J. Phys. C: Solid State Phys. **21** (1988) L1209–L1211. Printed in the UK

LETTER TO THE EDITOR

A step-by-step Bogoliubov transformation method for diagonalising a kind of non-Hermitian effective Hamiltonian

Nguyen Ba An
Centre for Theoretical Physics, Academy of Sciences of Vietnam, Nghia Do, Tu Liem, Ha Noi, Vietnam

Received 25 August 1988

Abstract. A method is presented for diagonalising a non-Hermitian Hamiltonian that can be used to plot the giant polariton dispersion relation.

As has long been known, the quadratic form of the Hermitian-Hamiltonian-like expression

$$H = \sum_k [E_k a_k^+ a_k + \omega_k c_k^+ c_k + g_k (a_k^+ c_k + c_k^+ a_k)] \tag{1}$$

with a and c being bosonic operators and E, ω and g being real quantities can be exactly diagonalised (Bogoliubov and Tjablikov 1949) into

$$H = \sum_{k;\, \nu=1,2} \Omega_{\nu k} \alpha_{\nu k}^+ \alpha_{\nu k} \tag{2}$$

which describes new eigenstates of the system of two kinds of quasi-particles interacting with each other. Should these quasi-particles be the photon and the exciton, their new eigenstates will be the polariton (Hopfield 1958, Agranovich 1959). The reversible $a(c)$-to-α transformation is performed by means of the Bogoliubov coefficients $u_{\nu k}$ and $v_{\nu k}$ (Nguyen 1983)

$$\alpha_{\nu k} = u_{\nu k} c_k + v_{\nu k} a_k \qquad \alpha_{\nu k}^+ = u_{\nu k} c_k^+ + v_{\nu k} a_k^+ \tag{3}$$

$$c_k = \sum_{\nu=1,2} u_{\nu k} \alpha_{\nu k} \qquad c_k^+ = \sum_{\nu=1,2} u_{\nu k} \alpha_{\nu k}^+ \tag{4}$$

$$a_k = \sum_{\nu=1,2} v_{\nu k} \alpha_{\nu k} \qquad a_k^+ = \sum_{\nu=1,2} v_{\nu k} \alpha_{\nu k}^+ \tag{5}$$

$$u_{\nu k} = [1 + g_k^2/(\Omega_{\nu k} - E_k)^2]^{-1/2} \tag{6}$$

$$v_{\nu k} = g_k u_{\nu k}/|\Omega_{\nu k} - E_k| \tag{7}$$

$$\Omega_{\nu k} = \tfrac{1}{2}\{\omega_k + E_k + (-1)^\nu [(E_k - \omega_k)^2 + 4g_k^2]^{1/2}\}. \tag{8}$$

The polariton theory is quite adequate for interpreting many optical phenomena at low light intensity, such as resonant electronic Raman scattering on donors (Nguyen *et al* 1979, 1980, Nguyen and Nguyen 1986, Ulbrich *et al* 1981), light-by-light scattering (Nguyen *et al* 1981) etc. However, when intense laser beams are applied the polariton concept needs to be treated with more precision (Haken and Schensle 1972, Liu 1983, Advjugin *et al* 1983, Nguyen 1988a, b). An effective Hamiltonian for polaritons at high light intensity (the so-called giant polaritons (Haken *et al* 1972)) was constructed by Liu (1983). Unfortunately, Liu's effective Hamiltonian H_{eff}, being Hermitian, is not self-consistent. Thus, Liu's equation (16*b*) cannot be derived from his H_{eff} as is required. Another appropriate effective Hamiltonian of bosonic operators, which will allow us to generate the physically correct set of equations of motion has since been provided by Nguyen (1988b). In that work the inconsistencies in the work of Liu (1983) and the incorrectness of Avdjugin (1983) have also been discussed. It should be noted that to be self-consistent the effective Hamiltonian of Nguyen (1988b) must be a non-Hermitian one, which is, in general, of the form

$$H = \sum_k \{[E_k a_k^+ a_k + \omega_k c_k^+ c_k + g_k(a_k^+ c_k + c_k^+ a_k)] + f_k a_k^+ c_k\}. \tag{9}$$

In the work of Nguyen (1988b) f_k is proportional to the exciton density, but here its concrete expression is not specified. Clearly, the polariton picture at a high level of excitation is connected with the diagonalisation of such a non-Hermitian Hamiltonian (9). This Letter attempts to present a possible method for diagonalising the non-Hermitian Hamiltonain given in (9), which now acquires a more generalised mathematical meaning.

It is apparent that the bracketed part of (9) is exactly diagonalised into (2) by the transformations given in (3). As for the last non-Hermitian term of (9), using (4, 5) to express it in terms of α and α^+ and then combining them with the diagonalised part we obtain (9) in the form:

$$H = \sum_{k;\nu=1,2} \left(\Omega_{\nu k} + f_k u_{\nu k} v_{\nu k}\right) \alpha_{\nu k}^+ \alpha_{\nu k} + \sum_{k;\nu,\nu'=1,2}' f_k u_{\nu k} v_{\nu' k} \alpha_{\nu k}^+ \alpha_{\nu' k}. \tag{10}$$

The superior prime in the symbol Σ excludes the terms with $\nu = \nu'$. Summing (10) over ν and ν' we obtain:

$$\begin{aligned} H = \sum_k \{&[(\Omega_{1k} + f_k u_{1k} v_{1k})\alpha_{1k}^+ \alpha_{1k} + (\Omega_{2k} + f_k u_{2k} v_{2k})\alpha_{2k}^+ \alpha_{2k}] \\ &+ f_k u_{1k} v_{2k}(\alpha_{1k}^+ \alpha_{2k} + \alpha_{2k}^+ \alpha_{1k}) \\ &+ f_k(u_{2k} v_{1k} - u_{1k} v_{2k})\alpha_{2k}^+ \alpha_{1k}\}. \end{aligned} \tag{11}$$

Using the following transformations:

$$\Omega_{1k} + f_k u_{1k} v_{1k} \to \omega_k^{(1)} \qquad \Omega_{2k} + f_k u_{2k} v_{2k} \to E_k^{(1)} \tag{12}$$

$$f_k u_{1k} v_{2k} \to g_k^{(1)} \qquad f_k(u_{2k} v_{1k} - u_{1k} v_{2k}) \to f_k^{(1)} \tag{13}$$

$$\alpha_{1k} \to c_k^{(1)} \qquad \alpha_{1k}^+ \to (c_k^{(1)})^+ \qquad \alpha_{2k} \to a_k^{(1)},\ \alpha_{2k}^+ \to (a_k^{(1)})^+ \tag{14}$$

equation (11) then becomes

$$\begin{aligned} H^{(1)} = \sum_k \{&[E_k^{(1)}(a_k^{(1)})^+ a_k^{(1)} + \omega_k^{(1)}(c_k^{(1)})^+ c_k^{(1)} + g_k^{(1)}((a_k^{(1)})^+ c_k^{(1)} \\ &+ (c_k^{(1)})^+ a_k^{(1)})] + f_k^{(1)}(a_k^{(1)})^+ c_k^{(1)}\}. \end{aligned} \tag{15}$$

We see that equation (15), like (9), remains non-Hermitian with another non-Hermitian term proportional to $f_k^{(1)}$. Taking into account the explicit expressions of $u_{\nu k}$ and $v_{\nu k}$ from (6) and (7) as well as (8) and (13) we can prove that

$$f_k^{(1)} = f_k[1 + 4g_k^2/(\omega_k - E_k)^2]^{-1/2}. \tag{16}$$

For the value of $\boldsymbol{k}$ such that $\omega_k = E_k$ (perfect resonance), $f_k^{(1)}$ must vanish and $H^{(1)}$ becomes a Hermitian expression which will be exactly diagonalised by again applying Bogoliubov transformations. Otherwise, $|f_k^{(1)}|$ is non-zero but always less than $|f_k|$ (see (16)). In this case, again performing the steps from (9) to (15), we could have $H^{(2)}$ containing the non-Hermitian term proportional to $f_k^{(2)}$, and so on. It is not difficult to verify that at the nth step in the application of Bogoliubov transformations in sequence we have

$$f_k^{(n)} = f_k^{(n-1)}[1 + 4(g_k^{(n-1)})^2/(\omega_k^{(n-1)} - E_k^{(n-1)})^2]^{-1/2}. \tag{17}$$

Equation (17) reveals that $|f_k|$, $|f_k^{(1)}|$, $|F_k^{(2)}|$, . . . form a decreasing series, which should at some step either vanish or converge with any desired accuracy. This means that the non-Hermitian Hamiltonian of the initial form given in (9) will become more and more Hermitian as successive Bogoliubov transformations are performed. Thus, for any value of $\boldsymbol{k}$, we might diagonalise H either exactly or with any derived accuracy. Using such a method, the non-Hermitian effective Hamiltonian formalism developed in (Nguyen 1988b) could be used to plot the giant polariton dispersion relation in the whole range of the wavevector $\boldsymbol{k}$. The numerical calculation, as a rule, must be carried out by a computer. Given the nature of the diagonalisation procedure, we call our method described above a step-by-step Bogoliubov transformation. It is a procedure that might find various applications other than that to the giant polariton problem.

The author is grateful to Professor Nguygen Van Hieu and Professor Dao Vong Duc for their constant support.

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