Patterns Formed by Growing Sandpiles

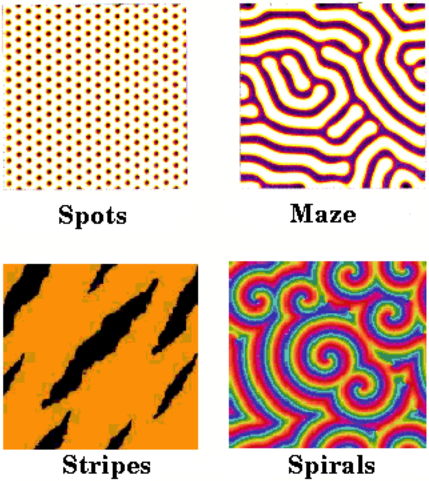
Deepak Dhar, Tridib Sadhu and Samarth Chandra

Tata Institute of Fundamental Research
Mumbai, INDIA

NSPCS08, Seoul, July1-4,2008

Outline

Complex patterns in nature



Theoretical models

One can get complex patterns from
simple, local, deterministic evolution rules.
e.g. Conway's Game of Life.

Example The rule $x_i' = [x_{i-1} + x_{i+1}](mod 2)$

000000000000*0000000000
0000000000*0*000000000
000000000*000*00000000
00000000*0*0*0*0000000
0000000*0000000*000000
000000*0*00000*0*00000
00000*000*000*000*0000
0000*0*0*0*0*0*0*0*000

Definition of Abelian sandpiles

Complex patterns in sandpile models.

Structure of Identity.

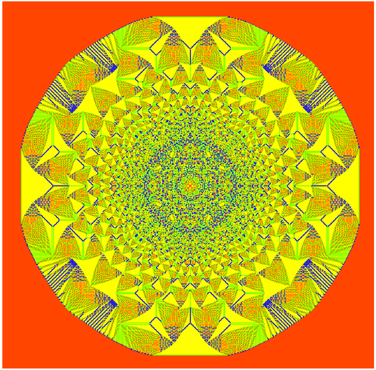
Evolution from special initial unstable states.

Rule for forming patterns:

Add N particles at one site, and relax.

Deterministic patterns.

This is what we study here.



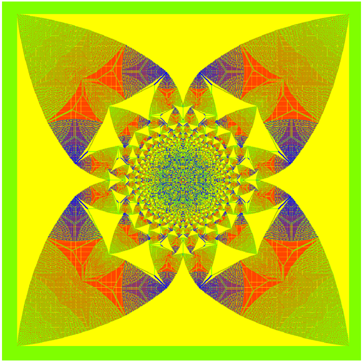


Figure: Patterns produced by adding 400000 particles at the origin, on a square lattice ASM, with initial state (a) all 0 (b) all 2.
Color code 0, 1, 2, 3 = R,B,G,Y

Outline

Characterizing the asymptotic pattern

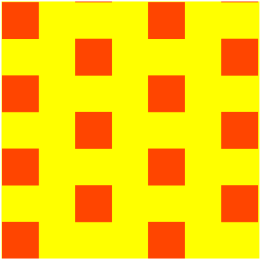
How do we characterize a complex pattern like this?

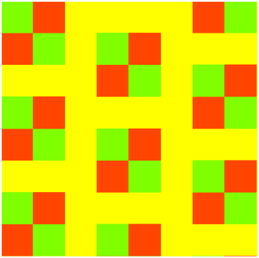
- A pixel by pixel description
- list of all patches and colors
- the rule for generating the pattern
- ?

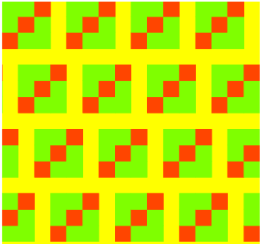
S. Ostojic (2003).

- Diameter $\sim \sqrt{N}$
- Proportionate growth.
- Periodic height pattern in each patch. [ignoring Transients]
- Reduced coordinates $\xi = x/\sqrt{N}, \eta = y/\sqrt{N}$
  coarse-grained density $\rho(\xi, \eta)$ is constant within a patch.
- Define
  $\phi(\xi, \eta) = Lim_{N \to \infty}(1/N)[\ \#$ of topplings at $(\xi, \eta)]$
  Then,
  ϕ is a quadratic function of ξ, η in each patch.

Examples of periodic patterns in patches







Outline

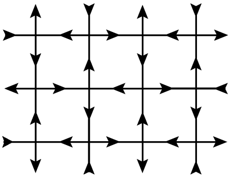
For the square lattice, the number of different patches is infinite, and not easy to characterize.

Other lattices, or backgrounds?

The F-Lattice.

Two arrows in and two out at each vertex.

Allowed stable ASM heights are 0 and 1.



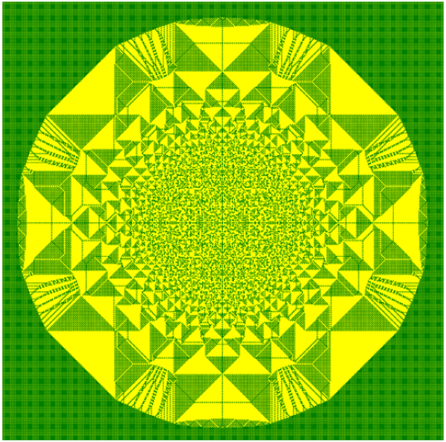


Figure: Pattern produced by adding 10^5 particles at the origin, on the F-lattice with initially empty lattice.

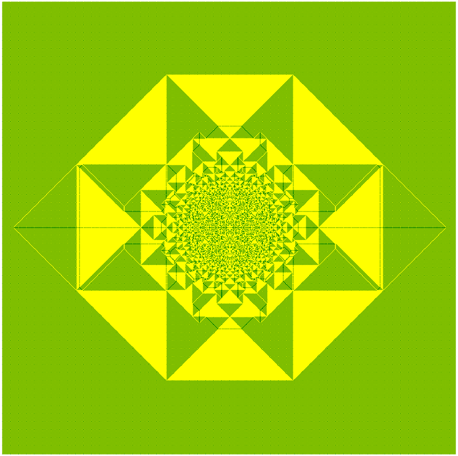


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard.

Characterizing the pattern on the F-lattice

Back-ground density 1/2

- Only two types of patches: densities 1/2 and 1.
- All boundaries are straight lines: slopes $0, \pm 1$, or ∞
- Each patch is 3- or 4-sided polygon

It is useful to look at the adjacency graph of the pattern.
Convenient to think of triangular patches as degenerate quadrilaterals.
Planar graph with coordination number 4, made of quadrilaterals.
(Except the outside patch has eight neighbors.)
The adjacency graph is a square lattice wedge of angle 4π.

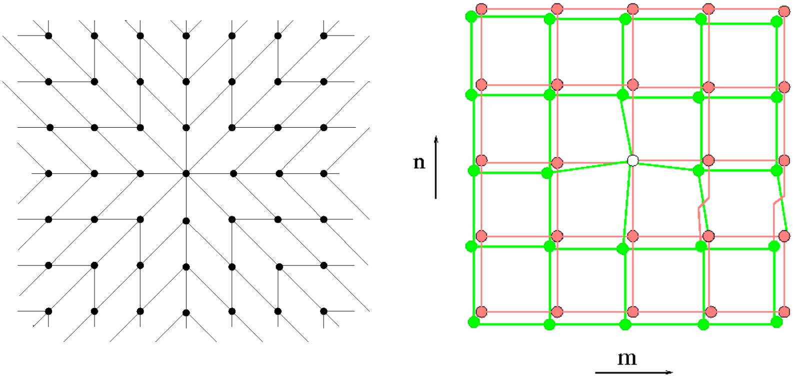


Figure: (a) Adjacency graph of the pattern. (b) representation as a square lattice wedge of wedge angle 4π.

Most easily seen by $1/z^2$ transform of the picture.
Patches are assigned integer labels (m, n).
Graph is bipartite. Patch is dense, iff $m + n =$ odd.

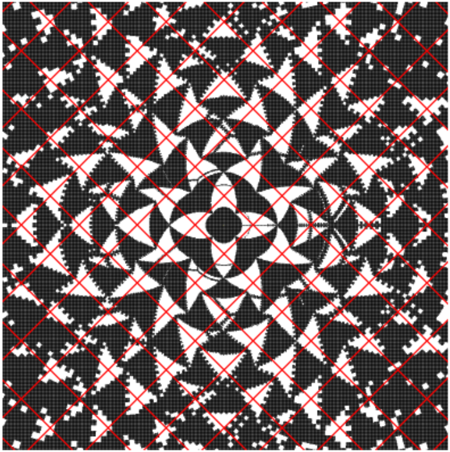
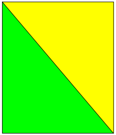


Figure: $z' = 1/z^2$ transform of original figure.

Outline

The F-lattice pattern can be obtained by using square tiles of variable size.



Mean excess density in the patch $=1/4$.
Equation of lines of the Bounding square of the pattern are

$$\xi = \pm 1; \qquad \eta = \pm 1$$

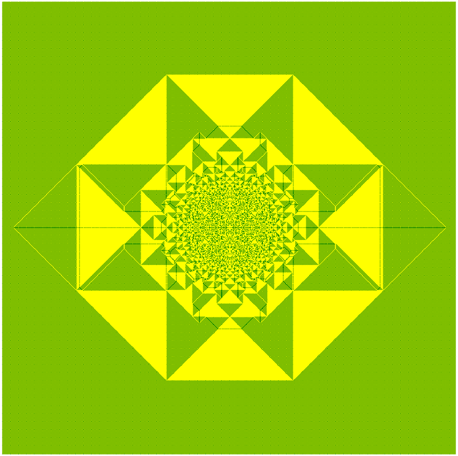


Figure: Pattern produced by adding $2x10^5$ particles at the origin, on the F-lattice with initial background being checkerboard.

Consider a dense patch P adjacent to a light patch P'.
ontinuity of ϕ and $\partial_\xi \phi$ and $\partial_\eta \phi$ at the boundary gives

$$\phi_P(\xi,\eta) = \phi_{P'}(\xi,\eta) + (1/4)[\ell_\perp(\xi,\eta)]^2$$

We parameterize ϕ for dense patches as

$$\begin{aligned}\phi_P(\xi,\eta) &= \frac{1}{8}(m_P+1)\xi^2 + \frac{1}{4}n_P\xi\eta + \frac{1}{8}(1-m_P)\eta_P^2 \\ &\quad + d_P\xi + e_P\eta + f_P\end{aligned}$$

Matching conditions relate m, n, d, e, f at neighboring patches. These can be simplified to show that $d_{m,n}$ and $e_{m,n}$ both satisfy the equation.

$$\psi_{m+1,n+1} + \psi_{m+1,n-1} + \psi_{m-1,n+1} + \psi_{m-1,n-1} - 4\psi_{m,n} = 0,$$

This has a solution $\psi(z) \sim z^{1/2}$. Near the origin $\phi \sim -\frac{1}{4\pi} \log r$. This implies that

$$d_{m,n} + ie_{m,n} \sim K\sqrt{m+in}$$

These set of linear equations can be solved numerically exactly. Then we can determine all the equations of all the boundaries.

$$\begin{aligned} e_{m+2,n} - e_{m,n} &= \frac{1}{2}\eta_{m,n} \\ d_{m-2,n} - d_{m,n} &= \frac{1}{2}\xi_{m,n} \end{aligned}$$

Thus, we have determine the coordinates of the vertices of all the patches.

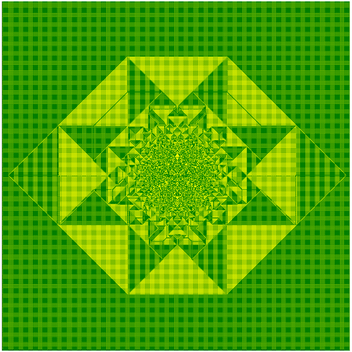
Outline

The arguments only depend on the existence of only two types of patches, and straight line boundaries.
These can be found (by trial and error) in other cases also.
Then the asymptotic pattern is identical.

Some examples:

F-lattice with density 5/8.

Initially all sites (i, j) with $i + j = 0 (mod 2)$
or $(i, j) = (0, 1)(mod 4)$ or $(i, j) = (2, 3)(mod 4)$ occupied.



We have obtained the same asymptotic pattern for some other backgrounds on the Manhattan lattice with density 1/2

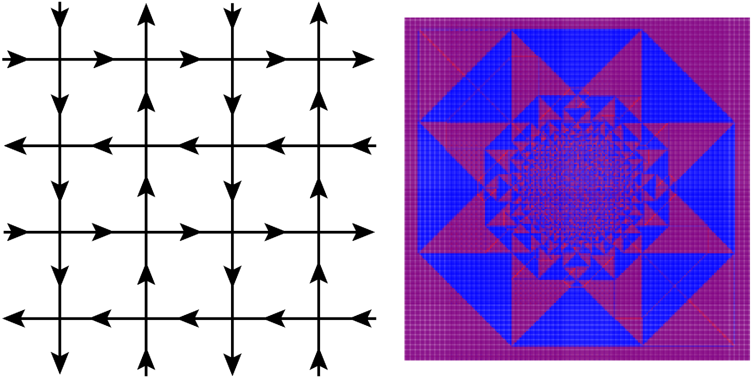


Figure: Manhattan lattice with initial checker board state, 10^5 particles.

Outline

- We can characterize quantitatively patterns when only two types of patches allowed.
- Prove that the pattern is has 8-fold rotational symmetry.
- In some cases, (e.g. F-lattice with background empty), the inner pattern can be quantified.
- Extention to more general patterns is needed.
- Criteria for what periodic patterns are allowed are not known.
- Theory of approximating functions with piecewise parabolic approximants?

References

M. C. Cross, P. C. Hohenberg, Rev of Mod Phys. **65**, 851 (1993).

S. Ostojic, Physica A **318** 187 (2003).

S. Ostojic, Diploma thesis (2002), Ecole Polytechnique Federale de Lausanne.

Deepak Dhar, T. Sadhu and Samarth Chandra, in preparation.

Thank You.