

Motion on Taub-NUT Manifold as Motion of Charged Particle in Magnetic Monopole

Free motion of particle on the so-called Taub-NUT space has resemblance to motion of charged particle in magnetic monopole background. The Lagrangian is

$$L = \frac{1}{2} \left(1 + \frac{1}{r}\right) \dot{\mathbf{r}}^2 + \frac{1}{2} \left(1 + \frac{1}{r}\right)^{-1} \left(\dot{\psi} + \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}}\right)^2 \quad (1)$$

where \mathbf{A} is a vector potential so that

$$\nabla \times \mathbf{A} = -\frac{\mathbf{r}}{r^3} \quad (2)$$

ψ is a cyclic coordinate (of period 4π), so its conjugate momentum \hat{q} is conserved.

Let us consider trajectories with a definite momentum \hat{q} . Effective Lagrangian governing such trajectories is

$$L' = \frac{1}{2} \left(1 + \frac{1}{r}\right) \dot{\mathbf{r}}^2 - \frac{\hat{q}^2}{2} \left(1 + \frac{1}{r}\right) + \hat{q} \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} \quad (3)$$

This Lagrangian resembles that of charged particle in the background of magnetic monopole, although kinetic term is more complicated and a new potential term $\sim \hat{q}^2$ is added. Conserved energy of this dynamics is

$$E = \frac{1}{2} \left(1 + \frac{1}{r}\right) (\mathbf{v}^2 + \hat{q}^2). \quad (4)$$

Modify the angular momentum of the above monopole problem to find a new conserved angular momentum \mathbf{J} . Furthermore, this case admits one more set of conserved quantity of the form,

$$\mathbf{K} = \left(1 + \frac{1}{r}\right) \mathbf{v} \times \mathbf{J} + \gamma(E - \hat{q}^2) \frac{\mathbf{r}}{r} \quad (5)$$

Find the correct value of the constant γ .

Use these conserved quantities, \hat{q} , E , \mathbf{J} , \mathbf{K} and show that all trajectories are conic sections that lie on a cone whose tip lies exactly at origin $\mathbf{r} = 0$. In other words, any trajectory is an intersection between a cone (with its tip at origin) and a plane. Is there any bound orbit?

Repeat the exercise for a modified Lagrangian where a potential term is added

$$\tilde{L}' = \frac{1}{2} \left(1 + \frac{1}{r}\right) \dot{\mathbf{r}}^2 - \frac{a^2}{1 + 1/r} - \frac{\hat{q}^2}{2} \left(1 + \frac{1}{r}\right) + \hat{q} \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} \quad (6)$$

for some positive constant a^2 . All of above conserved quantity exists here as well with slight modification of their forms. Write down analog of \mathbf{J} , \mathbf{K} and show that trajectories are again conic sections. Find bound orbits and describe them in some detail.