MCKAY QUIVERS AND TERMINAL QUOTIENT SINGULARITIES IN DIMENSION 3

SEUNG-JO JUNG

Abstract. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. For such G, the quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein and has a terminal singularity. The singular variety X has the economic resolution which is "close to being crepant". In this paper, we prove that the economic resolution of the quotient variety $X = \mathbb{C}^3/G$ is isomorphic to the birational component of a moduli space of θ-stable representations of the McKay quiver for a suitable GIT parameter θ. We conjecture that the moduli space is irreducible.

1. Introduction

The motivation of this work stems from the philosophy of the *McKay correspondence*, which says that if a finite group G acts on a variety M, then the crepant resolutions of the quotient variety M/G have information of the G-equivariant geometry of M [20].

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. A G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a *G-constellation* if its global sections $\mathrm{H}^0(\mathcal{F})$ are isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module. In particular, the structure sheaf of a G-invariant subscheme $Z \subset \mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$ as a G-module, which is called a *G-cluster*, is a G-constellation. It is known that G-clusters are θ-stable G-constellations for a particular choice of GIT stability parameter θ [8].

For a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$, Ito and Nakamura [9] introduced $G\text{-}\mathrm{Hilb}\,\mathbb{C}^2$ which is the fine moduli space parametrising G-clusters and proved that $G\text{-}\mathrm{Hilb}\,\mathbb{C}^2$ is the minimal resolution of $\mathbb{C}^2/G$. In the celebrated paper [1], Bridgeland, King and Reid proved that for a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, $G\text{-}\mathrm{Hilb}\,\mathbb{C}^3$ is a crepant resolution of the quotient variety $\mathbb{C}^3/G$. Also Craw and Ishii [2] showed that in the case of a finite abelian group $G \subset \mathrm{SL}_3(\mathbb{C})$, *any* projective crepant resolution can be realised as the fine moduli space of θ-stable G-constellations for a suitable stability parameter θ.

For a finite abelian group $G \subset \mathrm{GL}_n(\mathbb{C})$ and a generic GIT parameter $\theta \in \Theta$, Craw, Maclagan and Thomas [4] showed that the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. So the irreducible component

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is birational to the quotient variety $\mathbb{C}^n/G$. The component Y_θ is called the *birational component*[1] of $\mathcal{M}_\theta$.

On the other hand, it is shown [16, 19] that a 3-fold cyclic quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if G is of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. In this case, X have a preferred toric resolution, called the *economic resolution*. For the group G of type $\frac{1}{r}(1, a, r-a)$, G-Hilb $\mathbb{C}^3$ is smooth and isomorphic to the economic resolution of X if and only if $a = 1$ or $r-1$ as shown in [13]. Kędzierski [12] proved that there exists a Weyl chamber $\mathfrak{C}$ in Θ such that the normalization of the birational component Y_θ of the moduli space of θ-stable G-constellations is isomorphic to the economic resolution Y of $X = \mathbb{C}^3/G$. To show this, he found a suitable family over the economic resolution Y and a chamber $\mathfrak{C}$ such that G-constellations in the family are θ-stable for $\theta \in \mathfrak{C}$. His original description of stability parameters is a set of inequalities, but one can show that his stability parameters form an open Weyl chamber and this is easy to describe using the A_{r-1} root system.

Main results. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^a, \epsilon^{r-a})$ where ϵ is a primitive rth root of unity. The quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein and has terminal singularities. Moreover, the singular variety $X = \mathbb{C}^3/G$ has no crepant resolution. However, there exist economic resolutions which are close to being crepant (see Section 5.7 in [19]). The economic resolution can be obtained by a toric method, which is called weighted blowups.

In this paper, we prove that the economic resolution Y is isomorphic to an irreducible component of the moduli space of G-equivariant sheaves on $\mathbb{C}^3$. More precisely, we have the following theorem.

Theorem 1.1 (Corollary 7.2)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. The economic resolution Y of $X = \mathbb{C}^3/G$ is isomorphic to the birational component Y_θ of the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations for a suitable parameter θ.*

To prove this, we introduce *generalized G-graphs* and *round down functions*. A *generalized* G-graph Γ is a generalized version of Nakamura G-graph in [17]. A G-graph corresponds to a torus invariant G-constellation. We define a toric affine open set $U(\Gamma)$ associated to a G-graph Γ and a family of G-constellations over $U(\Gamma)$. These give us a local chart of the moduli space of θ-stable McKay quiver representations for suitable parameter θ. On the other hand, the round down functions are related to weighted blowups. For each step of the weighted blowups, we define three *round down functions*, that are maps between

[1]This component is also called the coherent component.

monomial lattices. The round down functions are used for finding admissible G-graphs, which define the universal family over the economic resolution Y.

Moreover, we prove that our stability parameters form an open Weyl chamber, which coincides with the chamber in [12]. Moreover, we can see that the chamber is a full chamber in the GIT stability parameter space.

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2. G-graphs and G-constellations

This section briefly reviews the notion of (generalized) G-graphs and G-iraffes which are introduced in [10].

2.1. G-constellations.

Definition 2.1. A *G-constellation* on $\mathbb{C}^3$ is a G-equivariant $\mathbb{C}[x,y,z]$-module $\mathcal{F}$ on $\mathbb{C}^3$, which is isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module.

The representation ring $R(G)$ of G is $\bigoplus_{\rho\in G^\vee}\mathbb{Z}\rho$. Define the GIT stability parameter space

$$\Theta = \left\{\theta \in \operatorname{Hom}_{\mathbb{Z}}(R(G),\mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right) = 0\right\}.$$

Definition 2.2. For a stability parameter $\theta \in \Theta$, we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \geq 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.
(ii) a G-constellation $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.
(iii) θ is *generic* if every θ-semistable object is θ-stable.

Let $\operatorname{Rep} G$ denote the McKay quiver representation space of G. Note that its coordinate ring is

$$\mathbb{C}[\operatorname{Rep} G] = \mathbb{C}\left[x_i, y_i, z_i \,\middle|\, i \in I\right] / I_G$$

where I_G is the ideal generated by the following quadrics:

$$\begin{cases} x_i y_{i+\alpha_1} - y_i x_{i+\alpha_2}, \\ x_i z_{i+\alpha_1} - z_i x_{i+\alpha_3}, \\ y_i z_{i+\alpha_2} - z_i y_{i+\alpha_3}. \end{cases} \tag{2.3}$$

King [14] constructed the fine moduli space of θ-stable quiver representations using Geometric Invariant Theory. We can use his theorem to construct the moduli space of G-constellations.

Theorem 2.4 (King [14]). *Assume that a parameter $\theta \in \mathbb{Q}^I$ is generic. The quasiprojective scheme*

$$\mathcal{M}_\theta := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\operatorname{Rep} G]_{\chi_\theta^n}\right)$$

is a fine moduli space of θ-stable G-constellations. Moreover, the variety $\mathcal{M}_\theta$ is projective over $\mathcal{M}_0$.

Let $\mathcal{M}_\theta$ denote the moduli space of θ-stable G-constellations. Ito and Nakajima [8] showed that G-$\operatorname{Hilb}\mathbb{C}^3$ is isomorphic to $\mathcal{M}_\theta$ if θ is in the following set:

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for nontrivial } \rho \neq \rho_0\right\}. \tag{2.5}$$

Let Z be a G-orbit in the algebraic torus $\mathbf{T} := (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$. Then $\mathrm{H}^0(\mathcal{O}_Z)$ is isomorphic to $\mathbb{C}[G]$, thus it is a G-constellation. Moreover, since Z is a free G-orbit, $\mathcal{O}_Z$ has no nonzero proper submodules. Hence it follows that $\mathcal{O}_Z$ is θ-stable for any parameter θ. Thus for any parameter θ, there exists a natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$.

Craw, Maclagan and Thomas [4] proved the following theorem.

Theorem 2.6 (Craw, Maclagan and Thomas [4]). *Let $\theta \in \Theta$ be generic. Then $\mathcal{M}_\theta$ has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover Y_θ satisfies the following properties:*

(i) *Y_θ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^3/G$.*
(ii) *Y_θ is projective over the quotient variety $\mathbb{C}^3/G$.*

$$\begin{tikzcd} Y_\theta \arrow[r, hook, "\text{irr.}"'] \arrow[d] & \mathcal{M}_\theta \arrow[d] \\ \mathbb{C}^3/G \arrow[r, hook, "\text{closed}"'] & \mathcal{M}_0 \end{tikzcd}$$

Remark 2.7. We call the unique irreducible component Y_θ of $\mathcal{M}_\theta$ the *birational component*. For generic $\theta \in \Theta$, Craw, Maclagan and Thomas [4] constructed the birational component Y_θ as GIT quotient of a reduced irreducible affine scheme by an algebraic torus. From this, it follows that Y_θ is irreducible and reduced. ♦

2.2. **Abelian group actions and toric geometry.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, i.e. G is the subgroup generated by the diagonal matrix $\operatorname{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3})$ where ϵ is a primitive rth root of unity. The group G acts naturally on $S := \mathbb{C}[x, y, z]$. Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$$

which is an overlattice of $\overline{L} = \mathbb{Z}^3$ of finite index. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$. Set $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The dual lattices $\overline{M}$ and M can be identified with Laurent monomials and G-invariant Laurent monomials, respectively. The embedding of G into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee$$

where $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ is the character group of G. Note that M is the kernel of the map wt.

Remark 2.8. There are two isomorphisms of abelian groups $L/\mathbb{Z}^3 \to G$ and $\overline{M}/M \to G^\vee$. ♦

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.

$$\overline{M}_{\geq 0} = \left\{ x^{m_1} y^{m_2} z^{m_3} \in \overline{M} \,\middle|\, m_1, m_2, m_3 \geq 0 \right\}.$$

For a set $A \subset \mathbb{C}[x^\pm, y^\pm, z^\pm]$, let $\langle A \rangle$ denote the $\mathbb{C}[x, y, z]$-submodule of $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ generated by A.

Let σ_+ be the cone in $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ generated by e_1, e_2, e_3, i.e. $\sigma_+ := \mathrm{Cone}(e_1, e_2, e_3)$.

2.3. **Generalized G-graphs.**

Definition 2.9. A *(generalized) G-graph* Γ is a subset of Laurent monomials in $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ satisfying:

(i) $\mathbf{1} \in \Gamma$.
(ii) $\mathrm{wt}\colon \Gamma \to G^\vee$ is bijective, i.e. for each weight $\rho \in G^\vee$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ whose weight is ρ.
(iii) if $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.
(iv) Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x, y, z in Γ.

For any Laurent monomial $\mathbf{m} \in \overline{M}$, let $\mathrm{wt}_\Gamma(\mathbf{m})$ denote the unique element $\mathbf{m}_\rho$ in Γ whose weight is $\mathrm{wt}(\mathbf{m})$.

Example 2.10. Let G be the group of type $\frac{1}{7}(1, 3, 4)$. Then

$$\Gamma_1 = \left\{ 1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2} \right\},$$

$$\Gamma_2 = \left\{ 1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z}, \tfrac{y^3}{z^2} \right\}$$

are G-graphs. In Γ_1, $\mathrm{wt}_{\Gamma_1}(x) = \frac{z}{y}$ and $\mathrm{wt}_{\Gamma_1}(y^3) = \frac{z^2}{y^2}$. ♦

As is defined in [17], for a generalized G-graph $\Gamma = \{\mathbf{m}_\rho\}$, define $S(\Gamma)$ to be the subsemigroup of M generated by $\dfrac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for all

$\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. Define a cone $\sigma(\Gamma)$ in $L_{\mathbb{R}} = \mathbb{R}^3$ as follows:

$$\begin{aligned}\sigma(\Gamma) &= S(\Gamma)^\vee \\ &= \left\{ \mathbf{u} \in L_{\mathbb{R}} \,\middle|\, \left\langle \mathbf{u}, \frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)} \right\rangle \geq 0 \quad \forall \mathbf{m}_\rho \in \Gamma,\ \mathbf{m} \in \overline{M}_{\geq 0} \right\}.\end{aligned}$$

Observe that:

(i) $\sigma(\Gamma) \subset \sigma_+$,
(ii) $\left(\overline{M}_{\geq 0} \cap M\right) \subset S(\Gamma)$,
(iii) $S(\Gamma) \subset \left(\sigma(\Gamma)^\vee \cap M\right)$.

Define an affine toric open set:

$$U(\Gamma) := \operatorname{Spec} \mathbb{C}[S(\Gamma)].$$

Note that the torus $\operatorname{Spec} \mathbb{C}[M]$ of $U(\Gamma)$ is isomorphic to $(\mathbb{C}^\times)^3/G$.

Definition 2.11. A generalized G-graph Γ is called a *G-iraffe* if the open set $U(\Gamma)$ has a torus fixed point.

Example 2.12. For the G-graphs in Example 2.10,

$$\begin{aligned}\sigma(\Gamma_1) &= \left\{ \mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m} \rangle \geq 0,\ \text{for all } \mathbf{m} \in \{\tfrac{y^5}{z^2}, \tfrac{z^3}{y^4}, \tfrac{xy}{z}\} \right\}, \\ &= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(3,2,5), \tfrac{1}{7}(1,3,4) \right), \text{ and} \\ \sigma(\Gamma_2) &= \left\{ \mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m} \rangle \geq 0,\ \text{for all } \mathbf{m} \in \{\tfrac{y^4}{z^3}, \tfrac{z^4}{y^3}, \tfrac{xz^2}{y^3}\} \right\}, \\ &= \operatorname{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(6,4,3) \right).\end{aligned}$$

In both cases, they are G-iraffes. One can see that $S(\Gamma_1) = \sigma(\Gamma_1)^\vee \cap M$ and $S(\Gamma_2) = \sigma(\Gamma_2)^\vee \cap M$. ♦

Lemma 2.13. *Let Γ be a G-graph. Define*

$$B(\Gamma) := \left\{ \mathbf{f}\cdot\mathbf{m}_\rho \,\middle|\, \mathbf{m}_\rho \in \Gamma,\ \mathbf{f} \in \{x,y,z\} \right\} \backslash \Gamma.$$

Then the semigroup $S(\Gamma)$ is generated as a semigroup by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$. In particular, $S(\Gamma)$ is finitely generated as a semigroup.

Remark 2.14. From Lemma 2.13, we know that $U(\Gamma)$ is an affine toric variety. ♦

2.4. **G-graphs and local charts.** Let Γ be a G-graph. Define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle,$$

then it can be seen that $C(\Gamma)$ is a torus invariant G-constellation. Note that $C(\Gamma)$ can be realised as follows: $C(\Gamma)$ is the $\mathbb{C}$-vector space with

a basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x,y,z]$ and whose $\mathbb{C}[x,y,z]$-action is given by

$$\mathbf{m} * \mathbf{m}_\rho = \begin{cases} \mathbf{m}\cdot\mathbf{m}_\rho & \text{if } \mathbf{m}\cdot\mathbf{m}_\rho \in \Gamma, \\ 0 & \text{if } \mathbf{m}\cdot\mathbf{m}_\rho \notin \Gamma, \end{cases}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$.

Any submodule $\mathcal{G}$ of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a $\mathbb{C}$-basis of $\mathcal{G}$. We give a combinatorial description of submodules of $C(\Gamma)$.

Lemma 2.15. *Let A be a subset of Γ. The following are equivalent.*

(i) *The set A forms a $\mathbb{C}$-basis of a submodule of $C(\Gamma)$.*
(ii) *If $\mathbf{m}_\rho \in A$ and $\mathbf{f} \in \{x,y,z\}$, then $\mathbf{f}\cdot\mathbf{m}_\rho \in \Gamma$ implies $\mathbf{f}\cdot\mathbf{m}_\rho \in A$.*

Let p be a point in $U(\Gamma)$. Then there exists the evaluation map

$$\mathrm{ev}_p \colon S(\Gamma) \to (\mathbb{C}, \times),$$

which is a semigroup homomorphism.

To assign a G-constellation $C(\Gamma)_p$ to the point p of $U(\Gamma)$, firstly consider the $\mathbb{C}$-vector space with basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x,y,z]$. Endow it with the following $\mathbb{C}[x,y,z]$-action,

$$\mathbf{m} * \mathbf{m}_\rho := \mathrm{ev}_p\left(\frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)}\right)\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho), \tag{2.16}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and an element $\mathbf{m}_\rho$ in Γ.

Definition 2.17. A G-graph is said to be *θ-stable* if the G-constellation $C(\Gamma)$ is θ-stable.

Proposition 2.18. *Let Γ be a G-iraffe, that is, $U(\Gamma)$ has a torus fixed point. Let Y_θ be the birational component in $\mathcal{M}_\theta$. For a generic θ, assume that $C(\Gamma)$ is θ-stable. Then $C(\Gamma)_p$ is θ-stable for any $p \in U(\Gamma)$. Thus there exists an open immersion*

$$U(\Gamma) = \operatorname{Spec}\mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta.$$

3. Weighted blowups and economic resolutions

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1,a,r-a)$ with a coprime to r, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^a, \epsilon^{r-a})$ where ϵ is a primitive rth root of unity. The quotient variety $X = \mathbb{C}^3/G$ has terminal singularities and has no crepant resolution. However, there exist a special kind of toric resolutions, which can be obtained by a sequence of weighted blowups. In this section, we review the notion of toric weighted blowups and define round down functions which are used for finding admissible G-iraffes.

3.1. **Weighted blowups and round down functions.** Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r-a)$ and set $\overline{L} = \mathbb{Z}^3 \subset L$. Consider two dual lattices $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ and $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$. Note that a (Laurent) monomial $\mathbf{m} \in \overline{M}$ is invariant under G if and only if $\mathbf{m}$ is in M. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and σ_+ the cone in $L_{\mathbb{R}}$ generated by e_1, e_2, e_3. Then $\mathrm{Spec}\,\mathbb{C}[\sigma_+^\vee \cap M]$ is the quotient variety $X = \mathbb{C}^3/G$. Set $v = \frac{1}{r}(1, a, r-a) \in L$, which corresponds to the exceptional divisor of the smallest discrepancy. (see Proposition 3.7). Define three cones

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v),$$

and define Σ to be the fan consisting of the three cones $\sigma_1, \sigma_2, \sigma_3$ and their faces. The fan Σ is the barycentric subdivision of σ_+ at v. Let Y_1 be the toric variety corresponding to the fan Σ together with the lattice L. Define $\varphi\colon Y_1 \to X$ to be the induced toric morphism, which is called *the weighted blowup* of X with weight $(1, a, r-a)$.

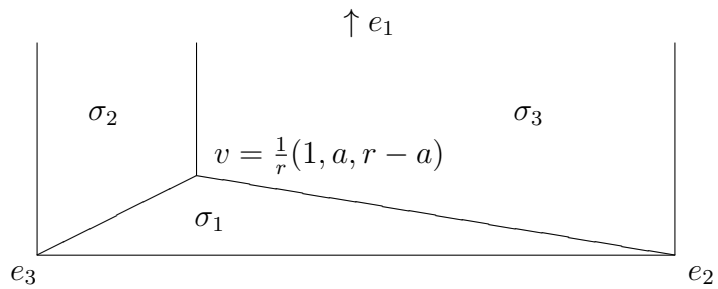

FIGURE 3.1. Weighted blowup of weight $(1, a, r-a)$

Let us consider the sublattice L_2 of L generated by e_1, v, e_3 and let us define $M_2 := \mathrm{Hom}_{\mathbb{Z}}(L_2, \mathbb{Z})$ with dual basis

$$\xi := xy^{-\frac{1}{a}}, \quad \eta := y^{\frac{r}{a}}, \quad \zeta := y^{\frac{a-r}{a}} z.$$

The lattice inclusion $L_2 \hookrightarrow L$ induces a toric morphism

$$\varphi\colon \mathrm{Spec}\,\mathbb{C}[\sigma_2^\vee \cap M_2] \to U_2 := \mathrm{Spec}\,\mathbb{C}[\sigma_2^\vee \cap M].$$

Since $\mathbb{C}[\sigma_2^\vee \cap M_2] \cong \mathbb{C}[\xi, \eta, \zeta]$ and the group $G_2 := L/L_2$ is of type $\frac{1}{a}(1, \overline{-r}, \overline{r-a})$ with eigencoordinates ξ, η, ζ, the open subset U_2 has a quotient singularity of type $\frac{1}{a}(1, \overline{-r}, \overline{r-a})$. Note that for $x^{m_1} y^{m_2} z^{m_3} \in \overline{M}_{\geq 0}$,

$$\varphi^*(x^{m_1} y^{m_2} z^{m_3}) = \xi^{m_1} \eta^{\frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3} \zeta^{m_3}.$$

Similarly, consider the sublattice L_3 of L generated by e_1, e_2, v. Let us define the lattice $M_3 := \mathrm{Hom}_{\mathbb{Z}}(L_3, \mathbb{Z})$ with basis

$$\xi_3 := xz^{-\frac{1}{r-a}}, \quad \eta_3 := yz^{\frac{-a}{r-a}}, \quad \zeta_3 := z^{\frac{r}{r-a}}.$$

Note that the open set $U_3 = \operatorname{Spec} \mathbb{C}[\xi_3, \eta_3, \zeta_3]$ has a singularity of type $\frac{1}{r-a}(1, \overline{a}, \overline{r-2a})$ with eigencoordinates ξ_3, η_3, ζ_3 with $G_2 := L/L_3$.

Lastly, consider the sublattice L_1 of L generated by v, e_2, e_3. Let us define $M_1 := \operatorname{Hom}_{\mathbb{Z}}(L_1, \mathbb{Z})$ with dual basis

$$\xi_1 := xz^{-\frac{1}{r-a}}, \quad \eta_1 := yz^{\frac{-a}{r-a}}, \quad \zeta_1 := z^{\frac{r}{r-a}}.$$

Since $\{v, e_2, e_3\}$ forms a $\mathbb{Z}$-basis of L, i.e. $G_1 = L/L_1$ is the trivial group, the open set $U_1 = \operatorname{Spec} \mathbb{C}[\xi_1, \eta_1, \zeta_1]$ is smooth.

Example 3.1. Let G be the group of type $\frac{1}{7}(1, 3, 4)$ as in Example 2.10. The fan of the weighted blowup of weight $(1, 3, 4)$ is shown in Figure 3.2.

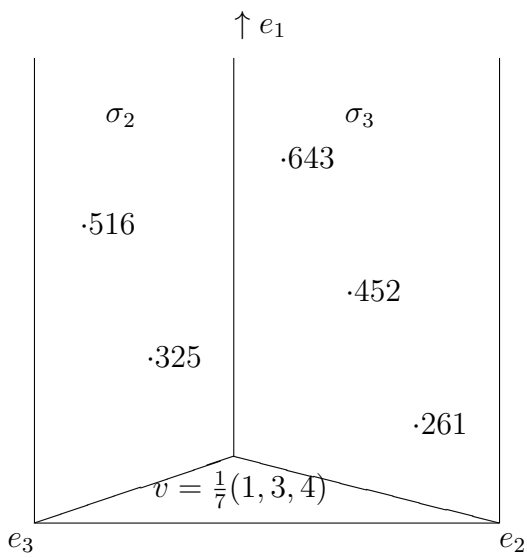

FIGURE 3.2. Weighted blowup of weight $(1, 3, 4)$

Let U_2 be the affine toric variety corresponding to the cone σ_2 on the left side of $v = \frac{1}{7}(1, 3, 4)$. Note that U_2 has a quotient singularity of type $\frac{1}{3}(1, 2, 1)$ with eigencoordinates $xy^{-\frac{1}{3}}, y^{\frac{7}{3}}, y^{-\frac{4}{3}}z$.

Let U_3 be the affine toric variety corresponding to the cone σ_3 on the left side of $v = \frac{1}{7}(1, 3, 4)$. Note that U_3 has a quotient singularity of type $\frac{1}{3}(1, 2, 1)$ with eigencoordinates $xz^{-\frac{1}{4}}, yz^{-\frac{3}{4}}, z^{\frac{7}{4}}$.

On the other hand, e_2, e_3, v form a $\mathbb{Z}$-basis of L, so that the affine toric variety corresponding to the cone generated by v, e_2, e_3 is smooth. ♦

Definition 3.2 (Round down functions)**.** With the notation above, the *left round down function* $\phi_2 \colon \overline{M} \to M_2$ of the weighted blowup with weight $(1, a, r-a)$ is defined by

$$\phi_2(x^{m_1} y^{m_2} z^{m_3}) = \xi^{m_1} \eta^{\lfloor \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \rfloor} \zeta^{m_3}.$$

where $\lfloor\ \rfloor$ is round down. In a similar manner, the *right round down function* $\phi_3\colon \overline{M} \to M_3$ of the weighted blowup with weight $(1, a, r-a)$ is defined by

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \xi_3^{m_1}\eta_3^{m_2}\zeta_3^{\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3\rfloor},$$

and the *central round down function* $\phi_1\colon \overline{M} \to M_1$ of the weighted blowup with weight $(1, a, r-a)$ by

$$\phi_1(x^{m_1}y^{m_2}z^{m_3}) = \xi_1^{\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3\rfloor}\eta_1^{m_2}\zeta_1^{m_3}.$$

Remark 3.3. Let ϕ_k be a round down function of the weighted blowup with weight $(1, a, r-a)$ as above for $k = 1, 2, 3$. For $\mathbf{m} \in \overline{M}$ and $\mathbf{n} \in M$, we have

$$\phi_k(\mathbf{m}\cdot\mathbf{n}) = \phi_k(\mathbf{m})\cdot\mathbf{n},$$

because M_k contains M as the lattice of G_k invariant monomials, especially, $\mathbf{n}$ is in M_k. Thus the weight of $\phi_k(\mathbf{m}\cdot\mathbf{n})$ and the weight of $\phi_k(\mathbf{m})$ are the same in terms of the G_k action. ♦

Remark 3.4. Davis, Logvinenko, and Reid [6] introduce a related construction in a more general setting. ♦

Lemma 3.5. *Let ϕ_k be a round down function of the weighted blowup with weight $(1, a, r-a)$ as above for $k = 1, 2, 3$. Let $\mathbf{m} \in \overline{M}$ be a Laurent monomial of weight j. Then we have the following:*

(i) $\phi_2(y\cdot\mathbf{m}) = \phi_2(\mathbf{m})$, *when* $0 \le j < r-a$.
(ii) $\phi_3(z\cdot\mathbf{m}) = \phi_3(\mathbf{m})$, *when* $0 \le j < a$.
(iii) $\phi_1(x\cdot\mathbf{m}) = \phi_1(\mathbf{m})$, *when* $0 \le j < r-1$.

Proof. Let $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3}$ be a Laurent monomial of weight j. To prove (i), assume that $0 \le j < r-a$. This means that

$$0 \le \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \lfloor\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3\rfloor < \frac{r-a}{r}.$$

Thus $\phi_2(y\cdot\mathbf{m}) = \phi_2(x^{m_1}y^{m_2+1}z^{m_3}) = \phi_2(x^{m_1}y^{m_2}z^{m_3})$.

The assertions (ii) and (iii) can be proved similarly. □

3.2. **Economic resolutions.** Let $v_i := \frac{1}{r}(i, \overline{ai}, \overline{r-ai})$ be a lattice point in L for each $1 \le i \le r-1$. The quotient variety $X = \mathbb{C}^3/G$ has a certain toric resolution which was introduced by Danilov [5] (see [19]).

Definition 3.6. For the group $G \subset \mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(1, a, r-a)$, the *economic resolution* of $\mathbb{C}^3/G$ is the toric variety obtained by the consecutive weighted blowups $v_1, v_2, \ldots, v_{r-1}$ from the quotient variety $X = \mathbb{C}^3/G$.

Let $\varphi\colon Y \to X = \mathbb{C}^3/G$ be the economic resolution. Let E_i denote the exceptional divisor of φ corresponding to the lattice point v_i for each $1 \le i < r$. From toric geometry, we have the following proposition (see [19]).

Proposition 3.7. *With the notation as above, the economic resolution Y has the following properties:*

(i) *Y is smooth and projective over X.*
(ii) *$K_Y = \varphi^*(K_X) + \sum_{1\le i<r} \frac{i}{r} E_i$. In particular, each discrepancy is $0 < \frac{i}{r} < 1$.*

Remark 3.8. From the fan of Y, we can see that Y can be covered by three open sets U_2, U_3 and U_1, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r-a)$, respectively. Note that U_2 and U_3 are isomorphic to the economic resolutions for the singularity of $\frac{1}{a}(1, \overline{-r}, \overline{r-a})$, of $\frac{1}{r-a}(1, \overline{a}, \overline{-r})$, respectively. ♦

Example 3.9. Let G be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.10. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.

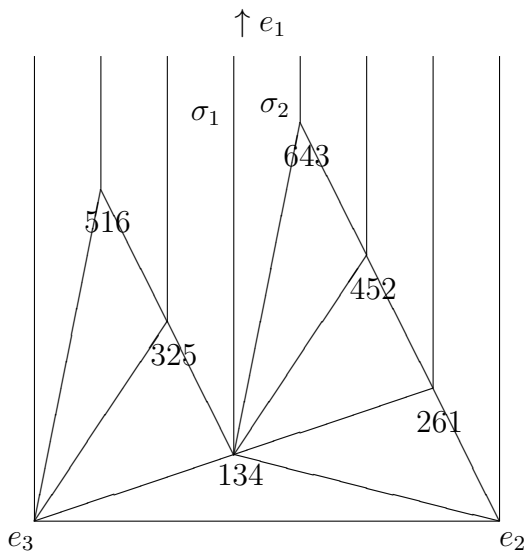

FIGURE 3.3. Fan of the economic resolution for $\frac{1}{7}(1,3,4)$

Let U_2 be the toric variety corresponding to the fan consisting of the cones on the left side of $v = \frac{1}{7}(1,3,4)$. Note that U_2 is the economic resolution of the quotient $\frac{1}{3}(1,2,1)$ which is G_2-Hilb $\mathbb{C}^3$, where G_2 is of type $\frac{1}{3}(1,2,1)$.

Let U_3 be the toric variety corresponding to the fan consisting of the cones on the right side of $v = \frac{1}{7}(1,3,4)$. Note that U_3 is the economic resolution of the quotient $\frac{1}{4}(1,3,1)$ which is G_3-Hilb $\mathbb{C}^3$, where G_3 is of type $\frac{1}{4}(1,3,1)$. ♦

4. Moduli interpretations of economic resolutions

This chapter contains our main theorem. Section 5 explains how to find an admissible set $\mathfrak{G}$ of G-iraffes. To find G-iraffes, we use the round down functions introduced in Section 3.1. Section A describes the universal families over the birational component Y_θ using G-iraffes. In Section 6, we show that there exists a stability parameter θ such that G-iraffes in $\mathfrak{G}$ are θ-stable.

5. How to find admissible G-iraffes

5.1. **G-iraffes for $\frac{1}{r}(1, r-1, 1)$.** Let G be the finite subgroup in $\mathrm{GL}_3(\mathbb{C})$ of $\frac{1}{r}(1, r-1, 1)$ type, i.e. $a = 1$ or $r-1$. Kędzierski [11] proved that for $G \subset \mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(1, r-1, 1)$, G-$\mathrm{Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$.

Theorem 5.1 (Kędzierski [11])**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with $a = 1$ or $r-1$. Then G-$\mathrm{Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$. In particular, G-$\mathrm{Hilb}\,\mathbb{C}^3$ is nonsingular and irreducible.*

For each $0 \le i \le r$, set $v_i = \frac{1}{r}(i, r-i, i)$. The fan corresponding to G-$\mathrm{Hilb}\,\mathbb{C}^3$ consists of the following $2r-1$ maximal cones and their faces:

$$\begin{aligned} \sigma_i &= \mathrm{Cone}(e_1, v_{i-1}, v_i) && \text{for } 1 \le i \le r, \\ \sigma_{r+i} &= \mathrm{Cone}(e_3, v_{i-1}, v_i) && \text{for } 1 \le i \le r-1. \end{aligned}$$

Each maximal cone has a corresponding (Nakamura) G-graph:

$$\begin{aligned} \Gamma_i &= \{1, y, y^2, \dots, y^{i-1}, z, z^2, \dots, z^{r-i}\} && \text{for } 1 \le i \le r, \\ \Gamma_{r+i} &= \{1, y, y^2, \dots, y^{i-1}, x, x^2, \dots, x^{r-i}\} && \text{for } 1 \le i \le r-1, \end{aligned}$$

with $S(\Gamma_j) = \sigma_j^\vee \cap M$ for $1 \le j \le 2r-1$. From the fact that each cone σ_j is 3-dimensional, it is immediate that these G-graphs are G-iraffes.

Example 5.2. Let G be the finite group of type $\frac{1}{2}(1, 1, 1)$. Set $v = \frac{1}{2}(1, 1, 1)$. Note that the economic resolution Y of $X = \mathbb{C}^3/G$ is the weighted blowup of X with weight $(1, 1, 1)$. Then the maximal cones of Y are

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v),$$

and the corresponding G-iraffes Γ_i to σ_i are

$$\Gamma_1 = \{1, x, x^2\}, \qquad \Gamma_2 = \{1, y, y^2\}, \qquad \Gamma_3 = \{1, z, z^2\}.$$

Let us consider the left round down function ϕ_2, the right round down function ϕ_3 and the central round down function ϕ_1 corresponding to

the weighted blowup with weight $(1,1,1)$. Then

$$\begin{aligned}\Gamma_1 &= \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_1(\mathbf{m}) = \mathbf{1}\right\},\\ \Gamma_2 &= \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) = \mathbf{1}\right\},\\ \Gamma_3 &= \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_3(\mathbf{m}) = \mathbf{1}\right\}.\end{aligned}$$

♦

Example 5.3. Let G be the finite group of type $\frac{1}{3}(1,2,1)$. Set $v_1 = \frac{1}{3}(1,2,1)$ and $v_2 = \frac{1}{3}(2,1,2)$. In this example, let ξ, η, ζ be the coordinates of $\mathbb{C}^3$. Note that the economic resolution Y of $X = \mathbb{C}^3/G$ can be obtained by the sequence of the weighted blowups:

$$Y \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} X,$$

where φ_1 is the weighted blowup with weight $(1,2,1)$ and φ_2 is the toric morphism induced by the weighted blowup with weight $(2,1,2)$. The fan corresponding to Y consists of the following five maximal cones and their faces:

$$\begin{aligned}&\sigma_1 = \mathrm{Cone}(e_1, e_3, v_2), \quad \sigma_2 = \mathrm{Cone}(e_1, v_2, v_1), \quad \sigma_2 = \mathrm{Cone}(e_1, v_1, e_2),\\ &\sigma_4 = \mathrm{Cone}(e_3, e_2, v_1), \quad \sigma_5 = \mathrm{Cone}(e_3, v_1, v_2).\end{aligned}$$

The following

$$\begin{aligned}&\Gamma_1 = \{1, \eta, \eta^2\}, \quad \Gamma_2 = \{1, \eta, \zeta\}, \quad \Gamma_3 = \{1, \zeta, \zeta^2\},\\ &\Gamma_4 = \{1, \xi, \xi^2\}, \quad \Gamma_5 = \{1, \xi, \eta\}.\end{aligned}$$

are their corresponding G-iraffes. ♦

5.2. **G-iraffes for $\frac{1}{r}(1, a, r-a)$.** In this section, we assign a G-iraffe Γ_σ for each full dimensional cone in the fan of Y with $S(\Gamma_\sigma) = \sigma^\vee \cap M$

Let X be the quotient variety $\mathbb{C}^3/G$ where $G \subset \mathrm{GL}_3(\mathbb{C})$ is the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. Let $\varphi\colon Y \to X$ be the economic resolution of X. Then Y can be covered by U_2, U_3 and U_1, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r-a)$, respectively.

Assume σ is a full dimensional cone in the fan of Y. We have three cases:

(1) the cone σ is below the vector v.
(2) the cone σ is on the left side of the vector v.
(3) the cone σ is on the right side of the vector v.

Case (1) the cone σ is below the vector v. This means that the toric cone σ is smooth and that the toric affine open set U_σ is equal to U_1. Then consider the central round down function ϕ_1 of the weighted blowup with weight $(1, a, r-a)$. Now, for $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3} \in \overline{M}$

$$\phi_1(\mathbf{m}) = \mathbf{1} \quad \text{if and only if} \quad 0 \le m_1 \le r-1 \text{ and } m_2 = m_3 = 0.$$

Thus the set $\Gamma := \phi_1^{-1}(\mathbf{1}) = \{1, x, x^2, \dots, x^{r-1}\}$ is a G-graph with $S(\Gamma) = \sigma^\vee \cap M$. Since the corresponding cone $\sigma(\Gamma)$ of Γ is equal to σ, Γ is a G-iraffe.

Case (2) the cone σ is on the left side of v. Consider the left round down function ϕ_2. From the fan of the economic resolution, it follows that U_2 is isomorphic to the economic resolution Y_2 for the group $G_2 = \frac{1}{a}(1, -r, r)$ with eigencoordinates ξ, η, ζ. There exists a unique full dimensional cone σ' in the fan of Y_2.

Lemma 5.4. *Let σ be a full dimensional cone in the toric fan of Y on the left side of the lattice v and σ' the corresponding full dimensional cone in the fan of Y_2, where Y_2 is the economic resolution for the group $G_2 = \frac{1}{a}(1, -r, r)$. Assume that there exists a G_2-graph Γ' such that $S(\Gamma') = (\sigma')^\vee \cap M$. Define a set*

$$\Gamma := \left\{ \mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) \in \Gamma' \right\}.$$

Then Γ is a G-graph.

Proof. Firstly note that $\mathbf{1} \in \Gamma$ since $\phi_2(\mathbf{1}) = \mathbf{1} \in \Gamma'$. To show that Γ satisfies the second condition in Definition 2.9, let $\rho \in G^\vee$ be an irreducible representation of G. We have to show that there exists a unique monomial of weight ρ in Γ. Then there exists a positive integer i such that the weight of x^i is ρ. Consider the monomial $\phi_2(x^j)$ in M_2 and its weight χ in terms of the G_2-action. Since Γ' is a G_2-graph, there exists a unique element $\mathbf{k}_\chi$ whose weight is the same as the weight of $\phi_2(x^j)$. Then $\left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)}\right)$ is in the G_2-invariant monomial lattice M, so it is in the monomial lattice $\overline{M}$. From Remark 3.3, it follows that

$$\phi_2 \colon x^j \cdot \left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)}\right) \longmapsto \mathbf{k}_\chi,$$

i.e. $x^j \cdot \left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)}\right)$ is in Γ. To show uniqueness, assume that two Laurent monomials $\mathbf{m}, \mathbf{n}$ of the same weights are mapped into Γ'. From the fact that the weights of $\phi_2(\mathbf{m})$ and $\phi_2(\mathbf{n})$ are equal, it follows that $\phi_2(\mathbf{m}) = \phi_2(\mathbf{n})$. From Remark 3.3,

$$\phi_2(\mathbf{m}) = \phi_2\left(\mathbf{n} \cdot \frac{\mathbf{m}}{\mathbf{n}}\right) = \phi_2(\mathbf{n}) \cdot \frac{\mathbf{m}}{\mathbf{n}},$$

and hence $\mathbf{m} = \mathbf{n}$.

Lastly, to show Γ is connected, let $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3} \in \overline{M}$ be an arbitrary element in Γ, i.e. $\mathbf{k}_\chi := \phi_2(\mathbf{m}) \in \Gamma'$. Consider the following six cases:

(A) Suppose $\xi \cdot \mathbf{k}_\chi$ is in Γ', but $\xi \cdot \mathbf{k}_\chi \neq \phi_2(x \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor + \frac{r-1}{r}.$$

From this equation, it is easy to show that $\phi_2(\frac{\mathbf{m}}{y}) = \mathbf{k}_\chi$ and $\phi_2(x \cdot \frac{\mathbf{m}}{y}) = \xi \cdot \mathbf{k}_\chi$. Hence, we can see that there is a path from $\mathbf{m}$ to $x \cdot \frac{\mathbf{m}}{y}$ in Γ and that $\phi_2(x \cdot \frac{\mathbf{m}}{y}) = \xi \cdot \mathbf{k}_\chi$.

(B) Suppose $\frac{\mathbf{k}_\chi}{\xi}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\xi} \neq \phi_2(\frac{\mathbf{m}}{x})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 + \frac{r-1}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this equation, it is easy to see that $\phi_2(y \cdot \mathbf{m}) = \mathbf{k}_\chi$ and $\phi_2(\frac{y \cdot \mathbf{m}}{x}) = \frac{\mathbf{k}_\chi}{\xi}$. Hence, there is a path from $\mathbf{m}$ to $\frac{y \cdot \mathbf{m}}{x}$ in Γ and $\phi_2(\frac{y \cdot \mathbf{m}}{x}) = \frac{\mathbf{k}_\chi}{\xi}$.

(C) Suppose $\eta \cdot \mathbf{k}_\chi$ is in Γ', but $\eta \cdot \mathbf{k}_\chi \neq \phi_2(y \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{r-a}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this, it is easy to show that there exists a positive integer k_0 such that $\phi_2(y^k \cdot \mathbf{m}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2(y^{k_0+1} \cdot \mathbf{m}) = \eta \cdot \mathbf{k}_\chi$. Hence, we can see that there is a path from $\mathbf{m}$ to $y^{k_0+1} \cdot \mathbf{m}$ in Γ and we get $\phi_2\big(y^{k_0+1} \cdot \mathbf{m}\big) = \eta \cdot \mathbf{k}_\chi$.

(D) Suppose $\frac{\mathbf{k}_\chi}{\eta}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\eta} \neq \phi_2(\frac{\mathbf{m}}{y})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this, it is easy to see that there exists a positive integer k_0 such that $\phi_2(\frac{\mathbf{m}}{y^k}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2\left(\frac{\mathbf{m}}{y^{k_0+1}}\right) = \frac{\mathbf{k}_\chi}{\eta}$. Hence, there is a path from $\mathbf{m}$ to $\frac{\mathbf{m}}{y^{k_0+1}}$ in Γ and $\phi_2\left(\frac{\mathbf{m}}{y^{k_0+1}}\right) = \frac{\mathbf{k}_\chi}{\eta}$.

(E) Suppose $\zeta \cdot \mathbf{k}_\chi$ is in Γ', but $\zeta \cdot \mathbf{k}_\chi \neq \phi_2(z \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this, it is easy to see that there exists a positive integer k_0[2] such that $\phi_2(\frac{\mathbf{m}}{y^k}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and

[2]This integer k_0 is the maximal integer satisfying

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r}k \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

$\phi_2\Big(\frac{\mathbf{m}}{y^{k_0+1}}\Big) \neq \mathbf{k}_\chi$. Moreover, $\phi_2(z \cdot \frac{\mathbf{m}}{y^{k_0}}) = \zeta \cdot \mathbf{k}_\chi$. Hence, there is a path from $\mathbf{m}$ to $z \cdot \frac{\mathbf{m}}{y^{k_0}}$ in Γ and $\phi_2(z \cdot \frac{\mathbf{m}}{y^{k_0}}) = \zeta \cdot \mathbf{k}_\chi$.

(F) Suppose $\frac{\mathbf{k}_\chi}{\zeta}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\zeta} \neq \phi_2(\frac{\mathbf{m}}{z})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{r-a}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this, it is easy to see that there exists a positive integer k_0 such that $\phi_2(y^k \cdot \mathbf{m}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2\Big(y^{k_0+1} \cdot \mathbf{m}\Big) \neq \mathbf{k}_\chi$. Moreover, $\phi_2(\frac{y^{k_0} \cdot \mathbf{m}}{z}) = \frac{\mathbf{k}_\chi}{\zeta}$. From this, it follows that there is a path from $\mathbf{m}$ to $\frac{y^{k_0} \cdot \mathbf{m}}{z}$ in Γ and that $\phi_2(\frac{y^{k_0} \cdot \mathbf{m}}{z}) = \frac{\mathbf{k}_\chi}{\zeta}$.

□

In proving Lemma 5.4, we have also proved the following lemma.

Lemma 5.5. *With the notation as above, for a monomial* $\mathbf{k} \in \{\xi, \eta, \zeta\}$ *of degree 1 and any* $\mathbf{k}_\chi \in \Gamma'$*, there exist a monomial* $\mathbf{f} \in \{x, y, z\}$ *of degree 1 and an element* $\mathbf{m}_\rho \in \Gamma$ *such that*

$$\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho) = \mathbf{k} \cdot \mathbf{k}_\chi$$

with $\phi_2(\mathbf{m}_\rho) = \mathbf{k}_\chi$.

From Remark 3.3, it can be shown that

$$\mathrm{wt}_{\Gamma'}\big(\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho)\big) = \phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)\big),$$

as they are elements in Γ' of the same weight.

Remark 5.6. By Lemma 3.5, it can be seen that if a Laurent monomial $\mathbf{m}_\rho$ of weight j is in Γ with $0 \leq j < r - a$, then $y \cdot \mathbf{m}_\rho$ is in Γ. ♦

Proposition 5.7. *With notation and assumptions as for Lemma 5.4, for the G-graph* Γ*, we have* $S(\Gamma) = S(\Gamma')$*. In particular,* Γ *is a G-iraffe with* $S(\Gamma) = \sigma^\vee \cap M$.

Proof. Note that $S(\Gamma)$ is generated by $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$. Let $\mathbf{m}$ be a genuine monomial in $\overline{M}_{\geq 0}$ and $\mathbf{m}_\rho$ an element in Γ. From the definition of Γ, it follows that $\phi_2(\mathbf{m}_\rho)$ is in Γ', which is denoted by $\mathbf{k}_\chi \in \Gamma'$. Set $\mathbf{k}$ to be $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathbf{m}_\rho}$. It is easy to see that $\mathbf{k}$ is a genuine monomial in ξ, η, ζ because of the definition of the left round down function. Since $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ is G-invariant, from Remark 3.3, we have

$$\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} = \frac{\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)\big)} = \frac{\frac{\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho)}{\phi_2(\mathbf{m}_\rho)} \cdot \phi_2(\mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)\big)} = \frac{\mathbf{k} \cdot \mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k} \cdot \mathbf{k}_\chi)},$$

so we prove $S(\Gamma) \subset S(\Gamma')$. For the reverse inclusion, let $\frac{\mathbf{k}\cdot\mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)}$ be a generator of $S(\Gamma')$ with $\mathbf{k} \in \{\xi, \eta, \zeta\}$. It is sufficient to show that this generator is in $S(\Gamma)$. From Lemma 5.5, we can find $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$ satisfying $\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho) = \mathbf{k}\cdot\mathbf{k}_\chi$. Note that $\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi) = \phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)$. Thus we have

$$\frac{\mathbf{k}\cdot\mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)} = \frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\mathrm{wt}_{\Gamma'}\big(\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)\big)} = \frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)} = \frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)},$$

and we proved the proposition. □

Case (3) the cone σ is on the right side of v. We can get a similar result.

Corollary 5.8. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X = \mathbb{C}^3/G$. Then there exists a set $\mathfrak{G}$ of G-iraffes such that there is a bijective map $\Sigma_{\max} \to \mathfrak{G}$ sending σ to Γ_σ satisfying $S(\Gamma_\sigma) = \sigma^\vee \cap M$. In particular, $U(\Gamma)$ is smooth for $\Gamma \in \mathfrak{G}$.*

Proof. Note that the assertion holds if $a = 1$ or $r-1$ by Section 5.1. We use induction on r and a.

Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X = \mathbb{C}^3/G$ and σ an arbitrary element of $\Sigma_{\max}$. Then σ is either on the left side of the lattice $v = \frac{1}{r}(1, a, r-a)$, the right side of v, or below v.

For the case where σ is below v, define

$$\Gamma_\sigma := \{1, x, x^2, \ldots, x^{r-2}, x^{r-1}\}.$$

Then we have seen that Γ_σ is a G-iraffe with $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

If the cone σ is on the left side of v, then we have a unique 3-dimensional cone σ' in the fan of the economic resolution of $\frac{1}{a}(1, \overline{-r}, \overline{r})$ where $^{-}$ denotes the residue modulo a. Note that $\overline{-r}$ is strictly less than a. Using induction and Proposition 5.7, we prove that there exists a G-iraffe Γ_σ satisfying $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

The case where the cone σ is on the right side of v can be proved similarly. □

Example 5.9. Let G be the group of type $\frac{1}{7}(1, 3, 4)$ as in Example 2.10. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.

Let us define the following cones:

$$\sigma_1 := \mathrm{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(3,2,5)\right),$$

$$\sigma_2 := \mathrm{Cone}\left((1,0,0), \tfrac{1}{7}(6,4,3), \tfrac{1}{7}(1,3,4)\right).$$

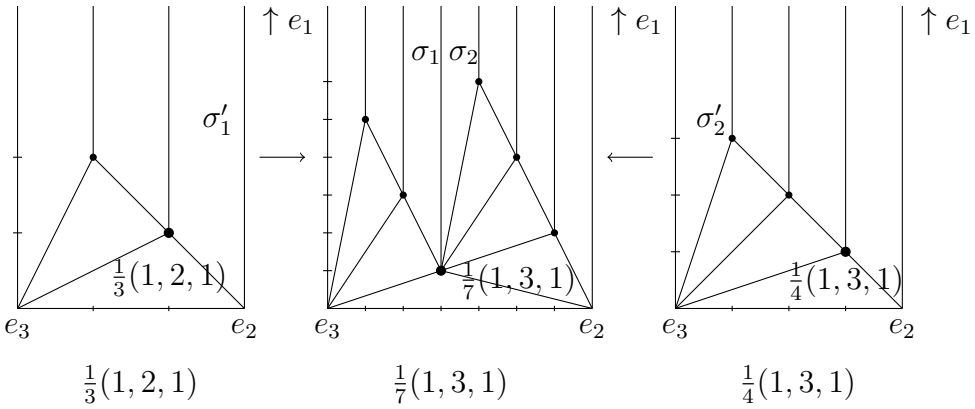

FIGURE 5.1. Recursion process for $\frac{1}{7}(1,3,4)$

We now calculate G-graphs associated to the cones σ_1 and σ_2. Note that the left side of the fan is the economic resolution of the quotient variety $\frac{1}{3}(1,2,1)$ which is G_2-Hilb $\mathbb{C}^3$, where G_2 is of type $\frac{1}{3}(1,2,1)$. Call the eigencoordinates ξ, η, ζ. Let σ_1' be the cone in the fan of G_2-Hilb $\mathbb{C}^3$ which corresponds to σ_1. Observe that the corresponding G_2-graph Γ_1' is

$$\Gamma_1' = \{1, \zeta, \zeta^2\},$$

and that the left round down function ϕ_2 is

$$\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor}\zeta^{m_3}.$$

Thus G-graph Γ_1 corresponding to σ_1 is

$$\begin{aligned}\Gamma_1 &\stackrel{\text{def}}{=} \{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \mid \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma_1'\} \\ &= \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\}.\end{aligned}$$

For the cone σ_2, note that the right side of the fan is the economic resolution of the quotient variety $\frac{1}{4}(1,3,1)$ which is G_3-Hilb $\mathbb{C}^3$, where G_3 is of type $\frac{1}{4}(1,3,1)$. Call the eigencoordinates α, β, γ. Let σ_2' be the cone in the fan of G_2-Hilb $\mathbb{C}^3$ which corresponds to σ_2. Observe that the corresponding G_3-graph Γ_2' is

$$\Gamma_2' = \{1, \beta, \beta^2, \beta^3\},$$

and that the right round down function ϕ_3 is

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \alpha^{m_1}\beta^{m_2}\gamma^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor}.$$

Thus the G-graph Γ_2 corresponding to σ_2 is

$$\begin{aligned}\Gamma_2 &\stackrel{\text{def}}{=} \{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \mid \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma_2'\} \\ &= \{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z^1}, \tfrac{y^3}{z^2}\}.\end{aligned}$$

From Example 2.12, $\sigma(\Gamma_1) = \sigma_1$ and $\sigma(\Gamma_2) = \sigma_2$. ♦

6. A chamber in the stability parameter space

This section proves that there exists a chamber $\mathfrak{C}$ such that the admissible G-iraffes in Section 5 are θ-stable for $\theta \in \mathfrak{C}$. In addition, we prove that the chamber $\mathfrak{C}$ coincides with the cone Kędzierski found and that the chamber is an open Weyl chamber. Moreover, it turns out that this chamber is a full chamber, i.e. the facets of $\mathfrak{C}$ form actually walls.

6.1. **Admissible chambers.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. We may assume $2a < r$. Let G_2 and G_3 be the groups of type $\frac{1}{a}(1, \overline{-r}, \overline{r})$ and of type $\frac{1}{r-a}(1, \overline{r}, \overline{-r})$, respectively. Note that for $k = 2$ or 3, the round down function ϕ_k induces a surjection $\phi_k \colon G^\vee \to G_k^\vee$.

The stability parameter space for G_k-constellations is

$$\Theta_k = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}\left(R(G_k), \mathbb{Q}\right) \,\middle|\, \theta\big(\mathbb{C}[G_k]\big) = 0\right\}$$

where $R(G_k)$ is the representation ring of G_k, i.e. $R(G_k) = \bigoplus_{\chi \in G_k^\vee} \mathbb{Z}\chi$. Let us assume that there exists a stability parameter $\theta^{(k)} \in \Theta_k$ such that the admissible G_k-graphs are $\theta^{(k)}$-stable. Take a GIT parameter $\theta_P \in \Theta$ satisfying the following system of linear equations:

$$\begin{cases} \theta^{(2)}(\chi) = \theta\big(\phi_2^{-1}(\chi)\big) & \text{for all } \chi \in G_2^\vee, \\ \theta^{(3)}(\chi') = \theta\big(\phi_3^{-1}(\chi')\big) & \text{for all } \chi' \in G_3^\vee. \end{cases} \tag{6.1}$$

Let us define a GIT parameter $\vartheta \in \Theta$ to be

$$\vartheta(\rho) = \begin{cases} -1 & \text{if } 0 \le \mathrm{wt}(\rho) < a\ , \\ 1 & \text{if } r-a \le \mathrm{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Note that $\vartheta\big(\phi_k^{-1}(\chi)\big) = 0$ for any $\chi \in G_k^\vee$ [3]. For a sufficiently large natural number m, set

$$\theta := \theta_P + m\vartheta. \tag{6.3}$$

We claim that the admissible G-iraffes are θ-stable.

Lemma 6.4. *Let θ be the parameter as above. For the set $\mathfrak{G}$ in Corollary 5.8, if Γ is in $\mathfrak{G}$, then Γ is θ-stable.*

Proof. Let Γ be a G-iraffe in $\mathfrak{G}$ and σ the corresponding cone to Γ. It suffices to show that $C(\Gamma)$ is θ-stable. We have three cases as in Section 5.2:

(1) the cone σ is below the vector v.
(2) the cone σ is on the left side of the vector v.

[3] One can see if any $\theta \in \Theta$ satisfies that $\theta\big(\phi_k^{-1}(\chi)\big) = 0$ for any $\chi \in G_k^\vee$, then θ must be a constant multiple of ϑ. This also explains the existence of a solution θ_P for (6.1).

(3) the cone σ is on the right side of the vector v.

In Case (1), we have only one G-iraffe

$$\Gamma = \{1, x, x^2, \dots, x^{r-2}, x^{r-1}\}.$$

By Lemma 2.15, any nonzero proper submodule $\mathcal{G}$ of $C(\Gamma)$ is given by the set

$$A = \{x^j, x^{j+1}, \dots, x^{r-2}, x^{r-1}\}$$

for some $1 \leq j \leq r-1$. Since m is sufficiently large, it follows that $\theta(\mathcal{G}) > 0$ so Γ is θ-stable.

We now prove the result in Case (2).

Let Γ be a G-iraffe with corresponding G_2-graph Γ'. Let $\mathcal{G}$ be a submodule of $C(\Gamma)$ whose $\mathbb{C}$-basis is $A \subset \Gamma$. Remark 5.6 and Lemma 2.15 imply that if $\mathbf{m}_\rho \in A$ for $0 \leq \mathrm{wt}(\mathbf{m}_\rho) < a$, then $\phi_2^{-1}\big(\phi_2(\mathbf{m}_\rho)\big) \subset A$. Thus $\vartheta(\mathcal{G}) \geq 0$ from the definition of θ as m is sufficiently large.

If $\vartheta(\mathcal{G}) > 0$, then since m is sufficiently large, it follows that $\theta(\mathcal{G}) > 0$.

If $\vartheta(\mathcal{G}) = 0$, then one can see that $A = \phi_2^{-1}\big(\phi_2(A)\big)$. Let us assume that $A = \phi_2^{-1}\big(\phi_2(A)\big)$. To show this, we prove that $\phi_2(A)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$ and that $\theta(\mathcal{G}) = \theta^{(2)}(\mathcal{G}')$. Since θ satisfies the system of linear equations (6.1), it suffices to show that $\phi_2(A)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$. Recall ξ, η, ζ are the coordinates of $\mathbb{C}^3$ with respect to the action of G_2. By Lemma 2.15, it is enough to show that if $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$, then $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho)$ in $\phi_2(A)$ for any $\mathbf{k} \in \{\xi, \eta, \zeta\}$ and $\mathbf{m}_\rho \in A$. Suppose $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$ for some $\mathbf{m}_\rho \in A$. By Lemma 5.5, there exists $\mathbf{m}_{\rho'}$ such that

$$\phi_2(\mathbf{f} \cdot \mathbf{m}_{\rho'}) = \mathbf{k} \cdot \mathbf{k}_\chi$$

with $\phi_2(\mathbf{m}_{\rho'}) = \phi_2(\mathbf{m}_\rho)$ for some $\mathbf{f} \in \{x, y, z\}$. In particular, $\mathbf{f} \cdot \mathbf{m}_{\rho'} \in \Gamma = \phi_2^{-1}(\Gamma')$. Since $A = \phi_2^{-1}\big(\phi_2(A)\big)$, we have $\mathbf{m}_{\rho'} \in A$, which implies $\mathbf{f} \cdot \mathbf{m}_{\rho'} \in A$ as A is a $\mathbb{C}$-basis of $\mathcal{G}$. Thus $\mathbf{k} \cdot \mathbf{k}_\chi$ is in $\phi_2(A)$. □

6.2. **Root system in A_{r-1}.** We review well known facts on the A_{r-1} root system. Let $I := \mathrm{Irr}(G)$ be identified with $\mathbb{Z}/r\mathbb{Z}$. As is well known, the following three are in 1-to-1 correspondence:

(1) Sets of simple roots Δ.
(2) Open Weyl Chambers $\mathfrak{C}$.
(3) Elements of $S_r := \big\{\omega \,\big|\, \omega \text{ is a permutation of } I\big\}$.

Let $\big\{\varepsilon_i \,\big|\, i \in I\big\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. Note that the indices are in $I = \mathbb{Z}/r\mathbb{Z}$. Define

$$\Phi := \big\{\varepsilon_i - \varepsilon_j \,\big|\, i, j \in I, i \neq j\big\}.$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated by Φ. Elements in Φ are called *roots*. For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$.

A chamber in stability parameter space. For each $i \in I$, let ρ_i denote the irreducible representation of G of weight i. Note that each root α can be considered as the support of a submodule of a G-constellation. In other words, α_i corresponds to the dimension vector of ρ_i. Thus in general root $\alpha = \sum_i n_i\alpha_i$ is the dimension vector of the representation $\oplus n_i\rho_i$. Abusing notation, let $\alpha = \sum_i n_i\alpha_i$ also denote the corresponding representation $\oplus n_i\rho_i$.

Let Δ be a set of simple roots. Define a subset $\mathfrak{C}$ of Θ associated to Δ as

$$\mathfrak{C} := \mathfrak{C}(\Delta) := \left\{\theta \in \Theta \,\middle|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta\right\}.$$

At this moment, $\mathfrak{C}(\Delta)$ is not necessarily a chamber in Θ because $\mathfrak{C}(\Delta)$ may contain nongeneric elements.

6.3. **Admissible sets of simple roots.** In this section, we define the *admissible set of simple roots* Δ_a for the group of type $\frac{1}{r}(1, a, r-a)$. The Weyl chamber $\mathfrak{C}_a$ corresponding to the admissible set of simple roots is equal to the GIT stability parameter cone in [12].

Remark 6.5. Kędzierski [12] described a cone of GIT parameters with a set of inequalities. One can easily see that this can be described using the root system A_{r-1}. He conjectured the cone is a full chamber. We prove that the conjecture is true. ♦

Firstly, we consider the case of $\frac{1}{r}(1, r-1, 1)$. Secondly, we define the admissible set of simple roots for $\frac{1}{r}(1, a, r-a)$ using a recursion process.

The case of $\frac{1}{r}(1, r-1, 1)$. From Theorem 5.1, we know that the economic resolution of the quotient variety $X = \mathbb{C}^3/G$ is isomorphic to G-Hilb $\mathbb{C}^3$ where G is of type $\frac{1}{r}(1, r-1, 1)$. Thus in this case, the G-iraffes are just Nakamura G-graphs which are θ-stable for $\theta \in \Theta_+$, where

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for } \rho \neq \rho_0\right\}.$$

In terms of the root system, $\theta(\alpha_i) > 0$ for nonzero $i \in I$. Note that $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. Thus the corresponding set of simple roots is

$$\Delta = \left\{\varepsilon_i - \varepsilon_{i-a} \in \Phi \,\middle|\, i \in I, i \neq 0\right\}.$$

Example 6.6. First, let us consider the group of type $\frac{1}{3}(1, 2, 1)$. Let $\left\{\varepsilon_j^L \,\middle|\, j = 0, 1, 2\right\}$ be the standard basis of $\mathbb{Q}^3$. Then the corresponding set of simple roots Δ^L is

$$\Delta^L = \left\{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\right\}.$$

Consider the group of type $\frac{1}{4}(1, 3, 1)$. Let $\left\{\varepsilon_k^R \,\middle|\, k = 0, 1, 2, 3\right\}$ be the standard basis of $\mathbb{Q}^4$. Then

$$\Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}$$

is the corresponding set of simple roots for type $\frac{1}{4}(1, 3, 1)$. ♦

The case of $\frac{1}{r}(1,a,r-a)$. Let G be the group of type $\frac{1}{r}(1,a,r-a)$. Let us assume that for $\frac{1}{a}(1,\overline{-r},\overline{r})$ and $\frac{1}{r-a}(1,\overline{a},\overline{-r})$ we have sets of simple roots Δ^L and Δ^R, respectively. Note that Δ^L is a set of simple roots in A_{a-1} and Δ^R is a set of simple roots in A_{r-a-1}. As in Section 6.2, let

$$\left\{\varepsilon_l^L \,\middle|\, l=0,1,\dots,a-1\right\}, \quad \left\{\varepsilon_k^R \,\middle|\, k=0,1,\dots,r-a-1\right\}$$

be the standard basis of $\mathbb{Q}^a$ and $\mathbb{Q}^{r-a}$, respectively. From the two sets of simple roots Δ^L and Δ^R, we construct a set Δ of simple roots in A_{r-1} as follows. Firstly, as in Section 6.2, let the standard basis $\left\{\varepsilon_i \,\middle|\, i\in I\right\}$ of $\mathbb{Q}^r$ be identified with the union of the two sets

$$\left\{\varepsilon_l^L \,\middle|\, l=0,1,\dots,a-1\right\} \text{ and } \left\{\varepsilon_k^R \,\middle|\, k=0,1,\dots,r-a-1\right\}$$

using the following identification:

$$\begin{aligned} \varepsilon_l^L &= \varepsilon_i \quad \text{with } i\equiv l \mod a, && r-a\le i<r,\\ \varepsilon_k^R &= \varepsilon_i \quad \text{with } i\equiv k \mod (r-a), && 0\le i<r-a. \end{aligned} \tag{6.7}$$

Secondly, with this identification, define a set Δ of simple roots

$$\Delta = \Delta^L \cup \left\{\varepsilon_{\lfloor \frac{r}{a}\rfloor a} - \varepsilon_{(r-2a)-\lfloor\frac{r-2a}{r-a}\rfloor(r-a)}\right\} \cup \Delta^R. \tag{6.8}$$

Note that Δ is actually a set of simple roots in A_{r-1}.

Remark 6.9. Note that if $\varepsilon_l^L-\varepsilon_k^L$ is a positive sum of simple roots in Δ^L, then the corresponding root of A_{r-1} is also a positive sum of simple roots in Δ. Moreover, $\varepsilon_l^L-\varepsilon_k^R$ can be written as a positive sum of simple roots in Δ: note that $\varepsilon_{\lfloor\frac{r}{a}\rfloor a}$ is identified with a vector ε^L and that $\varepsilon_{(r-2a)-\lfloor\frac{r-2a}{r-a}\rfloor(r-a)}$ is identified with a vector ε^R; since we add the root $\varepsilon_{\lfloor\frac{r}{a}\rfloor a}-\varepsilon_{(r-2a)-\lfloor\frac{r-2a}{r-a}\rfloor(r-a)}$ to Δ, $\varepsilon_l^L-\varepsilon_k^R$ is a positive sum of simple roots in Δ. ♦

Definition 6.10. With the notation as above, we call the set Δ of simple roots the *admissible set of simple roots* for $G=\frac{1}{r}(1,a,r-a)$, which is denoted by Δ_a. For the admissible set of simple roots, define

$$\mathfrak{C}_a := \left\{\theta\in\Theta \,\middle|\, \theta(\alpha)>0 \quad \forall\alpha\in\Delta_a\right\}$$

with considering roots $\alpha=\sum_i n_i\alpha_i$ as corresponding representations $\oplus n_i\rho_i$. We call $\mathfrak{C}_a$ the *admissible Weyl chamber* for $G=\frac{1}{r}(1,a,r-a)$.

As is stated in Section 6.2, note that a set of simple roots Δ_a is determined by and determines a permutation of $I=\mathbb{Z}/r\mathbb{Z}$. Indeed,

$$\Delta_a = \left\{\varepsilon_{\omega(i)}-\varepsilon_{\omega(i-a)} \,\middle|\, i\in I, i\neq 0\right\}$$

for a unique permutation $\omega\colon I\to I$.

Let $\{\theta_i\}_{i=1}^{r-1}$ be the dual basis of the GIT parameter space Θ with respect to $\{\alpha_i\}_{i=1}^{r-1}$, i.e. $\theta_i(\alpha_j)=\delta_{ij}$. Set $\theta_0=-\sum_{i=1}^{r-1}\theta_i$. As is standard,

we can present the rays of the Weyl chamber $\mathfrak{C}_a$ using this basis and the permutation ω: the rays are generated by the following vectors

$$\sum_{j=0}^{i-1} \left(\theta_{\omega(ja)+a} - \theta_{\omega(ja)}\right) \tag{6.11}$$

for $i = 1, 2, \ldots, r-1$. Thus any $\theta \in \mathfrak{C}_a$ is a positive linear sum of the vectors above in (6.11).

Example 6.12. Let G be the group of type $\frac{1}{7}(1,3,4)$. From the fan of the economic resolution of this case (see Example 3.9), the left and right sides are the economic resolutions of singularities of $\frac{1}{3}(1,2,1)$ and $\frac{1}{4}(1,3,1)$, respectively. By Example 6.6, we have two sets

$$\Delta^L = \{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\} \text{ and } \Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}.$$

As in the construction (6.8), the admissible set of simple roots is

$$\Delta_a = \{\varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \underline{\varepsilon_6 - \varepsilon_1}, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_0\},$$

where the underlined root is the added root as in (6.8). In terms of $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$,

$$\Delta_a = \{\alpha_4 + \alpha_1, \alpha_5 + \alpha_2, \underline{-\alpha_1 - \alpha_5 - \alpha_2}, \alpha_1 + \alpha_5, \alpha_2 + \alpha_6, \alpha_3\}.$$

Thus the set of parameters $\theta \in \Theta$ satisfying

$$\begin{array}{lll} \theta(\rho_4 \oplus \rho_1) > 0, & \theta(\rho_5 \oplus \rho_2) > 0, & \theta(\rho_1 \oplus \rho_5 \oplus \rho_2) < 0, \\ \theta(\rho_1 \oplus \rho_5) > 0, & \theta(\rho_2 \oplus \rho_6) > 0, & \theta(\rho_3) > 0 \end{array}$$

is the admissible Weyl chamber $\mathfrak{C}_a$ where ρ_i is the irreducible representation of G of weight i.

The corresponding permutation ω is

$$\omega = \begin{pmatrix} 0 & 3 & 6 & 2 & 5 & 1 & 4 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$$

i.e. $\omega(0) = 0$, $\omega(3) = 3$, $\omega(6) = 2$, etc. The rays of the Weyl chamber $\mathfrak{C}_a$ are the row vectors of the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

with the basis $\{\theta_i\}$. Note that for any $\theta \in \mathfrak{C}_a$, $\theta(\alpha_i)$ is negative if and only if $0 \le i < 3$. ♦

6.4. **An open Weyl chamber.** In this section, we prove that the stability parameters described in Section 6.1 form an open Weyl chamber. It follows that our stability parameters are the same as Kędzierski's in [12].

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. We may assume $2a < r$. Let G_2 and G_3 be the groups of type $\frac{1}{a}(1, \overline{-r}, \overline{r})$ and of type $\frac{1}{r-a}(1, \overline{r}, \overline{-r})$, respectively. To use recursion steps, assume that the admissible set of simple roots Δ^L and Δ^R give the full chambers $\mathfrak{C}^L$ and $\mathfrak{C}^R$. Let Δ_a be the admissible set of simple roots and $\mathfrak{C}_a$ the admissible Weyl chamber for $\frac{1}{r}(1, a, r-a)$.

We prove that $\mathfrak{C}_a$ is a full chamber such that the admissible G-iraffes are θ-stable for $\theta \in \mathfrak{C}_a$ by the following three steps.

Step 1. Firstly, we prove that for any $\theta \in \mathfrak{C}_a$, there exist $\theta^{(2)} \in \mathfrak{C}^L$ and $\theta^{(3)} \in \mathfrak{C}^R$ such that θ is a partial solution of the system of linear equations (6.1). Let θ be in $\mathfrak{C}_a$. Let us define $\theta^{(2)}$, $\theta^{(3)}$ to be

$$\begin{cases} \theta^{(2)}(\chi) = \theta\big(\phi_2^{-1}(\chi)\big) & \text{for } \chi \in G_2^\vee, \\ \theta^{(3)}(\chi') = \theta\big(\phi_3^{-1}(\chi')\big) & \text{for } \chi' \in G_3^\vee. \end{cases}$$

It suffices to show that $\theta^{(2)} \in \mathfrak{C}^L$ and $\theta^{(3)} \in \mathfrak{C}^R$. Let χ_l be a character of G_2 whose weight is l. Then

$$\phi_2^{-1}(\chi_l) = \left\{ \rho_i \in G^\vee \,\middle|\, 0 \le i < r,\ i \equiv l \mod a \right\},$$

by the definition of the left round down function, so the dimension vector of $\phi_2^{-1}(\chi_l)$ in terms of roots is

$$\sum_{\substack{0 \le i < r, \\ i \equiv l \bmod a}} \alpha_i = \sum_{\substack{0 \le i < r, \\ i \equiv l \bmod a}} (\varepsilon_i - \varepsilon_{i-a}) = \alpha_l^L.$$

Note that θ is positive on Δ_a. In particular θ is positive on the roots coming from Δ^L. From this, it follows that $\theta^{(2)}$ is in $\mathfrak{C}^L$. For $\theta^{(3)}$, we can prove the assertion in a similar way.

Step 2. Secondly, we prove that the vector ϑ in (6.2) is a ray of the chamber $\mathfrak{C}_a$. From this, it follows that any $\theta \in \mathfrak{C}_a$ can be written as the form (6.3) so admissible G-iraffes are θ-stable.

Let ϑ be the vector in (6.2). As is well known, ϑ is a ray of the Weyl chamber $\mathfrak{C}_a$ associated to the set of simple root Δ_a if and only if there exists a unique simple root α in Δ_a such that $\vartheta(\alpha)$ is positive and ϑ is zero on the other simple roots in Δ_a. A simple observation shows that ϑ is zero on the sets Δ^L and Δ^R with the identification (6.8). It remains to show that $\vartheta(\alpha)$ is positive for

$$\begin{aligned} \alpha &= \varepsilon_{\lfloor \frac{r}{a} \rfloor a} - \varepsilon_{(r-2a) - \lfloor \frac{r-2a}{r-a} \rfloor (r-a)} = \varepsilon_{\lfloor \frac{r}{a} \rfloor a} - \varepsilon_{r-2a} \\ &= \sum_{\rho_i \in \phi_2^{-1}(A)} \alpha_i + \alpha_{r-a}. \end{aligned}$$

for a subset A of $G^\vee$. Since $\vartheta(A) = 0$ and $\vartheta(\alpha_{r-a}) = 1$, we have $\vartheta(\alpha) = 1$.

Step 3. Lastly, we prove that the chamber is a full chamber. By Step 1 and Step 2, we prove that the Weyl chamber $\mathfrak{C}_a$ is a cone in Θ such that the admissible G-iraffes are θ-stable for $\theta \in \mathfrak{C}_a$. Considering the torus invariant G-constellations which x acts trivially on, it is immediate that the chamber structure in Θ is finer than the Weyl chamber structure of A_{r-1}. Therefore the admissible Weyl chamber is a full chamber in the stability parameter space Θ.

We have proved the following proposition:

Proposition 6.13. *For the set $\mathfrak{G}$ of G-iraffes in Corollary 5.8, there exists an open Weyl chamber $\mathfrak{C}_a \subset \Theta$ such that Γ is θ-stable if $\Gamma \in \mathfrak{G}$ and $\theta \in \mathfrak{C}_a$. Furthermore, the chamber $\mathfrak{C}_a$ is a full chamber in Θ.*

From Step 3, we make the following conjecture:

Conjecture 6.14. *The chamber structure of the GIT stability parameter space Θ of G-constellations coincides with the Weyl chamber structure of A_{r-1}.*

7. Main theorem

Theorem 7.1 (Main Theorem)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X = \mathbb{C}^3/G$. Then there exist a set $\mathfrak{G}$ of G-iraffes and $\theta \in \Theta$ such that:*

(i) *there exists a bijective map $\Sigma_{\max} \to \mathfrak{G}$ sending σ to Γ_σ with $S(\Gamma_\sigma) = \sigma^\vee \cap M$.*
(ii) *every Γ_σ is θ-stable if $\Gamma_\sigma \in \mathfrak{G}$.*

Thus Y is isomorphic to $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$. In particular, $U(\Gamma)$ is smooth for any $\Gamma \in \mathfrak{G}$.

Proof. Corollary 5.8 shows that there exists a set $\mathfrak{G}$ of G-iraffes satisfying the condition (i). For the set $\mathfrak{G}$, Lemma 6.4 shows that there exists a stability parameter θ satisfying the condition (ii). □

Corollary 7.2. *With the notation as Theorem 7.1, the economic resolution Y is isomorphic to the birational component Y_θ of the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations.*

Proof. The main theorem proves that the economic resolution Y is isomorphic to $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$. From Proposition 2.18, there exists an open immersion from Y to Y_θ. This open immersion is a closed embedding because both Y and Y_θ are projective over X. Since both Y and Y_θ are

3-dimensional and irreducible, this embedding is an isomorphism.

$$\begin{tikzcd} Y \cong \bigcup_{\Gamma\in\mathfrak{G}} U(\Gamma) \arrow[r, hook] \arrow[dr, "\text{proper}"'] & Y_\theta \arrow[d, "\text{proj.}"] \\ & X \end{tikzcd}$$

□

By the construction of this family, we have seen that elements in Γ form a $\mathbb{C}$-basis of the G-constellation over $p \in U(\Gamma)$.

Conjecture 7.3. *The moduli space $\mathcal{M}_\theta$ is irreducible. In particular, any θ-stable G-graph Γ is in the set $\mathfrak{G}$ in Theorem 7.1.*

If this conjecture holds, then the moduli space $\mathcal{M}_\theta$ is isomorphic to the economic resolution. In the case $G = \frac{1}{2k+1}(1, 2, 2k-1)$, we can prove that Conjecture 7.3 is true so $\mathcal{M}_\theta$ is isomorphic to the economic resolution for $\theta \in \mathfrak{C}_a$. We hope to establish this more generally in future work.

Appendix A. Universal families

In the previous sections, we assigned a θ-stable G-graph Γ_σ to each full dimensional cone σ of the fan of the economic resolution Y of $X = \mathbb{C}^3/G$, where G is of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. This section describes the *universal family* over the economic resolution Y.

Let ρ be an irreducible representation of G. From the data (σ, Γ_σ), for each full dimensional cone σ, there exists a unique Laurent monomial $\mathbf{m}_\sigma \in \Gamma_\sigma$ whose weight is ρ. The data $\{\mathbf{m}_\sigma\}$ is called the *canonical data* of ρ.

Remark A.1. This canonical data gives a line bundle, which is called a universal family over $Y_\theta = G\text{-Hilb}\,\mathbb{C}^3$ if $a = 1$ or $r-1$. ♦

Proposition A.2. *Let ρ be a fixed irreducible representation of G. The canonical data $\{\mathbf{m}_\sigma\}$ of ρ gives a line bundle $\mathcal{L}_\rho$ on Y satisfying $\mathcal{L}_\rho|_{U_\sigma} \cong \mathcal{O}_{U_\sigma}(\operatorname{div}\mathbf{m}_\rho^{-1})$. In other words, $\mathcal{L}_\rho$ is the line bundle corresponding to the Cartier divisor D_ρ defined by $D_\rho|_{U_\sigma} = \operatorname{div}\mathbf{m}_\rho^{-1}|_{U_\sigma}$ for all σ.*

Proof. From general toric geometry (see e.g. [3]), it suffices to show that $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection $\sigma \cap \sigma'$ for any two adjacent cones σ, σ'. Suppose that the intersection is the cone generated by $\mathbf{u}_1, \mathbf{u}_2 \in L$ and then it should be shown that $\langle \mathbf{u}_i, \frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} \rangle$ is zero for $i = 1, 2$. Set $\mathbf{m}_\sigma = x^{m_1} y^{m_2} z^{m_3}$ and $\mathbf{m}_{\sigma'} = x^{m'_1} y^{m'_2} z^{m'_3}$. There are four cases:

(1) Both σ and σ' are cones in either the left side or the right side.
(2) One of them is the cone on the central side and the other is the cone on the central side of the left side.

(3) One of them is the cone on the central side and the other is the cone on the central side on the right side.
(4) One of them is the most right cone of the left side and the other is the most left cone of the right side.

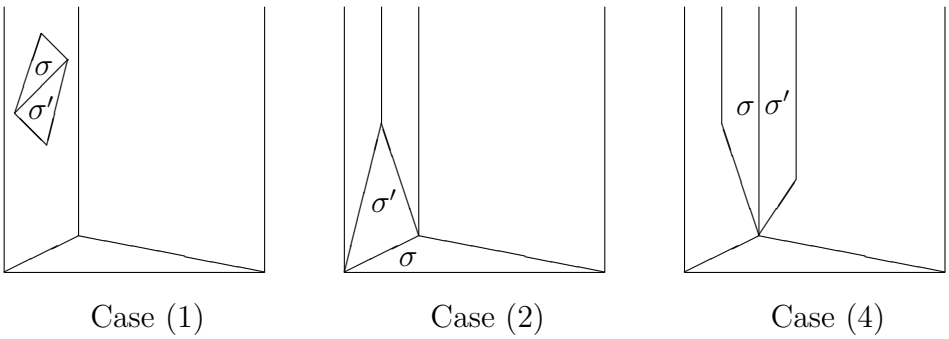

FIGURE A.1. Four cases for two full dimensional cones in the fan of Y

Case (1). Assume that the cones are on the left side. Let ϕ_2 be the left round down function of the weighted blowup with weight $(1, a, r-a)$. Since the weights of $\mathbf{m}_\sigma$ and $\mathbf{m}_\sigma$ are equal to ρ, $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} = \phi_2(\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}})$. By induction on r, it follows that $\langle \mathbf{u}_i, \frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} \rangle = 0$.

Case (2). Assume that σ is the cone on the central side and that σ' is the cone on the central side of the left side. Note that the G-graph for σ is $\{1, x, x^2, \dots, x^{r-1}\}$ and that the G-graph for σ' is

$$\left\{ \mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) \in \{1, \xi, \dots, \xi^{a-1}\} \right\}.$$

Thus, with the fact that both $\mathbf{m}_\sigma$ and $\mathbf{m}_{\sigma'}$ have the same weights,

$$\begin{aligned} \mathbf{m}_\sigma &= x^{m_1} && \text{for some } 0 \le m_1 < r, \\ \mathbf{m}_{\sigma'} &= x^{m_1'} y^{m_2'} && \text{for some } 0 \le m_1' < a \quad \text{with } m_1' + a m_2' = m_1. \end{aligned}$$

Since $\sigma \cap \sigma' = \mathrm{Cone}\left((0,0,1), \frac{1}{r}(1, a, r-a)\right)$, the Laurent monomial $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection.

Case (3). Case (3) is similar to Case (2).

Case (4). Assume that σ is the most right cone in the left side and that σ' is the most left cone in the right side. Note that $\sigma \cap \sigma'$ is the cone generated by $(1,0,0), \frac{1}{r}(1, a, r-a)$. Similarly to Case (2), note that

$$\mathbf{m}_\sigma = y^{m_2} z^{m_3}, \quad \mathbf{m}_{\sigma'} = y^{m_2'} z^{m_3'}$$

with $a m_2 + (r-a) m_3 = a m_2' + (r-a) m_3'$. Hence it follows that $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection. $\square$

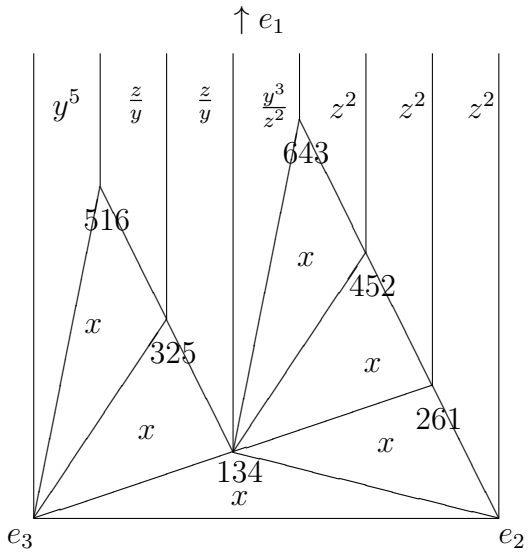

FIGURE A.2. Elements of weight 1 in Γ_σ for $\frac{1}{7}(1,3,4)$

Remark A.3. For the trivial representation ρ_0, $\mathbf{1}$ is in every G-graph and hence the line bundle for the trivial representation is $\mathcal{O}_Y$. The direct sum of all such line bundles

$$\mathcal{L} = \bigoplus_{\rho \in G^\vee} \mathcal{L}_\rho$$

is a *gnat family* in the sense of [15], which is the same family in [12]. ♦

Example A.4. Let G be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.10. Let ρ be the irreducible representation of G with weight 1. Consider the line bundle $\mathcal{L}_\rho$ as in Proposition A.2. In Figure A.2, the monomial in a maximal cone σ is a unique element in Γ_σ whose weight is 1. ♦

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Mathematics Institute, University of Warwick,
Coventry, CV4 7AL, England
E-mail: s-j.jung@warwick.ac.uk