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Antiphase dynamics: a generalized reference model

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Abstract

We generalize the reference model [Optics Comm. 107 (1994) 245] of a multimode laser to include an arbitrary loss rate k. As a result, the decay rates and the oscillation frequencies of the laser relaxation are both k-dependent. Perfect antiphased dynamics is shown to remain, no matter what value is assumed by k. However, depending on k, the oscillatory component of the dynamics may be quenched either partially or completely. For a given k, there exists a cut-off mode number and a set of critical laser parameters separating domains where oscillations occur from domains where there are no oscillations.

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Since antiphase dynamics (AD) was observed experimentally [1–7] in free-running lasers described by the Tang, Statz and deMars (TSD) rate equations [8], a series of theoretical papers has been published [9–18] to elucidate the nature and properties of AD. So far, in most lasers exhibiting AD the population inversion lifetime divided by the photon lifetime, k, is large ($k \sim 10^4$ in Nd:YAG lasers and $k \sim 10^6$ in $LiNdP_4O_{12}$ lasers). This fact gave rise to the assumption that AD is in part due to this large value of k. This induced the introduction of a perturbation scheme based on the small parameter $\epsilon = 1/\sqrt{k}$. Such an ϵ-based approach yielded good agreement with experiments for lasers with large k. However, there are lasers, for example the Ti:sapphire laser or Nd-doped fiber lasers, which can also be described by the TSD equations but have smaller k, typically 10^2 to 10^3 or less. The question which arises naturally is whether finite losses destroy AD? Further, it is known [11] that if k is large, the laser dynamics is characterized by two widely separated time scales: fast relaxation oscillations $\mathcal{O}(\sqrt{k})$ and slow decays $\mathcal{O}(1)$. This is the basis for the multiple time expansion of the previous work based on the ϵ expansion. The effect of finite k is to yield oscillations and decay rates which may be of same order of magnitude or, in some circumstances, oscillations might be quenched by decay processes. A first result in this direction was obtained recently [19] where it was shown that for finite k, there exists a cut-off mode number $N_{\text{cut-off}}$ such that for $N \geq N_{\text{cut-off}}$, one of the two relaxation oscillation frequencies vanishes. In this Communication, we develop an exactly soluble model to study analytically the new physics induced by the finite loss in the dynamics of multimode Fabry-Pérot lasers.

Let i_p, d, and d_p, respectively, be the modal intensity, the space averaged population inversion and the population inversion grating in the steady state of an N-mode Fabry-Pérot laser described by the TSD equations with relative modal gains γ_p, modal intensity losses k_p and pump w. The pump is normalized to the single-mode threshold pump and the gains to the first lasing mode gain: $w \geq w_1 = 1$, $\gamma_p \leq \gamma_1 = 1$. Linear stability of the steady state solution is determined by solving the equation

$$(\mathscr{M} - \lambda \mathscr{I}) Z(\lambda) \equiv \mu(\lambda) Z(\lambda) = 0 \tag{1}$$

for the eigenvalues λ and the eigenvectors $Z(\lambda) = \text{col}\{D(\lambda), I_I(\lambda), D_I(\lambda), I_2(\lambda), D_2(\lambda), \dots, I_N(\lambda), D_N(\lambda)\}$ of the matrix

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.$\mathscr{N}$, where $I_T(\lambda)=\sum_{p=1}^{N} I_p(\lambda)$ and $D_T(\lambda)=\sum_{p=1}^{N} D_p(\lambda)$. The variables $D(\lambda)$, $I_p(\lambda)$ and $D_p(\lambda)$ are the linearized deviations from the steady state. In Eq. (1), $\mu(\lambda)$ is a $2N+1$ by $2N+1$ matrix

$$\mu(\lambda)=\begin{pmatrix} M_1(\lambda) & Q_2 & \dots & Q_{N-1} & Q_N \\ R_2 & M_2(\lambda) & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{N-1} & O & \dots & M_{N-1}(\lambda) & O \\ R_N & O & \dots & O & M_N(\lambda) \end{pmatrix},$$

in which O is the 2×2 zero matrix,

$$M_1(\lambda)=\begin{pmatrix} \lambda+U & 1 & -A_1 \\ -B & \lambda & k_1 A_1 \\ -C & G & \lambda+U \end{pmatrix},\quad M_p(\lambda)=\begin{pmatrix} \lambda & k_p A_p \\ -d\gamma_p/2 & \lambda+U \end{pmatrix},$$

$$R_p=\begin{pmatrix} -k_p A_p & 0 & 0 \\ -A_p/2 & d_p & 0 \end{pmatrix},\quad Q_p=\begin{pmatrix} 0 & A_1-A_p \\ 0 & k_p A_p - k_1 A_1 \\ (\gamma_p-1)G & 0 \end{pmatrix},$$

where $p=2, 3, \dots, N$, and

$$A_p=\gamma_p i_p,\quad B=\sum_{q=1}^{N} k_q A_q,\quad C=\tfrac{1}{2}\sum_{q=1}^{N} A_q,\quad G=(N-\tfrac{1}{2})d-\sum_{q=1}^{N}\frac{1}{\gamma_q},\quad U=1+2C.$$

In general, Eq. (1) must be solved for each mode number N separately except in the reference model (RM) [20] where the double limit $\gamma_p=1$ and $k_p=k\to\infty$ is taken. We now generalize the RM by setting $\gamma_p=1$ but letting $k_p\equiv k$ remain arbitrary. It turns out that this model can still be solved analytically for arbitrary N because the elements of Q_p become identically zeros, so that μ becomes a triangular matrix. In addition, $M_2=\dots=M_N$, $R_2=\dots=R_N$, so that Eq. (1) possesses 5 distinct eigenvalues. Three of them, λ_1, λ_2, and λ_3, are the roots of the cubic $\det M_1(\lambda) = \lambda^3+a_2\lambda^2+a_1\lambda+a_0 = 0$ and the two remaining roots, λ_4 and λ_5, which are $N-1$ degenerate, are the roots of the quadratic $\det M_2(\lambda) = \lambda^2+b_1\lambda+b_0 = 0$. The coefficients a_j are

$$a_0=(kA/2)[AN(2G-1)+2U(N-G)],$$

$$a_1=U^2+A[k(N-G)-AN/2],\quad a_2=2U,$$

while $b_0=kdA/2$ and $b_1=U$, with $A=2(d-1)/[2N-(2N-1)d]$. Making use of the properties of the steady state solution $d<2N/(2N-1)$ and $1<w-d$ [11,17,18], we can prove analytically that all the coefficients of the equations $\det M_1(\lambda) = 0$ and $\det M_2(\lambda) = 0$ are positive in the whole parameter range. This, by virtue of the Descartes' rule of signs, guarantees the negativity of Re λ_j. The steady state solutions are therefore stable: any small departure from them relaxes via either exponential decay or damped oscillations. There are at most two relaxation oscillation frequencies, Ω_R and Ω_L. The occurrence of relaxation oscillation at the frequency Ω_R depends on the sign of the discriminant of the cubic Q and oscillations at the frequency Ω_L depend on the sign of the discriminant of the quadratic Δ.

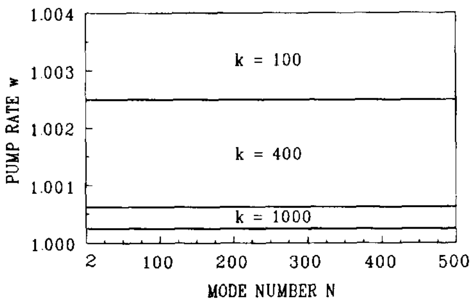

Fig. 1. Phase-diagram of the sign of Q in the (N,w) plane for different values of k. In the domain above the curve the oscillation at frequency Ω_R occurs.

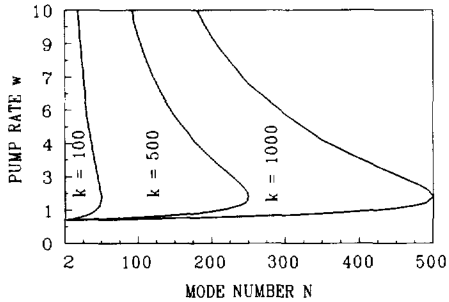

Fig. 2. Phase-diagram of the sign of Δ in the (N,w) plane for different values of k. The domain within which the oscillations at frequency Ω_L occur is surrounded by the curve.

In Fig. 1 we plot the boundary $Q=0$ in the (N,w)-plane for different values of k. The domain above the line corresponds to $Q<0$ and gives rise to oscillations with frequency $\Omega_R=(2\pi)^{-1}|\mathrm{Im}\ \lambda_{2,3}|$. The effect of k manifests itself as follows.

- For a given k there exists a critical pump w_R such that the oscillation at Ω_R only occur for $w>w_R$.
- The larger k, the closer w_R is to 1. In the RM limit $k\to\infty$, we have $w_R\to 1$ implying that the oscillation at Ω_R always occurs. The possible disappearance of the oscillation at Ω_R thus finds its origin in the finiteness of k.

From the results of Fig. 1, it is clear that the critical pump parameter w_R is close to unity. An analytic approximation of w_R can be derived by assuming $w=1+\eta$ where $\eta\ll 1$. In this case $\det M_1(\lambda)$ factorizes in leading order in η as

$$\det M_1(\lambda)=(\lambda+1)\left[\lambda^2+\left(1+\frac{4N\eta}{2N+1}\right)\lambda+k\eta\right],$$

which gives

$$w_R=1+\frac{1}{4(k-2N/(2N+1))} \tag{2}$$

in excellent agreement with Fig. 1. The condition $\eta\ll 1$ implies $k\gg 2N/(2N+1)$. This is hardly a constraint for today's lasers since $0.8\le 2N/(2N+1)\le 1$.

The boundary $\Delta=0$ is shown in Fig. 2. The domain within the curve corresponds to $\Delta<0$ and gives rise to another oscillation $\Omega_L=(2\pi)^{-1}|\mathrm{Im}\ \lambda_{4,5}|$. The effect of k is as follows.

- The larger k, the wider the domain surrounded by the curve. In the RM limit $k\to\infty$ such domain fills the whole physical quadrant ($w>1$, $N\ge 2$) implying that the oscillation at Ω_L always occurs. The possible disappearance of the oscillation at Ω_L is thus also a finite k effect.
- For a given k, there is a cut-off mode number $N_{\text{cut-off}}$ beyond which there is no oscillation at Ω_L. A closer look at Fig. 2 suggests the relation $N_{\text{cut-off}}\simeq k/2$.
- For each mode number $N<N_{\text{cut-off}}$ there exist two critical pumps w_{L1} and w_{L2} such that there is no oscillation at Ω_L if either $1<w\le w_{L1}$ or $w\ge w_{L2}$.
- For a given pair of k and w, there exists a critical number N_L such that oscillation at Ω_L is possible only if $N<N_L$.

Analytic expressions for $N_{\text{cut-off}}$, $w_{L1,L2}$ and N_L can be derived when the mode number N is large. To leading order in $\mathcal{O}(1/N)$, Δ is

$$\Delta=(w-\alpha)^2+\alpha(2-\alpha),\quad \alpha=(2k+1)/(2N). \tag{3}$$

Clearly, Eq. (3) implies that for $\alpha\le 2$, i.e., $N\ge N_{\text{cut-off}}=(2k+1)/4\simeq k/2$, we have $\Delta\ge 0$ for any w: oscillation at Ω_L is impossible. This confirms the result obtained numerically for $N_{\text{cut-off}}$. For $\alpha>2$, the two critical values of the pump parameter are given by

$$w_{L1}=\alpha-\sqrt{\alpha(\alpha-2)},\quad w_{L2}=\alpha+\sqrt{\alpha(\alpha-2)}. \tag{4}$$

Note that $w_{L1,L2}$ depend on the ratio k/N rather than on k and on N separately if $k\gg 1$. Eq. (3) also yields an expression for N_L

$$N_L=(2k+1)(w-1)/w^2. \tag{5}$$

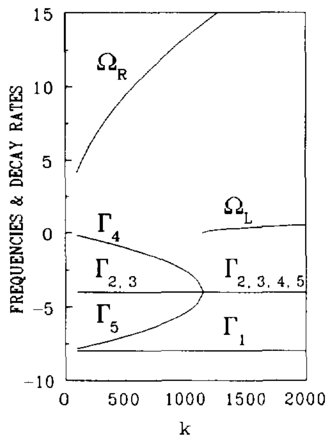

Fig. 3. Relaxation oscillation frequencies and decay rates as functions of $k \geq 100$ for $w = 8$ and $N = 250$.

From this expression, it follows that N_L reaches its maximum at $w = 2$, i.e.

$$\operatorname{Max} N_L = (2k+1)/4 = N_{\text{cut-off}}, \tag{6}$$

as it should be. Formulae (4)–(6) agree very well with the numerical result for $N \geq 10$. For $N < 10$, the numerical simulations indicate that formula (4) gives an upper bound for w_{L1} and a lower bound for w_{L2}.

Our results predict the quenching of oscillations at Ω_R and/or Ω_L. In lasers, however, k is greater than 10^2 as of now. Hence, only the disappearance of Ω_L is of interest. Indeed, to observe the disappearance of Ω_R either the laser should be pumped very close to the threshold ($w < w_R \leq 1.0025$ for $k \geq 10^2$) or the loss should be too small at a reasonable pump rate ($k \leqslant k_R \simeq \{3.1, 1.9, \dots\}$ for $w = \{1.1, 1.2, \dots\}$. None of these conditions are realistic at present. Fig. 3 is a plot of $\Omega_{R,L}$ and $\Gamma_j = \operatorname{Re} \lambda_j$ as functions of $k \geq 100$ for $w = 8$ and $N = 250$. We see the existence of a critical loss k_L,

$$k_L = U^2/2dA,$$

in terms of which Ω_L is

$$\Omega_L = (1/\pi)\sqrt{dA(k-k_L)/2}.$$

For $k \leq k_L$, Ω_L disappears and $\lambda_{4,5}$ bifurcates into two distinct decay rates Γ_4 and Γ_5. For the parameters used in Fig. 3, $k_L = 1140$. The vanishing of Ω_L could therefore be expected in lasers like the Ti:sapphire laser where $k = 10^2$ to 10^3. In general, the decay rates are k-dependent and satisfy the relation

$$2(\Gamma_4 + \Gamma_5) = \Gamma_1 + \Gamma_2 + \Gamma_3 = -2U.$$

In the presence of oscillations at both Ω_R and Ω_L this relation simplifies to

$$2\Gamma_L = \tfrac{1}{2}\Gamma_1 + \Gamma_R = -U,$$

where $\Gamma_4 = \Gamma_5 = \Gamma_L$ and $\Gamma_2 = \Gamma_3 = \Gamma_R$. Furthermore, for small mode numbers ($N < 10$), $|\Gamma_L|$ is close to though smaller than $|\Gamma_R|$. However, for large mode numbers ($N \geq 10$) one has

$$\Gamma_R = \Gamma_L = \tfrac{1}{2}\Gamma_1 \simeq -w/2.$$

Therefore, when a large number of modes lase, the damping rates are proportional to the pump rate.

We complete this analysis by determining the eigenvectors of the matrix $\mathscr{M}$ from which the phase coherence of the lasing modes is obtained. For $j = 1, 2, 3$ we get

$$Z_1(\lambda_j) = \operatorname{col}\left\{ -\frac{\lambda_j + U + AG}{R_j}, 1, -\frac{2(\lambda_j + U)G + NA}{R_j}, \frac{1}{N}, -\frac{2(\lambda_j + U)G + NA}{NR_j}, \dots, \frac{1}{N}, -\frac{2(\lambda_j + U)G + NA}{NR_j} \right\}. \tag{7}$$

with $R_j = 2(\lambda_j + U)^2 - NA^2$, while for $j = 4,5$ we have

$$\begin{array}{lcl} Z_2(\lambda_j) & = & \mathrm{col}\{0,0,0,2(\lambda_j+U)/d,1,0,0,\dots,0,0\}, \\ Z_3(\lambda_j) & = & \mathrm{col}\{0,0,0,0,0,2(\lambda_j+U)/d,1,\dots,0,0\}, \\ \vdots & \vdots & \qquad\qquad\vdots \\ Z_N(\lambda_j) & = & \mathrm{col}\{0,0,0,0,0,0,0,\dots,2(\lambda_j+U)/d,1\}. \end{array} \tag{8}$$

The structure of the eigenvectors (7) corresponding to $\lambda_{1,2,3}$ indicates that they describe an inphased dynamics since they contribute to the total intensity (second component of the eigenvector) and all modal intensities have equal weight $1/N$. On the contrary, the eigenvectors (8) corresponding to $\lambda_{4,5}$ describe an antiphased dynamics irrespectively of how the λ_j depend on k and on the other parameters since they do not contribute at all to the total intensity. Hence perfect AD, i.e. a zero contribution from the eigenvectors (8) to the total intensity, is not affected by the magnitude of k but solely depends on the gain degeneracy $\gamma_p = 1$. More explicitly, the dynamics governed by (7) and (8) can be expressed as follows. The time evolution of a modal intensity I_p is a linear superposition $\sum_{j=1}^{3} c_j \exp(\lambda_j t) + \sum_{j=4}^{5}\sum_{q=2}^{N} c_{jq} u_{qp}(\lambda_j)\exp(\lambda_j t)$, where the c_j and the c_{jq} are determined by the initial condition and $u_{qp}(\lambda_j) = (\delta_{qp} - \delta_{1p})(\lambda_j + U)$. The total intensity is simply given by $N\sum_{j=1}^{3} c_j \exp(\lambda_j t)$. That is, the modal intensity contributions proportional to $\exp(\lambda_{4,5} t)$ completely cancel out each other, but those proportional to $\exp(\lambda_{1,2,3} t)$ add up to a nonzero contribution. Such self-organized dynamics takes place for arbitrary k and is independent of the initial condition.

In conclusion, we find that AD is not related to the limit $k \to \infty$. Its physical origin rests in the gain symmetries among the lasing modes. In the present model, perfect AD occurs because the modal gains are identical. If this degeneracy is removed, a partial AD subsists, as shown in Ref. [16], leading to a nonzero but very small contribution of the modal intensities to the total intensity proportional to $\exp(\lambda_{4,5} t)$. The consequence of finite loss is the appearance of a set of critical quantities w_R, $w_{L1,L2}$, N_L, $N_{\text{cut-off}}$ and $k_{R,L}$ which separate physical domains where the laser relaxation undergoes oscillations from domains where oscillations are quenched either partially or completely. Since from (2) and (4) it follows that $w_R < w_{L1} < w_{L2}$, the following sequence appears for increasing pumping rate:

- For $1 < w \le w_R$ there is no internal relaxation oscillation at all.
- For $w_R < w \le w_{L1}$ the relaxation oscillates with a single frequency Ω_R only.
- For $w_{L1} < w < w_{L2}$ the lower frequency Ω_L emerges and the relaxation oscillates with both the frequencies Ω_R and Ω_L.
- For $w_{L2} \le w$ the oscillation with Ω_L dies whereas that with Ω_R survives.

Finally, let us stress that AD is not necessarily bound to the existence of oscillations, damped or not, periodic or chaotic. It also takes place in the domain where all characteristic roots are real since the structure of the eigenvectors (6) and (7) does not depend on whether the λ_j are real or complex.

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