A NOTE ON ELEPHANTS AND GIT CHAMBERS

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Abstract. In this note, we investigate the GIT chambers for G-constellations where $G \subset \mathrm{GL}_3(\mathbb{C})$ is the group of type $\frac{1}{r}(b, 1, r-1)$. We conjecture that there exists a "strong connection" between the chamber structure and the Weyl chamber structure of A_{r-1}.

1. Elephants for the economic resolution

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(b, 1, r-1)$. Consider the quotient variety $X = \mathbb{C}^3/G$.

Let D be the hyperplane section of X defined by $x = 0$, i.e. the Weil divisor defined by $x = 0$. One can see that

$$K_X + D \sim_{\mathbb{Q}} 0.$$

Thus D is an element[1] of the anticanonical system $|-K_X|$. Moreover, D is isomorphic to the quotient $\mathbb{C}^2$ by the group of type $\frac{1}{r}(1, r-1)$ so D has an A_{r-1} singularity.

Consider the economic resolution $\varphi\colon Y \to X = \mathbb{C}^3/G$. Let S be the strict transform of D. Then one can show that S is an element of the anticanonical system $|-K_Y|$ and that we have the following diagram:

$$\begin{array}{ccc} S & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ D & \hookrightarrow & X, \end{array}$$

where the vertical morphism $S \to D$ is the minimal resolution of D.

It is well known [1,6] that the minimal resolution of A_{r-1} singularities is isomorphic to the moduli space of θ-stable A-constellations for a generic parameter θ where $A \subset \mathrm{SL}_2(\mathbb{C})$ is the group of type $\frac{1}{r}(1, -1)$. Moreover, the chamber structure of the GIT stability parameter space for A-constellations coincides with the Weyl chamber structure of type A_{r-1}. We expect that the morphism $Y \to X$ might have a modular description as moduli spaces of G-constellations.

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[1]Elements of the anticanonical system of a variety X are called *elephants* of X.

2. Root system in A_{r-1}

We review well known facts on the A_{r-1} root system. Let $I := \mathrm{Irr}(G)$ be identified with $\mathbb{Z}/r\mathbb{Z}$. As is well known, the following three are in 1-to-1 correspondence:

(1) Sets of simple roots Δ.
(2) Open Weyl Chambers $\mathfrak{C}$.
(3) Elements of $S_r := \{\omega \mid \omega \text{ is a permutation of } I\}$.

Let $\{\varepsilon_i \mid i \in I\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. Note that the indices are in $I = \mathbb{Z}/r\mathbb{Z}$. Define

$$\Phi := \{\varepsilon_i - \varepsilon_j \mid i, j \in I, i \neq j\}.$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated by Φ. Elements in Φ are called *roots*. For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$. For any root α, one can see that $\langle \alpha, \alpha \rangle = 2$. Note that

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is the root system of A_{r-1} and the Weyl group of this root system is the group generated by simple reflections

$$s_i \colon \alpha \mapsto \alpha - \frac{\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$

It is easy to see that

$$s_i(\varepsilon_k - \varepsilon_l) = \varepsilon_{\omega_i(k)} - \varepsilon_{\omega_i(l)},$$

where ω_i is the (adjacent) transposition in S_r

$$\omega_i(j) = \begin{cases} i+1 & \text{if } j = i, \\ i & \text{if } j = i+1, \\ j & \text{otherwise.} \end{cases}$$

Thus the Weyl group can be thought as the group of permutations of I.

Here, we consider roots as dimension vectors:

(i) α_i is the dimension vector of the vertex simple at the vertex ρ_i;
(ii) the dimension vector of the vertex simple at the trivial representation ρ_0 is $-\sum_{i \neq 0} \alpha_i$.

The stability parameter space Θ can be identified with the dual space of $\mathfrak{h}^*$. Let ω be a permutation of I. As is customary (see e.g. [4]), define a set of simple roots and an open Weyl chamber associated to

ω:

$$\Delta(\omega) := \left\{\varepsilon_{\omega(i)} - \varepsilon_{\omega(i-a)} \in \Phi \,\middle|\, i \in I, i \neq 0\right\},$$
$$\mathfrak{C}(\omega) := \left\{\theta \in (\mathfrak{h}^*)^* \,\middle|\, \theta\big(\varepsilon_{\omega(i)} - \varepsilon_{\omega(i-a)}\big) > 0 \quad \forall i \in I, i \neq 0\right\}.$$

In particular, for the identity permutation of I, the corresponding simple roots Δ_+ and Weyl chamber $\mathfrak{C}_+$ are

$$\Delta_+ = \left\{\varepsilon_i - \varepsilon_{i-a} \in \Phi \,\middle|\, i \in I, i \neq 0\right\} = \left\{\alpha_i \,\middle|\, i \in I, i \neq 0\right\},$$
$$\mathfrak{C}_+ = \left\{\theta \in (\mathfrak{h}^*)^* \,\middle|\, \theta\big(\alpha_i\big) > 0 \quad \forall i \in I, i \neq 0\right\},$$

which is the cone Θ_+ for G-Hilb.

A chamber in stability parameter space. For each $i \in I$, let ρ_i denote the irreducible representation of G of weight i. Note that each root α can be considered as the support of a submodule of a G-constellation. In other words, α_i corresponds to the dimension vector of ρ_i. Thus in general root $\alpha = \sum_i n_i\alpha_i$ is the dimension vector of the representation $\oplus n_i\rho_i$. Abusing notation, let $\alpha = \sum_i n_i\alpha_i$ also denote the corresponding representation $\oplus n_i\rho_i$.

Let Δ be a set of simple roots. Define a subset $\mathfrak{C}$ of Θ associated to Δ as

$$\mathfrak{C} := \mathfrak{C}(\Delta) := \left\{\theta \in \Theta \,\middle|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta\right\}.$$

At this moment, $\mathfrak{C}(\Delta)$ is not necessarily a chamber in Θ because $\mathfrak{C}(\Delta)$ may contain nongeneric elements.

3. Chamber structures and elephants

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(b, 1, -1)$, with b coprime to r, which is the same group as before but taking another primitive rth root of unity. In this section, we investigate the chamber structure of the GIT parameter space of G-constellations.

Let ρ_i be the irreducible representation of G whose weight is i. We can identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$.

Recall the McKay quiver of G is the quiver whose vertex set is I with the $3r$ following arrows:

$$x_i\colon i \to i + b,$$
$$y_i\colon i \to i + 1,$$
$$z_i\colon i \to i - 1,$$

for each $i \in I$. The representation of the McKay quiver of G with commutation relations is the representation of the McKay quiver whose dimension vector is $(1, \dots, 1)$ satisfying the following relations:

$$\begin{cases} x_i y_{i+b} = y_i x_{i+1}, \\ x_i z_{i+b} = z_i x_{i-1}, \\ y_i z_{i+1} = z_i y_{i-1}. \end{cases}$$

Let $A \subset \mathrm{SL}_2(\mathbb{C})$ be of type $\frac{1}{r}(1,-1)$ with coordinates y, z. The McKay quiver of A is the quiver whose vertex set is I with the $2r$ following arrows:

$$y_i \colon i \to i+1,$$
$$z_i \colon i \to i-1,$$

for each $i \in I$. The representation of the McKay quiver of A with commutation relations is the representation of the McKay quiver whose dimension vector is $(1, \dots, 1)$ satisfying the following relations:

$$y_i z_{i+1} = z_i y_{i-1} \quad \text{for all } i \in I.$$

Note that the GIT parameter space Θ of G-constellations can be identified with

$$\Theta = \left\{ \theta = (\theta^i) \in \mathbb{Q}^r \,\middle|\, \sum \theta^i = 0 \right\},$$

which is also the GIT parameter space of A-constellations. Furthermore, we have the following proposition.

Proposition 3.1. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(b,1,-1)$ and $A \subset \mathrm{SL}_2(\mathbb{C})$ the finite subgroup of type $\frac{1}{r}(1,-1)$. Then the chamber structure of the GIT parameter space of G-constellations is finer than, or equal to, that of A-constellations.*

Proof. It suffices to show that a wall of the GIT parameter space of A-constellations is also a wall of the GIT parameter space of G-constellations.

Let θ be a stability parameter on a wall of the GIT parameter space of A-constellations. This means that there exists a θ-semistable A-constellation $\mathcal{F}$ such that it is not θ-stable, i.e. there exists a $\mathbb{C}[y,z]$-submodule $\mathcal{G}$ with $\theta(\mathcal{G}) = 0$.

Note that we have a natural identification between A-constellations and G-constellations whose x-action is zero. Thus $\mathcal{F}$ can be thought of as a G-constellation and $\mathcal{G}$ is a $\mathbb{C}[x,y,z]$-submodule of $\mathcal{F}$ with $\theta(\mathcal{G}) = 0$. As it is easy to see that $\mathcal{F}$ is θ-semistable G-constellation, it proves that θ is also on a wall of the GIT parameter space of G-constellations. □

Note that the chamber structure of GIT parameter space of A-constellations is the same as the Weyl chamber structure of A_{r-1}.

Conjecture 3.2. *The chamber structure of the GIT stability parameter space Θ of G-constellations coincides with the Weyl chamber structure of A_{r-1}.*

Let θ be a generic element of the GIT parameter space of G-constellations. By Proposition 3.1, θ is generic in the GIT parameter space of A-constellations so there exists an open Weyl chamber $\mathfrak{C}$ such that $\theta \in \mathfrak{C}$. Let ω be the corresponding element in S_r.

Let us consider the space of G-constellations $\operatorname{Rep} G$ and the space of A-constellations $\operatorname{Rep} A$. Consider the reductive group

$$\mathrm{GL}(\delta) := \prod_{i \in I} \mathbb{C}^\times$$

acting on $\operatorname{Rep} G$ and $\operatorname{Rep} A$ as basis change. The moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations is

$$\mathcal{M}_\theta = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\operatorname{Rep} G]_{\chi_\theta^n} \right).$$

Let $\operatorname{Rep}^s G$ be the θ-stable locus in $\operatorname{Rep} G$ and $\operatorname{Rep}^s A$ the θ-stable locus in $\operatorname{Rep} A$. We can identify $\operatorname{Rep} A$ with the closed subvariety of $\operatorname{Rep} G$ defined by $x_0 = \cdots = x_{r-1} = 0$ and $\operatorname{Rep}^s A$ with the closed subvariety $\widetilde{S}_\theta$ of $\operatorname{Rep}^s G$ defined by $x_0 = \cdots = x_{r-1} = 0$.

Since $\widetilde{S}_\theta$ is a $\mathrm{GL}(\delta)$-invariant closed set, and $\mathcal{M}_\theta$ is a geometric quotient, the inclusion $\widetilde{S}_\theta \subset \operatorname{Rep}^s G$ induces an inclusion $S_\theta \subset \mathcal{M}_\theta$

$$\begin{array}{ccc} \widetilde{S}_\theta & \hookrightarrow & \operatorname{Rep}^s G \\ \downarrow & & \downarrow \\ S_\theta & \hookrightarrow & \mathcal{M}_\theta \end{array}$$

where S_θ is the closed subvariety of $\mathcal{M}_\theta$ parametrising G-constellations on which x acts trivially. Note that the variety S_θ is isomorphic to the moduli space of θ-stable A-constellations.

Let D be the hyperplane section of $\mathbb{C}^3/G$ defined by $x = 0$. Then D is isomorphic to $\mathbb{C}^2/A$ and has an A_{r-1} singularity as in Section 1. Since $\mathcal{M}_0$ is isomorphic to $\mathbb{C}^3/G$ by Proposition **??**, we have the following diagram

$$\begin{array}{ccccc} S_\theta & \underset{\text{codim.1}}{\hookrightarrow} & Y_\theta & \underset{\text{irr.}}{\hookrightarrow} & \mathcal{M}_\theta \\ \downarrow & & \downarrow & & \downarrow \\ D & \underset{\text{codim.1}}{\hookrightarrow} & \mathbb{C}^3/G & = & \mathcal{M}_0 \end{array}$$

where the vertical morphisms are the canonical projective morphisms induced by GIT quotients. As is known, the morphism $S_\theta \to D$ is the minimal resolution of D.

References

[1] T. Bridgeland, A.King, M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.

[2] H. Cassens, P. Slodowy, *On Kleinian singularities and quivers*, Singularities (Oberwolfach, 1996), 263–288, Progr. Math., **162**, Birkhäuser, Basel, 1998.

[3] V. Danilov, *Birational geometry of three-dimensional toric varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 971–982, 1135; English translation: Math. USSR-Izv. **21** (1983), no. 2, 269–280.

[4] J. Humphreys, *Introduction to Lie algebras and representation theory*, Second printing, revised. Graduate Texts in Mathematics, **9**. Springer-Verlag, New York-Berlin, 1978.

[5] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser.(2) **45** (1994), no. 180, 515–530.

[6] P. Kronheimer, *The construction of ALE spaces as a hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683.

[7] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

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