Magnetic Monopoles and Motion of Charged Particles under Its Influence

Dirac's Monopole

Dirac's magnetic monopole is a hypothetical object that carries a magnetic charge. Usual particle we know of, such as electrons and nucleons, carries electric charges only, and Maxwell's equations incorporate only electric charges and electric currents. However, one could imagine a modified Maxwell's equation as follows

$$\nabla \cdot \mathbf{B} = 0 \qquad \rightarrow \qquad \nabla \cdot \mathbf{B} = \rho_m \tag{1}$$

Look up Jackson's electrodynamics and review basic facts about magnetic monopole and Dirac's quantization condition. The latter asserts that magnetic charge and electric charge are both quantized and imply that $\int \rho_m$ should be an integer multiple of $2\pi/e$ where e is the unit electric charge.

When ρ_m is a pointlike source of charge $-4\pi/e$ at origin $\mathbf{r} = 0$, one finds a magnetic field

$$\mathbf{B} = -\frac{\mathbf{r}}{er^3} \tag{2}$$

In real world, similar magnetic monopoles are expected to exist, according to some grand unified theories, although their number density is very very low so that we are unlikely to encounter one.

Trajectories and Conserved Quantities

Consider an electrically charged particle of charge q moving under the influence of such a magnetic monopole. Equation of motion for a particle of charge q and mass $m = 1$ in this background is

$$\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B} \tag{3}$$

Show that this equation of motion not only conserve the total energy but also some form of angular momentum. Write down these conserved quantities explicitly. Look for conserved angular momentum of the form

$$\mathbf{J} = \mathbf{r} \times \mathbf{v} + \beta\frac{\mathbf{r}}{r} \tag{4}$$

and determine the constant β. The expression for the conserved angular momentum is due to Poincare, late 19th century.

Use this to show that each trajectory lie on a cone whose tip lies exactly at origin $\mathbf{r} = 0$. Draw typical trajectories and discuss whether a closed orbit is possible.

Lagrangian Formulation

The above motion of electrically charged particle can be studied starting with a Lagrangian,

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 + \hat{q}\dot{\mathbf{r}} \cdot \mathbf{A} \tag{5}$$

where $\hat{q} = q/e$ and $\mathbf{A}(\mathbf{r})$ is the "vector potential" associated with $e\mathbf{B}$ such that

$$e\mathbf{B} = \nabla \times \mathbf{A} \tag{6}$$

$\mathbf{A}$ cannot be well-defined everywhere, since $\nabla \cdot \nabla \times \mathbf{A} = 0$, which contradict the above modified Maxwell equation at origin. Look up a textbook and write down a possible form of $\mathbf{A}$ which is well-defined except along z-axis. This form is good enough for solving this Lagrangian dynamics. Discuss what it means to use different forms of $\mathbf{A}$ with the same $\nabla \times \mathbf{A}$. Show that the action

$$\int dt\ L \tag{7}$$

is invariant under the spatial rotation. This is why there is a conserved angular momentum. Derive conserved angular momentum and write it in terms of canonical variables. Finally derive the equation of motion and show it is identical to the usual one, already given above.

Quantum Scattering

This is not an easy problem to solve exactly. Either try to solve it approximately by using a perturbation method, or dig up existing paper on the subject and try to compare the result to the classical result above.

Compute scattering amplitude of an electrically charged particle off the vector potential by solving Schrödinger equation,

$$\frac{1}{2}\left(\mathbf{p} - \hat{q}\mathbf{A}(\mathbf{r})\right)^2 \Psi = i\frac{\partial}{\partial t}\Psi \tag{8}$$

Or try the following two-component Schrödinger equation,

$$\frac{1}{2}\left(\sigma^a(p^a - \hat{q}A^a(\mathbf{r})\right)^2 \Psi = i\frac{\partial}{\partial t}\Psi \tag{9}$$

with three Pauli's matrices, σ^a, and two-component wavefunction Ψ. Use approximation if you must, such as the partial wave analysis, or even truncate the problem to S-wave sector alone.