morphism $U(\Gamma) \to D(\Gamma) \subset \mathcal{M}_\theta$ induces an isomorphism from the torus $\operatorname{Spec}\mathbb{C}[M]$ onto the torus T of Y_θ. In other words, $U(\Gamma)$ contains the torus T of Y_θ. Let ψ denote the restriction of the morphism to $\operatorname{Spec}\mathbb{C}[M]$. First note that T represents G-constellations whose support is in $\mathbf{T} = (\mathbb{C}^\times)^3$. Let p be a point in the torus $\operatorname{Spec}\mathbb{C}[M] \subset U(\Gamma)$ with the corresponding free G-orbit Z. By Lemma 5.4, the G-constellation $C(\Gamma)_p$ over p is isomorphic to $\mathcal{O}_Z$. Thus ψ maps $\operatorname{Spec}\mathbb{C}[M]$ into T. On the other hand, Lemma 2.12 shows that any G-constellation whose support is a free G-orbit Z in $\mathbf{T}$ is isomorphic to $\mathcal{O}_Z$. From this, it follows that ψ is a bijective morphism between the two tori. As ψ is a group homomorphism by the construction of $C(\Gamma)_p$, ψ is an isomorphism between $\operatorname{Spec}\mathbb{C}[M]$ and T.

Remember that $U(\Gamma)$ is reduced and irreducible as it is defined by an affine semigroup algebra $\mathbb{C}[S(\Gamma)]$. Note that $U(\Gamma)$ is in the component $Y_\theta \cap D(\Gamma)$ because $U(\Gamma)$ is a closed subset of $D(\Gamma)$ containing T. Since both are of the same dimension, $U(\Gamma)$ is equal to $Y_\theta \cap D(\Gamma)$. Thus there exists an open immersion from $U(\Gamma)$ to Y_θ. □

6. G-iraffes and torus fixed points in Y_θ

In this section, we present a 1-to-1 correspondence between the set of torus fixed points in Y_θ and the set of θ-stable G-iraffes.

For a genuine monomial $\mathbf{m} \in \overline{M}_{\geq 0}$, let $\mathbf{m}_{(\rho)}$ denote the path induced by $\mathbf{m}$ in the McKay quiver from the vertex ρ. In other words, $\mathbf{m}_{(\rho)}$ is the linear map induced by the action of the monomial $\mathbf{m}$ on the vector space $\mathbb{C}\rho$.

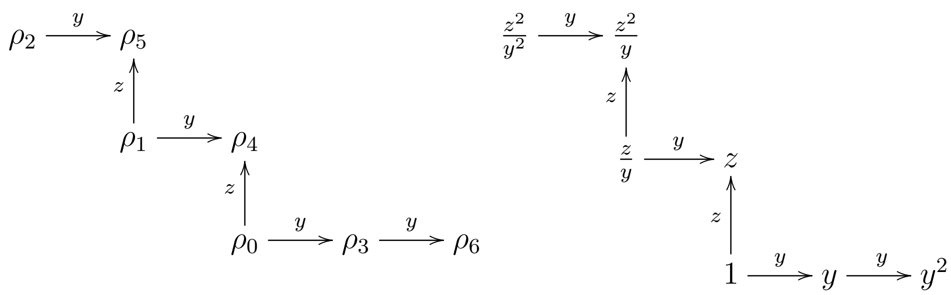
An *undirected path* in the McKay quiver is a path in the underlying graph of the McKay quiver. For a G-constellation $\mathcal{F}$, an undirected path in the McKay quiver is said to be *defined* if the linear maps corresponding to the opposite-directed arrows in the path are nonzero in $\mathcal{F}$.

Definition 6.1. A defined undirected path in the McKay quiver is *of type* $\mathbf{m}$ for a Laurent monomial $\mathbf{m} \in \overline{M}$ where $\mathbf{m}$ is the Laurent monomial obtained by forgetting outgoing vertices.

Example 6.2. Let G be the group of type $\frac{1}{7}(1,3,4)$. Consider the G-graph

$$\Gamma = \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\}.$$

The torus invariant G-constellation $C(\Gamma)$ has the following configurations:



where the marked arrows are nonzero and the others are all zero. The path from 1 to y^2 is induced by y^2 at ρ_0, whose type is y^2. The undirected path from ρ_2 to ρ_4 is a defined undirected path of type $\frac{y^2}{z}$ because the path consists of nonzero linear maps:

$$\rho_2 \xrightarrow{y} \rho_5 \xleftarrow{z} \rho_1 \xrightarrow{y} \rho_4.$$

However, the following undirected path of the same type $\frac{y^2}{z}$ from ρ_2 to ρ_4

$$\rho_2 \xrightarrow{y} \rho_5 \xrightarrow{y} \rho_1 \xleftarrow{z} \rho_4$$

is not defined because the third arrow is zero in $C(\Gamma)$. ♦

Remark 6.3. Let $\mathbf{p}$ be a nonzero path induced by a genuine monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ from ρ_i. If $\mathbf{q}$ is a path induced by a genuine monomial $\mathbf{n} \in \overline{M}_{\geq 0}$ from ρ_i with the condition that $\mathbf{n}$ divides $\mathbf{m}$, then the path $\mathbf{q}$ is nonzero. ♦

Lemma 6.4. *Let $\mathcal{F}$ be a torus invariant G-constellation. Then there are no defined (undirected) cycles of type $\mathbf{m}$ with $\mathbf{m} \neq \mathbf{1}$.*

Proof. For a contradiction, suppose that there is a defined cycle of type $\mathbf{m} \neq \mathbf{1}$. Then $\mathbf{m}$ is a G-invariant Laurent monomial.

We may assume that the cycle is a cycle around ρ_0 of type $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3}$. A point $(t_1, t_2, t_3) \in \mathbf{T} = (\mathbb{C}^\times)^3$ acts on the cycle by a scalar multiplication of ${t_1}^{m_1}{t_2}^{m_2}{t_3}^{m_3}$. Since $\mathbf{m} \neq \mathbf{1}$, there exists $t \in \mathbf{T}$ such that ${t_1}^{m_1}{t_2}^{m_2}{t_3}^{m_3} \neq 1$, i.e. $t^*(\mathcal{F})$ is not isomorphic to $\mathcal{F}$. Therefore $\mathcal{F}$ is not torus invariant. □

In Section 5, we proved that if Γ is a θ-stable G-iraffe, then $C(\Gamma)$ is a torus invariant G-constellation over Y_θ and the corresponding point is fixed by its algebraic torus. Clearly, two different G-iraffes Γ, Γ' give non-isomorphic G-constellations $C(\Gamma)$, $C(\Gamma')$. Moreover, we now prove that for any torus fixed point $p \in Y_\theta$, the corresponding G-constellation is isomorphic to $C(\Gamma)$ for some G-iraffe Γ.

Let p be a torus fixed point in Y_θ. There exists a one parameter subgroup

$$\lambda^u \colon \mathbb{C}^\times \longrightarrow T \subset Y_\theta$$

with $\lim_{t\to 0} \lambda^u(t) = p$. Since Y_θ is the fine moduli space of θ-stable G-constellations, we have a family $\mathcal{U}$ of θ-stable G-constellations over $\mathbb{A}^1_{\mathbb{C}}$ with the following property: for nonzero $s \in \mathbb{A}^1_{\mathbb{C}}$ and the point $q := \lambda^u(s)$, the G-constellation $\mathcal{U}_s$ over s is isomorphic to $\mathcal{O}_Z$ where Z is the free G-orbit in $\mathbf{T}$ corresponding to the point q. In particular, the support of the G-constellation $\mathcal{U}_s$ is in the torus $\mathbf{T} = (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$.

Let $\mathcal{F}$ be the θ-stable G-constellation over $0 \in \mathbb{A}^1$. Let us define a subset Γ of Laurent monomials to be

$$\Gamma = \left\{ \mathbf{m} \in \overline{M} \;\middle|\; \begin{array}{l} \exists \text{ a defined nonzero undirected} \\ \text{path in } \mathcal{F} \text{ of type } \mathbf{m} \text{ from } \rho_0 \end{array} \right\}.$$

Firstly, we prove that Γ is a G-graph. Clearly, Γ contains $\mathbf{1}$. Since θ is generic and $\mathcal{F}$ is θ-stable, there exists a nonzero undirected defined path from ρ_0 to ρ so there is a Laurent monomial $\mathbf{m}_\rho$ in Γ for each $\rho \in G^\vee$. The Laurent monomial $\mathbf{m}_\rho$ is unique: suppose there exists a defined path of type $\mathbf{n}_\rho$ from ρ_0 to ρ, and then there exists a defined cycle of type $\frac{\mathbf{m}_\rho}{\mathbf{n}_\rho}$ at ρ_0, which implies $\mathbf{n}_\rho = \mathbf{m}_\rho$ by Lemma 6.4. It remains to show the condition (c) of Definition 4.1. We need the following lemma:

Lemma 6.5. *With the notation as above, let* $\mathbf{p}$ *and* $\mathbf{q}$ *be two defined (undirected) paths of the same type* $\mathbf{m}$ *from* ρ *to* ρ' *for some Laurent monomial* $\mathbf{m} \in \overline{M}$. *Then, in* $\mathcal{F}$,

$$\mathbf{p} * e_\rho = \mathbf{q} * e_\rho$$

where e_ρ *is a basis of* $\mathbb{C}\rho$.

Proof. Firstly, note that if $\mathbf{m}$ is a genuine monomial, then the assertion follows from the $\mathbb{C}[x,y,z]$-module structure.

Let $\mathbf{m}$ be a Laurent monomial. There exists a genuine monomial $\mathbf{n} \in \overline{M}_{\geq 0}$ so that $\mathbf{n} \cdot \mathbf{m}$ is a genuine monomial with $\mathbf{n}$ nonzero on $\lambda^u(\mathbb{C}^\times)$. Since two paths $\mathbf{p} * e_\rho$ and $\mathbf{q} * e_\rho$ are of type $\mathbf{m} \cdot \mathbf{n}$, we have

$$\mathbf{n}_{(\rho')} * \mathbf{p} * e_\rho = \mathbf{n}_{(\rho')} * \mathbf{q} * e_\rho. \tag{6.6}$$

Since (6.6) implies $\mathbf{p} * e_\rho = \mathbf{q} * e_\rho$ in the G-constellation $\mathcal{U}_s$ for nonzero $s \in \mathbb{A}^1$, the assertion is proved by flatness of the family $\mathcal{U}$. □

To show that Γ satisfies the condition (c) of Definition 4.1, suppose that $\mathbf{m}\cdot\mathbf{n}\cdot\mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$. We need to show that $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$. By the definition of Γ, there exist nonzero (undirected) paths $\mathbf{p}$ of type $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho$ and $\mathbf{q}$ of type $\mathbf{m}_\rho$. By Lemma 6.5, it follows that the defined undirected path $\mathbf{m}_{(\rho'')} * \mathbf{n}_{(\rho')} * \mathbf{q}$ is nonzero as it is of the same type as bp. This implies that the defined undirected path $\mathbf{n}_{(\rho')} * \mathbf{q}$ is nonzero. Thus $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.

Proposition 6.7. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite cyclic group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. For a generic parameter θ, there is a 1-to-1 correspondence between the set of torus fixed points in the birational component Y_θ and the set of θ-stable G-iraffes.*

Proof. From the argument above, we have shown that there exists a G-graph Γ for each torus fixed point p. Using Lemma 6.5, one can easily show that $C(\Gamma)$ is actually isomorphic to $\mathcal{F}$ as a G-constellation. In particular, $C(\Gamma)$ lies over $p \in Y_\theta$, and hence $U(\Gamma)$ contains the torus fixed point p. Thus Γ is a G-iraffe.

Let Γ be a θ-stable G-iraffe. By Proposition 5.7 and Lemma 5.4, we can see that $C(\Gamma)$ lies over Y_θ for a G-graph Γ if Γ is a G-iraffe. Thus we have a torus fixed point p representing the isomorphism class of $C(\Gamma)$. □

Corollary 6.8. *Let Γ be a G-graph. Then $C(\Gamma)$ lies over the birational component Y_θ if and only if Γ is a G-iraffe.*

Theorem 6.9. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G-constellations. Assume that $\mathfrak{G}$ is the set of all θ-stable G-iraffes.*

(i) *The birational component Y_θ of $\mathcal{M}_\theta$ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$.*
(ii) *The normalization of Y_θ is isomorphic to the normal toric variety whose toric fan consists of the full dimensional cones $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{G}$ and their faces.*

Proof. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. Consider the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3).$$

Let Y_θ be the birational component of the moduli space of θ-stable G-constellations and Y_θ^ν the normalization of Y_θ. Let Y denote the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$. Define the fan Σ in $L_\mathbb{R}$ whose full dimensional cones are $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{G}$. One can see that the corresponding toric variety $Y^\nu := X_\Sigma$ is the normalization of Y.

Since Y_θ^ν is a normal toric variety, it is covered by toric affine open sets U_i with the torus fixed point p_i in U_i. Let q_i be the image of p_i under the normalization. As each q_i is a torus fixed point, it follows from Proposition 6.7 that there is a (unique) G-iraffe $\Gamma_i \in \mathfrak{G}$ with $C(\Gamma_i)$ isomorphic to the G-constellation represented by q_i.

By Proposition 5.7, for each G-iraffe $\Gamma \in \mathfrak{G}$, there is an open immersion of $U(\Gamma)$ into Y_θ. Thus we have an open immersion $\psi \colon Y \to Y_\theta$ and the image $\psi(Y)$ contains all torus fixed points of Y_θ.

The induced morphism $\psi^\nu \colon Y^\nu \to Y_\theta^\nu$ is an open embedding. Note that the numbers of full dimensional cones are the same. Thus ψ^ν should be an isomorphism. This proves (ii).

To show (i), suppose that $Y_\theta \backslash \psi(Y)$ is nonempty so it contains a torus orbit O of dimension $d \geq 1$. Since the normalization morphism is torus equivariant and surjective, there exists a torus orbit O' in $Y_\theta^\nu = Y^\nu$ of dimension d which is mapped to the torus orbit O. At the same time, from the fact that Y^ν is the normalization of Y and that the normalization morphism is finite, it follows that the image of O' is a torus orbit of dimension d, so the image is O. Thus O is in $\psi(Y)$, which is a contradiction. □

Corollary 6.10. *With notation as Theorem 6.9, Y_θ is a normal toric variety if and only if $S(\Gamma) = \sigma(\Gamma)^\vee \cap M$ for all $\Gamma \in \mathfrak{G}$.*

7. Example

Let G be the finite group of type $\frac{1}{7}(1,3,4)$. Firstly, consider the following G-graphs:

$$
\begin{aligned}
\Gamma_1 &:= \Big\{1, z, z^2, z^3, z^4, z^5, z^6\Big\},\\
\Gamma_2 &:= \Big\{1, y, z, z^2, z^3, z^4, z^5\Big\},\\
\Gamma_3 &:= \Big\{1, y, y^2, z, z^2, z^3, \tfrac{y^2}{z}\Big\},\\
\Gamma_4 &:= \Big\{1, \tfrac{y^3}{z^2}, \tfrac{y^2}{z}, \tfrac{y^3}{z}, y, y^2, z\Big\},\\
\Gamma_5 &:= \Big\{1, y, y^2, \tfrac{z}{y}, z, \tfrac{z^2}{y^2}, \tfrac{z^2}{y}\Big\},\\
\Gamma_6 &:= \Big\{1, y, y^2, y^3, y^4, \tfrac{z}{y}, z\Big\},\\
\Gamma_7 &:= \Big\{1, y, y^2, y^3, y^4, y^5, y^6\Big\},\\
\Gamma_8 &:= \Big\{1, x, x^2, x^3, z, xz, x^2z\Big\},\\
\Gamma_9 &:= \Big\{1, x, x^2, y, z, xz, x^2z\Big\},\\
\Gamma_{10} &:= \Big\{1, x, y, y^2, z, xz, \tfrac{y^2}{z}\Big\},\\
\Gamma_{11} &:= \Big\{1, x, x^2, y, xy, x^2y, y^2\Big\},\\
\Gamma_{12} &:= \Big\{1, x, y, xy, y^2, y^3, y^4\Big\}.\\
\Gamma_{13} &:= \Big\{1, x, x^2, x^3, x^4, x^5, x^6\Big\},
\end{aligned}
\tag{7.1}
$$

Secondly, consider the cone $\mathfrak{C}$ in Θ generated by the row vectors of the following matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For each $0 \leq i \leq 7$, let v_i denote the lattice point $\frac{1}{7}(\overline{5i}, i, 7-i)$ where $\overline{}$ denotes the residue modulo 7. One can check that all G-iraffes in (7.1) are θ-stable for any $\theta \in \mathfrak{C}$ and that each Γ_i corresponds to the cone σ_i where:

$$\sigma_i := \begin{cases} \text{Cone}\left(e_1, v_{8-i}, v_{7-i}\right) & \text{if } 1 \leq i \leq 7, \\ \text{Cone}\left(v_3, v_{15-i}, v_{14-i}\right) & \text{if } 8 \leq i \leq 10, \\ \text{Cone}\left(e_3, v_{14-i}, v_{13-i}\right) & \text{if } 11 \leq i \leq 12, \\ \text{Cone}\left(e_2, e_3, v_3\right) & \text{if } i = 13. \end{cases}$$

Moreover, by a direct calculation, it can be shown that

$$S(\Gamma_i) = \sigma_i^\vee \cap M.$$

Thus every affine piece $U(\Gamma)$ is normal and the fan corresponding to the birational component Y_θ is shown in Figure ★ .

Appendix A. Example: G-graphs which are not G-iraffes

In [18] Nakamura assumed that $U(\Gamma)$ has a torus fixed point for any Nakamura G-graph Γ, i.e. every G-graph in his sense is a G-iraffe. His assumption implies that every torus invariant G-cluster lies over the birational component of G-Hilb. However, Craw, Maclagan and Thomas [5] showed that there exists a torus invariant G-cluster which is not over the birational component.

Example A.1 (Craw, Maclagan and Thomas [5])**.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{14}(1,9,11)$. Note that G is isomorphic to $\frac{1}{7}(1,2,4) \times \frac{1}{2}(1,1,1)$. Consider the monomial ideal

$$I = \langle y^2z, xz^2, xy^2, x^2y, yz^2, x^2z, x^4, y^4, z^4 \rangle$$

and the corresponding Nakamura G-graph

$$\Gamma = \{1, x, x^2, x^3, y, y^2, y^3, z, z^2, z^3, xy, xz, yz, xyz\}.$$

Craw, Maclagan and Thomas [5] showed that this ideal does not lie over the birational component using Gröbner basis techniques.

We show this by proving the G-graph Γ is not a G-iraffe. One can calculate the semigroup $S(\Gamma)$ and notice that $S(\Gamma)$ is generated as a subsemigroup in M by $\frac{xy^2}{z^3}, \frac{yz^2}{x^3}, \frac{x^2z}{y^3}, \frac{y^2z}{x}$. Note that

$$\frac{xy^2}{z^3} \cdot \frac{yz^2}{x^3} \cdot \frac{x^2z}{y^3} = 1$$

and hence $\frac{xy^2}{z^3} \in S(\Gamma) \cap (S(\Gamma))^{-1} \neq \{\mathbf{1}\}$. Thus $U(\Gamma)$ does not have a torus fixed point. Indeed, the cone $\sigma(\Gamma)$ is the cone generated by $\frac{1}{14}(7,7,7)$ so it is not a full dimensional cone. Therefore the G-cluster $C(\Gamma) = \mathbb{C}[x,y,z]/I$ does not lie over the birational component. ♦

Remark A.2. Craw, Maclagan, and Thomas [5] provided an equivalent condition using Gröbner basis for a monomial ideal to be over the birational component. In the terms of G-iraffes, the condition is equivalent for a Nakamura G-graph to be a G-iraffe. ♦

Example A.3 (Reid)**.** Let $G \subset \mathrm{SL}_4(\mathbb{C})$ be the finite subgroup of type $\frac{1}{30}(1,6,10,13)$ with coordinates x,y,z,t. Consider the monomial ideal

$$I = \left\langle \begin{matrix} x^6, x^3y, x^3t, x^2z, x^2t^2, xy^2, xyt, xzt, xt^3, \\ y^5, y^4z, y^3t, y^2zt, yz^2, yt^2, z^3, z^2t, zt^2, t^4 \end{matrix} \right\rangle$$

and the corresponding Nakamura G-graph

$$\Gamma = \left\{ \begin{matrix} 1, x, x^2, x^3, x^4, x^5, y, y^2, y^3, y^4, z, z^2, \\ t, t^2, t^3, xy, x^2y, xz, xz^2, xt, x^2t, xt^2, \\ yz, y^2z, y^3z, yt, y^2t, zt, xyz, yzt \end{matrix} \right\}.$$

Note that $\frac{y^2zt}{x^5}, \frac{x^3y}{t^3}, \frac{x^2t^2}{y^3z}$ are in the semigroup $S(\Gamma)$ and

$$\frac{y^2zt}{x^5} \cdot \frac{x^3y}{t^3} \cdot \frac{x^2t^2}{y^3z} = 1.$$

Thus $\frac{y^2zt}{x^5} \in S(\Gamma) \cap (S(\Gamma))^{-1} \neq \{\mathbf{1}\}$. Thus $U(\Gamma)$ does not have a torus fixed point. Therefore the G-cluster $C(\Gamma) = \mathbb{C}[x,y,z,t]/I$ does not lie over the birational component. ♦

Remark A.4. Reid used the ideal in Example A.3 to provide a case where G-Hilb has a 5-dimensional component even if G is a subgroup of $\mathrm{GL}_4(\mathbb{C})$. ♦

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