Two-curve Green's function for 2-SLE

Dapeng Zhan

Michigan State University

Random Conformal Geometry and Related Fields
June 18-22, 2018, KIAS

2-SLE

A multiple 2-SLE$_\kappa$ ($\kappa \in (0,8)$) is a pair of random curves (η_1, η_2) in a simply connected domain D connecting two pairs of boundary points $(a_1, b_1; a_2, b_2)$ such that conditioning on any curve, the other is a chordal SLE$_\kappa$ curve in a complement domain. If $\kappa \in (0,4]$, η_1 and η_2 are disjoint; if $\kappa \in (4,8)$, η_1 and η_2 may or may not intersect.

A 2-SLE arises naturally as interacting flow lines in imaginary geometry, as scaling limit of some lattice model with alternating boundary conditions, and as two exploration curves of a CLE.

It is known that a 2-SLE$_\kappa$ exists for any $\kappa \in (0,8)$ and any admissible connection pattern $(D; a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_2)$, and its law is unique. Moreover, the marginal law of either η_1 or η_2 is that of an hSLE$_\kappa$, i.e., hypergeometric SLE$_\kappa$ with the other pair of points as force points.

Two-curve Green's Function

A two-curve Green's function for a 2-SLE$_\kappa$: (η_1, η_2) at $z_0 \in D$ is the limit

$$\lim_{r \downarrow 0} r^{-\alpha} \mathbb{P}[\mathrm{dist}(z_0, \eta_j) < r, j = 1, 2]$$

for some suitable α. We need to find the correct exponent α, prove the convergence of the limit, and find the explicit formula for the limit.

We can ask the similar question for a point $z_1 \in \partial D \setminus \{a_1, b_1, a_2, b_2\}$ assuming that ∂D is smooth near z_1.

Works on one-curve Green's Function

- Lawler, '09: conformal radius version of Green's function for chordal SLE.
- Lawler-Rezaei, '15: Euclidean distance Green's function for chordal SLE.
- Lawler, '15: boundary point Green's function for chordal SLE.
- Alberts-Kozdron-Lawler, '12: Green's function for radial SLE.
- Lenells-Viklund, '17: Green's function for $\mathrm{SLE}_\kappa(\rho)$ and hSLE.
- Lawler-Werness, '13: two-point Green's function for chordal SLE.
- Rezaei-Zhan, '16: multi-point Green's function for chordal SLE.
- Mackey-Zhan, '17: multi-point estimate for radial SLE.

Main Results

Throughout, we fix $\kappa \in (0,8)$. Sometimes we require that $\kappa \in (4,8)$. A constant depends only on κ. Define two exponents:

$$\alpha_0 = \frac{(12-\kappa)(\kappa+4)}{8\kappa}, \quad \alpha_1 = \frac{2}{\kappa}(12-\kappa).$$

The α_0 appeared in the work [Miller-Wu], where $2-\alpha_0$ was shown to be the Hausdorff dimension of the double points of SLE_κ for $\kappa \in (4,8)$.

Let F be the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma,\cdot)$ with $\alpha = \frac{4}{\kappa}$, $\beta = 1-\frac{4}{\kappa}$, $\gamma = \frac{8}{\kappa}$, defined by

$$F(x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n,$$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ if $n \geq 1$. The series has radius 1. With these particular parameters α, β, γ, F extends continuously to $x=1$, and is positive on $[0,1]$. Such F was used to define hSLE_κ.

Interior Point Green's Function

Let D be a simply connected domain with four distinct boundary points (prime ends): a_1, b_1, a_2, b_2 such that b_1 and b_2 do not lie on the same connected component of $\partial D \setminus \{a_1, a_2\}$. Define $G^0_{D;a_1,b_1;a_2,b_2}$ on D by the following. If $D = \mathbb{D} = \{|z| < 1\}$ and $z_0 = 0$, then

$$G^0_{\mathbb{D};a_1,b_1;a_2,b_2}(0) = (|a_1 - b_1||a_2 - b_2|)^{\frac{8}{\kappa}-1}(|a_1 - a_2||b_1 - b_2|)^{\frac{4}{\kappa}} \times F\Big(\frac{|a_1 - b_2||a_2 - b_1|}{|a_1 - a_2||b_1 - b_2|}\Big)^{-1}.$$

In general, if f maps D conformally onto $\mathbb{D}$ and takes z_0 to 0, then

$$G^0_{D;a_1,b_1;a_2,b_2}(z_0) = |f'(z_0)|^{\alpha_0} G^0_{\mathbb{D};f(a_1),f(b_1);f(a_2),f(b_2)}(0).$$

Main Theorems

Theorem

Let (η_1, η_2) *be a* 2-SLE$_\kappa$ *with connection pattern* $(D; a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_2)$. *There exist constants* $C_0, \beta_0 > 0$ *such that for any* $z_0 \in D$, *with* $R := \mathrm{dist}(z_0, \partial D)$,

$$\mathbb{P}[\mathrm{dist}(z_0, \eta_j) < r, j = 1, 2] = r^{\alpha_0} C_0 G^0_{D; a_1, b_1; a_2, b_2}(z_0)(1 + O(r/R)^{\beta_0}).$$

This implies that

$$\mathbb{P}[\mathrm{dist}(z_0, \eta_j) < r, j = 1, 2] \lesssim \left(\frac{r}{R}\right)^{\alpha_0}.$$

If $\kappa \in (4, 8)$, *then there is a constant* $C_0' > 0$ *such that*

$$\mathbb{P}[\mathrm{dist}(z_0, \eta_1 \cap \eta_2) < r] = r^{\alpha_0} C_0' G^0_{D; a_1, b_1; a_2, b_2}(z)(1 + O(r/R)^{\beta_0}).$$

Boundary Point Green's Function

Define another function $G^1_{D;a_1,b_1;a_2,b_2}$ on the analytic part of $\partial D \setminus \{a_1,b_1,a_2,b_2\}$ by the following. If $D=\mathbb{H}=\{\operatorname{Im} z>0\}$, $z_1=0$ and $a_1,b_1,a_2,b_2\in\mathbb{R}\setminus\{0\}$, then

$$\begin{aligned}G^1_{\mathbb{H};a_1,b_1;a_2,b_2}(0)=&(|a_1-b_1||a_2-b_2|)^{\frac{8}{\kappa}-1}(|a_1-a_2||b_1-b_2|)^{\frac{4}{\kappa}}\\&\times|a_1b_1a_2b_2|^{1-\frac{12}{\kappa}}F\Big(\frac{|a_1-b_2||a_2-b_1|}{|a_1-a_2||b_1-b_2|}\Big)^{-1}.\end{aligned}$$

In general, if f maps D conformally onto $\mathbb{H}$ and takes z_1 to 0, then

$$G^1_{D;a_1,b_1;a_2,b_2}(z_1)=|f'(z_1)|^{\alpha_1}G^1_{\mathbb{H};f(a_1),f(b_1);f(a_2),f(b_2)}(0).$$

Theorem

There exist constants $C_1, C_1', \beta_1 > 0$ *such that if* $D = \mathbb{H}$, $z_1 \in \mathbb{R} \setminus \{a_1, b_1, a_2, b_2\}$, *then with* $R := \mathrm{dist}(z_1, \{a_1, b_1, a_2, b_2\})$,

$$\mathbb{P}[\mathrm{dist}(z_1, \eta_j) < r, j = 1, 2] = r^{\alpha_1} C_1 G^1_{\mathbb{H};a_1,b_1;a_2,b_2}(z_1)(1 + O(r/R)^{\beta_1});$$

if $\kappa \in (4, 8)$, *then*

$$\mathbb{P}[\mathrm{dist}(z_1, \eta_1 \cap \eta_2) < r] = r^{\alpha_1} C_1' G^1_{\mathbb{H};a_1,b_1;a_2,b_2}(z_1)(1 + O(r/R)^{\beta_1}).$$

For a general D *and an analytic point* $z_1 \in \partial D \setminus \{a_1, b_1, a_2, b_2\}$, *we have*

$$\lim_{r \downarrow 0} r^{-\alpha_1} \mathbb{P}[\mathrm{dist}(z_1, \eta_j) < r, j = 1, 2] = C_1 G^1_{D;a_1,b_1;a_2,b_2}(z_1);$$

and when $\kappa \in (4, 8)$,

$$\lim_{r \downarrow 0} r^{-\alpha_1} \mathbb{P}[\mathrm{dist}(z_1, \eta_1 \cap \eta_2) < r] = C_1' G^1_{D;a_1,b_1;a_2,b_2}(z_1).$$

Commuting hSLEs

We focus on the interior case, and suppose $D = \mathbb{D}$ and $z_0 = 0$. Because of the reversibility of SLE_κ, we may assume that η_j grows from a_j to b_j, $j = 1, 2$. If η_j disconnects 0 from b_j at some time T_j, then it will not get closer to 0 after T_j. So it suffices to consider the portions of η_1 and η_2 before separating 0 from b_1 and b_2. We may parametrize these portions of η_1 and η_2 using radial parametrization (viewed from 0). Then they become two radial Loewner curves such that

- η_1 is an hSLE_κ in $\mathbb{D}$ from a_1 to b_1 with force points a_2 and b_2; and η_2 is likewise.
- η_1 and η_2 commute with each other in the sense that if τ_2 is a stopping time for η_2 that happens before T_2, then conditional on $\eta_2|_{[0,\tau_2]}$, η_1 up to hitting $\eta_2[0, \tau_2]$ is an hSLE_κ from a_1 to b_1 in a complement domain of $\eta_2[0, \tau_2]$ in D with force points $\eta_2(\tau_2)$ and b_2; and η_2 is likewise.

Commuting radial $\mathrm{SLE}_\kappa(2,2,2)$

There are other pairs of random curves in $\mathbb{D}$ starting from a_1 and a_2 that satisfy similar commutation relations. One of them is the commuting pair of radial $\mathrm{SLE}_\kappa(2,2,2)$ curves. More specifically, there is a pair (η_1,η_2) such that

- η_1 is a radial $\mathrm{SLE}_\kappa(2,2,2)$ curve in $\mathbb{D}$ from a_1 to 0 with force points b_1, a_2, b_2; and η_2 is likewise.
- If τ_2 is a stopping time for η_2, then conditional on $\eta_2|_{[0,\tau_2]}$, η_1 is a radial $\mathrm{SLE}_\kappa(2,2,2)$ curve from a_1 to 0 in a complement domain of $\eta_2[0,\tau_2]$ in D with force points $\eta_2(\tau_2), a_1, b_1$; and η_2 is likewise.

4-SLE

These η_1 and η_2 both end at 0 and do not intersect with each other at other points. Given (η_1, η_2), if we further draw chordal SLE_κ curves γ_1 and γ_2 in two complement domains of $\eta_1 \cup \eta_2$ in D from b_1 and b_2, respectively, to 0, then $(\eta_1, \eta_2, \gamma_1, \gamma_2)$ form a 4-SLE_κ with connection pattern $(\mathbb{D}; a_j \to 0, b_j \to 0, j = 1, 2)$: if we condition on any three of them, the remaining curve is a chordal SLE_κ curve. It can be understood as a 2-SLE_κ with connection pattern $(\mathbb{D}; a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_2)$ conditioned on the event that both curves pass through 0.

Comparing Laws

Let $\mathbb{P}_2$ denote the joint law of the radial Loewner driving functions of a 2-SLE$_\kappa$ with connection pattern $(\mathbb{D}; a_1 \to b_1, a_2 \to b_2)$. Let $\mathbb{P}_4$ denote the joint law of the radial Loewner driving functions of the curves starting from a_1 and a_2 of a 4-SLE$_\kappa$ with connection pattern $(\mathbb{D}; a_j \to 0, b_j \to 0, j = 1, 2)$. Using Girsanov Theorem and some study of two-time-parameter martingales, we can conclude that $\mathbb{P}_2$ is locally absolutely continuous w.r.t. $\mathbb{P}_4$, and derive the local Radon-Nikodym derivatives.

Two-parameter Filtration

Here is the setup. Let $\Sigma = \bigcup_{0<T\leq\infty} C([0,T),\mathbb{R})$. For $f \in \Sigma$, let T_f denote its lifetime, which may be finite or infinite. For each $t \geq 0$, let

$$\mathcal{F}_t := \sigma(\{f \in \Sigma : T_f > s, f(s) \in A\} : 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R})).$$

Then we get a filtration $(\mathcal{F}_t)_{t\geq 0}$. We mainly work on the space Σ^2, and understand that $\mathbb{P}_2$ and $\mathbb{P}_4$ are probability measures on Σ^2.

The first and second coordinates of Σ^2 respectively generate filtrations $(\mathcal{F}_t^1)_{t\geq 0}$ and $(\mathcal{F}_t^2)_{t\geq 0}$. Let $\mathcal{Q} = [0,\infty)^2$ be the first quadrant with partial order: $\underline{t} = (t_1, t_2) \leq \underline{s} = (s_1, s_2)$ if $t_1 \leq s_1$ and $t_2 \leq s_2$. Then we get an $\mathcal{Q}$-indexed filtration $(\mathcal{F}_{\underline{t}})_{\underline{t}\in\mathcal{Q}}$ by $\mathcal{F}_{(t_1,t_2)} = \mathcal{F}_{t_1}^1 \vee \mathcal{F}_{t_2}^2$. An $(\mathcal{F}_{\underline{t}})$-stopping time is a function $\underline{T} : \Sigma^2 \to \mathcal{Q}$ such that $\{\underline{T} \leq \underline{t}\} \in \mathcal{F}_{\underline{t}}$, $\forall \underline{t} \in \mathcal{Q}$.

Radon-Nikodym Derivative

Let $\mathcal{R}$ denote the set of $(t_1, t_2) \in \mathcal{Q}$ such that t_j is less than the lifetime of η_j, $j = 1, 2$, and $\eta_1[0, t_1] \cap \eta_2[0, t_2] = \emptyset$. We have $\mathbb{P}_4$-a.s. $\mathcal{R} = \mathcal{Q}$ but $\mathbb{P}_2$-a.s. $\mathcal{R} \subsetneqq \mathcal{Q}$. For $(t_1, t_2) \in \mathcal{R}$, let $D(t_1, t_2)$ denote the connected component of $\mathbb{D} \setminus (\eta_1[0, t_1] \cup \eta_2[0, t_2])$ that contains 0. It has four boundary points: $\eta_1(t_1), b_1, \eta_2(t_2), b_2$ in the cw or ccw order.

Lemma

Let $M_{4\to 2}(t_1, t_2) = G_{D(t_1,t_2);\eta_1(t_1),b_1;\eta_2(t_2),b_2}(0)^{-1}$ for $(t_1, t_2) \in \mathcal{R}$. Then for any $(\mathcal{F}_{\underline{t}})$-stopping time $\underline{T}$,

$$\frac{d\mathbb{P}_2|\mathcal{F}_{\underline{T}} \cap \{\underline{T} \in \mathcal{R}\}}{d\mathbb{P}_4|\mathcal{F}_{\underline{T}} \cap \{\underline{T} \in \mathcal{R}\}} = \frac{M_{4\to 2}(\underline{T})}{M_{4\to 2}(\underline{0})}.$$

In particular,

$$\mathbb{P}_2[\underline{T} \in \mathcal{R}] = \mathbb{E}_4[M_{4\to 2}(\underline{T})] * G_{\mathbb{D};a_1,b_1;a_2,b_2}(0).$$

We will apply the lemma to some suitable stopping times. One natural choice is $\underline{T} = \underline{T}^r := (T_1^r, T_2^r)$, where T_j^r is the first time (in the radial parametrization) that η_j reaches the disc $\{|z| \le r\}$. Then the event that $\operatorname{dist}(z_1, \eta_j) < r$, $j = 1, 2$, agrees with the event that $\underline{T}^r \in \mathcal{R}$. However, it is not easy to compute $\mathbb{E}_4[M_{4\to 2}(\underline{T}^r)]$. We know almost nothing about T_j^r except that $-\log(r) \ge T_j^r \ge -\log(4r)$. We will consider different stopping times instead.

Following Lawler's approach on the Green's function for chordal SLE, we will first use conformal radius instead of Euclidean distance. One obstacle here is that the conformal radius of $\mathbb{D} \setminus (\eta_1[0, t_1] \cup \eta_2[0, t_2])$ viewed from 0 is comparable to $\min\{\operatorname{dist}(0, \eta_1[0, t_1]), \operatorname{dist}(0, \eta_2[0, t_2]),$ not to $\max\{\operatorname{dist}(0, \eta_1[0, t_1]), \operatorname{dist}(0, \eta_2[0, t_2])$. The latter quantity is what we are really interested in. The way that we overcome this is to let η_1 and η_2 grow simultaneously (but with random speeds).

Simultaneous Growth

We now assume that b_1 and b_2 are opposite points in $\mathbb{D}$ viewed from 0, i.e., $b_1 + b_2 = 0$. This assumption is not critical because if it is not the case, we may always grow η_1 or η_2 for some small piece and map the remaining domain back to the unit disc such that the images of b_1 and b_2 are opposite points on the circle. With the assumption, there exists a unique continuous and strictly increasing function $\underline{u} = (u_1, u_2) : [0, T^u) \to \mathcal{R}$ with the properties that $\underline{u}(0) = \underline{0}$, and for any $0 \le t < T^u$,

(I) b_1 and b_2 are conformal opposite points in $D(u_1(t), u_2(t))$ viewed from 0.

(II) The conformal radius of $D(u_1(t), u_2(t))$ viewed from 0 is e^{-t}.

Moreover, the curve $\underline{u}$ can not be extended to T^u with (I) and (II).

Simultaneous Growth

At any time $t \in [0, T^u)$, from Koebe's 1/4 Theorem, we get

$$\min\{\mathrm{dist}(0, \eta_1[0, u_1(t)]), \mathrm{dist}(0, \eta_2[0, u_2(t)])\} \asymp e^{-t}.$$

By Beurling's estimate, $\mathrm{dist}(0, \eta_1[0, u_1(t)]) \asymp \mathrm{dist}(0, \eta_2[0, u_2(t)])$. So

$$\max\{\mathrm{dist}(0, \eta_1[0, u_1(t)]), \mathrm{dist}(0, \eta_2[0, u_2(t)])\} \asymp e^{-t}.$$

This means $\mathbb{P}_2[\mathrm{dist}(z_1, \eta_j) < r, j = 1, 2] \asymp \mathbb{P}_2[T^u > -\log(r)]$.

Extend $\underline{u}$ to $[0, \infty)$ such that $\underline{u}(t) = \lim_{t \uparrow T^u} \underline{u}(t)$ if $t \geq T^u$. It turns out that for any $t \geq 0$, $\underline{u}(t)$ is an $(\mathcal{F}_{\underline{t}})$-stopping time. Then

$$\mathbb{P}_2[T^u > -\log(r)] = \mathbb{P}_2[\underline{u}(-\log(r)) \in \mathcal{R}]$$

$$= \mathbb{E}_4[M_{4\to 2}(\underline{u}(-\log(r)))] * G_{\mathbb{D}; a_1, b_1; a_2, b_2}(0).$$

In order to compute $\mathbb{E}_4[M_{4\to 2}(\underline{u}(t))]$, we will work on a system of SDEs.

We now assume that (η_1, η_2) are commuting radial $\mathrm{SLE}_\kappa(2,2,2)$ curves so that the joint law of their driving functions is $\mathbb{P}_4$. For each $(t_1, t_2) \in \mathcal{R}$, let $g_{(t_1,t_2)}$ be the conformal map from $D(t_1, t_2)$ to $\mathbb{D}$, which fixes 0 and satisfies $g'_{(t_1,t_2)}(0) > 0$. Let $\mathrm{m}(t_1, t_2) = \log(g'_{(t_1,t_2)}(0))$. Let $\widehat{a}_j(t_1, t_2) = g_{(t_1,t_2)}(\eta_j(t_j))$ and $\widehat{b}_j(t_1, t_2) = g_{(t_1,t_2)}(b_j)$, $j = 1, 2$.

We may find continuous real valued functions W_1, V_1, W_2, V_2 on $\mathcal{R}$ such that $\widehat{a}_j = e^{iW_j}$ and $\widehat{b}_j = e^{iV_j}$. By symmetry, we may assume that $W_1 > V_1 > W_2 > V_2 > W_1 - 2\pi$. Let $\cot_2(x)$ denote the function $\cot(x/2)$. Because of the covering radial Loewner equation, we find that V_1 and V_2 satisfy differential equations:

$$\partial_{t_j} V_k = \partial_{t_j} \mathrm{m} * \cot_2(V_k - W_j), \quad j, k \in \{1, 2\}.$$

ODE for V_j^u

We will focus on the values of W_j and V_j, $j = 1, 2$, along the curve $\underline{u}$, and write $X^u(t)$ for $X(u_1(t), u_2(t))$. By Assumption (ii) we know that $\mathrm{m}^u(t) \equiv t$. So

$$\partial_{t_1} \mathrm{m}\,|_{\underline{u}(t)} * u_1'(t) + \partial_{t_2} \mathrm{m}\,|_{\underline{u}(t)} * u_2'(t) = 1.$$

We have ODEs for V_j^u, $j = 1, 2$:

$$\partial_t V_j^u(t) = \sum_{k=1}^{2} \partial_{t_k} \mathrm{m}\,|_{\underline{u}(t)} * u_k'(t) * \cot_2(V_j^u - W_k^u).$$

Because of Assumption (i) we have $V_1^u - V_2^u \equiv \pi$. So $\partial_t V_1^u \equiv \partial_t V_2^u$. Let $Z_j = W_j - V_j \in (0, 2\pi)$, $j = 1, 2$. We then solve

$$\partial_{t_j} \mathrm{m}\,|_{\underline{u}(t)} * u_j'(t) = \frac{\sin(Z_j^u)}{\sin(Z_1^u) + \sin(Z_2^u)}, \quad j = 1, 2.$$

We write P_j for the RHS. Note that $P_1 + P_2 = 1$.

SDE for W_j^u

The SDEs for W_1^u and W_2^u are more involved. The statement is that there exist two independent Brownian motions $B_1^u(t)$ and $B_2^u(t)$ such that W_1^u and W_2^u satisfy

$$\begin{aligned} dW_j^u =& \sqrt{\kappa P_j} dB_j^u + P_{3-j} \cot_2(W_j^u - W_{3-j}^u) dt \\ & + P_j \Big[\cot_2(W_j^u - W_{3-j}^u) + \sum_{k=1}^{2} \cot_2(W_j^u - V_k) \Big] dt. \end{aligned}$$

Here the first drift term comes from the growth of η_{3-j}, and the drift terms in the second line comes from the drift terms for the radial $\mathrm{SLE}_\kappa(2,2,2)$ curve η_j. The fact that $B_1^u(t)$ and $B_2^u(t)$ are independent follows from the commutation relation between η_1 and η_2.

Diffusion Process

Then we get SDEs for Z_1^u and Z_2^u:

$$dZ_j^u = \sqrt{\frac{\kappa \sin(Z_j^u)}{\sin(Z_1^u) + \sin(Z_2^u)}} dB_j^u + \frac{4\cos(Z_j^u)}{\sin(Z_1^u) + \sin(Z_2^u)}, \quad j = 1, 2.$$

We have $Z_j^u \in (0, \pi)$. Let $Z_\pm^u = (Z_1^u \pm Z_2^u)/2$. Then $Z_+^u \in (0, \pi)$ and $Z_-^u \in (-\pi/2, \pi/2)$. Define $B_\pm^u(t)$ such that $B_\pm^u(0) = 0$ and

$$dB_\pm^u = \sqrt{\frac{\sin(Z_1^u)}{\sin(Z_1^u) + \sin(Z_2^u)}} dB_1^u \pm \sqrt{\frac{\sin(Z_2^u)}{\sin(Z_1^u) + \sin(Z_2^u)}} dB_2^u.$$

Then B_+^u and B_-^u are both Brownian motions. But they are not independent. Instead, $d\langle B_+^u, B_-^u \rangle_t = \cot(Z_+^u) \tan(Z_-^u) dt$.

Diffusion Process

We get the following SDEs for Z_+^u and Z_-^u:

$$dZ_+^u = \frac{\sqrt{\kappa}}{2} dB_+^u + 2\cot(Z_+^u)dt;$$

$$dZ_-^u = \frac{\sqrt{\kappa}}{2} dB_-^u - 2\tan(Z_-^u)dt.$$

After linearly scaling the time and space, we can make Z_+^u and Z_-^u into two radial Bessel processes, whose marginal transition density are known. Our task is to derive the joint transition density of them.

Diffusion Process

Let $X = \cos(Z_+^u)$ and $Y = \sin(Z_-^u)$. Then $X, Y \in (-1, 1)$, and satisfy the SDEs

$$dX = -\frac{\sqrt{\kappa}}{2}\sqrt{1 - X^2}dB_+^u - \Big(2 + \frac{\kappa}{8}\Big)X dt;$$

$$dY = +\frac{\sqrt{\kappa}}{2}\sqrt{1 - Y^2}dB_-^u - \Big(2 + \frac{\kappa}{8}\Big)Y dt;$$

$$d\langle X, Y\rangle_t = -\frac{\kappa}{4}XY dt.$$

Since $X^2 + Y^2 = 1 - \sin(Z_1^u)\sin(Z_2^u) < 1$, we see that $(X, Y) \in \mathbb{D}$. We will derive the transition density for the diffusion process (X, Y).

Transition Density

Suppose that the transition density $p_t((x,y),(x^*,y^*))$ for (X,Y) exists and is smooth. Then for any fixed $(x^*,y^*) \in \mathbb{D}$ and $t_0 > 0$, the process $M_t := p(t_0 - t, (X(t), Y(t)), (x^*, y^*))$, $0 \le t_0 < t$, is a martingale, which implies by Itô's formula that $p.((\cdot,\cdot),(x^*,y^*))$ satisfies the PDE:

$$-\partial_t p + \mathcal{L} p = 0, \tag{1}$$

where

$$\mathcal{L} := \frac{\kappa}{8}(1-x^2)\partial_x^2 + \frac{\kappa}{8}(1-y^2)\partial_y^2 - \frac{\kappa}{4} xy\partial_x\partial_y - (2+\frac{\kappa}{8})(x\partial_x + y\partial_y).$$

We note that if $f(x,y)$ on $\mathbb{D}$ is an eigenvector for $\mathcal{L}$ with eigenvalue λ, then $e^{\lambda t} f(x,y)$ is a solution of (1). We expect that p can be written as an infinite sum of such functions.

Eigenvectors

To derive the eigenvectors of $\mathcal{L}$, first note that for $n, m \in \mathbb{Z}$ with $n, m \geq 0$,

$$\begin{aligned}\mathcal{L}(x^n y^m) = &- \frac{\kappa}{8}(n+m)(n+m+\frac{16}{\kappa})x^n y^m + \\ &+ \frac{\kappa}{8}n(n-1)x^{n-2}y^m + \frac{\kappa}{8}m(m-1)x^n y^{m-2}.\end{aligned}$$

Let $\lambda_s = -\frac{\kappa}{8}s(s+\frac{16}{\kappa}) \leq 0$. Then $\mathcal{L}(x^n y^m)$ equals $\lambda_{n+m} * x^n y^m$ plus a polynomial of degree less than $n+m$. Thus, for any $n, m \geq 0$, we get a polynomial $P_{(n,m)}$ expressed as $x^n y^m$ plus a polynomial of degree less than $n+m$ such that $\mathcal{L}P_{(n,m)} = \lambda_{n+m}P_{(n,m)}$. For any fixed $n \geq 0$, any linear combinations of $P_{(s,n-s)}$, $0 \leq s \leq n$, is also an eigenvector of $\mathcal{L}$ of eigenvalue λ_n.

Orthogonal Polynomials

Let $\Psi(x,y) = (1-x^2-y^2)^{\frac{8}{\kappa}-1}$, and define the inner product

$$\langle f, g\rangle_\Psi := \int\int_{\mathbb{D}} f(x,y)g(x,y)\Psi(x,y)dxdy.$$

We find that for any smooth functions f and g on $\overline{\mathbb{D}}$,

$$\langle \mathcal{L}f, g\rangle_\Psi = \langle f, \mathcal{L}g\rangle_\Psi.$$

Thus, if f and g are eigenvectors of $\mathcal{L}$ with different eigenvalues, then they are orthogonal w.r.t. $\langle \cdot, \cdot\rangle_\Psi$. We may now derive a family of functions $v_{(n,s)}$, $0 \leq n < \infty$, $0 \leq r \leq n$, such that each $v_{(n,r)}$ is a linear combination of $P_{(s,n-s)}$, $0 \leq s \leq n$, and so is an eigenvector of $\mathcal{L}$ with eigenvalue λ_n, and all $v_{(n,s)}$ form an orthonormal basis w.r.t. $\langle \cdot, \cdot\rangle_\Psi$.

Orthogonal Polynomials

Using the theory of orthogonal polynomials of several variables, we may express $v_{(n,s)}$ in terms of Jacobi polynomials.

$$v_{n,j,1} = h_{n,j,1} P_j^{(\frac{8}{\kappa}-1,n-2j)}(2r^2-1) r^{n-2j} \cos((n-2j)\theta), \quad 0 \le 2j \le n,$$
$$v_{n,j,2} = h_{n,j,2} P_j^{(\frac{8}{\kappa}-1,n-2j)}(2r^2-1) r^{n-2j} \sin((n-2j)\theta), \quad 0 \le 2j \le n-1,$$

where $P_j^{(\frac{8}{\kappa}-1,n-2j)}$ are Jacobi polynomials of index $(\frac{8}{\kappa}-1, n-2j)$, (r,θ) is the polar coordinate of (x,y), and $h_{n,j,\sigma} > 0$ are normalization constants. Using the knowledge on Jacobi polynomials, we find that the series

$$\sum_{n=0}^{\infty} \sum_{s=0}^{n} \Psi(x^*,y^*) v_{(n,s)}(x,y) v_{(n,s)}(x^*,y^*) e^{\lambda_n t}$$

converges for any $t > 0$, and solves the PDE $-\partial_t + \mathcal{L} = 0$.

Transition Density

We now briefly explain why the limit $p_t((x,y),(x^*,y^*))$ is the transition density for (X,Y). Let (X,Y) start from $(x,y) \in \mathbb{D}$. We need to show that for any polynomial f of x,y and any $t>0$,

$$\mathbb{E}[f(X(t),Y(t))] = \int\int_{\mathbb{D}} f(x^*,y^*)p_t((x,y),(x^*,y^*))dx^*dy^*.$$

Express the RHS as $f(t,(x,y))$. It equals

$$\sum_{n,s} \langle f, v_{(n,s)}\rangle_\Psi * v_{(n,s)}(x,y)e^{\lambda_n t}.$$

Since there are only finitely many non-zero terms in the series, $f(t,(x,y))$ solves the PDE $-\partial_t + \mathcal{L} = 0$. By Itô's formula, for any fixed $t_0 > 0$, $M_t := f(t_0 - t, (X(t), Y(t)))$, $0 \le t \le t_0$, is a bounded martingale. Since $M_0 = f(t_0,(x,y))$ and $M_{t_0} = f(X(t_0), Y(t_0))$, we get $\mathbb{E}[f(X(t_0),Y(t_0))] = f(t_0,(x,y))$, as desired.

Invariant Density

Using $X = \cos_2(Z_1^u + Z_2^u)$ and $Y = \sin_2(Z_1^u - Z_2^u)$, we can then derive the transition density $p_t^Z((z_1, z_2), (z_1^*, z_2^*))$ for (Z_1^u, Z_2^u).

Using the orthogonality of $v_{(n,s)}$ w.r.t. $\langle \cdot, \cdot \rangle_\Psi$, we know that the leading term $C\Psi(x^*, y^*)$ in the series for $p_t((x, y), (x^*, y^*))$ is an invariant density for (X, Y). As t grows, the leading term stays constant, and other terms decay to zero. So the transition density for (X, Y) approaches the invariant density for (X, Y) as $t \to \infty$ despite of the initial value. After a coordinate change, we get an invariant density $p_\infty^Z(z_1^*, z_2^*)$ for (Z_1^u, Z_2^u), which is the limit of $p_t^Z((z_1, z_2), (z_1^*, z_2^*))$.

Change of Measures

The underlying probability for the above argument is $\mathbb{P}_4$. We now derive corresponding results for $\mathbb{P}_2$. Under $\mathbb{P}_2$, the lifetime T^u for (Z_1^u, Z_2^u) is a.s. finite. Using the lemma, we get

$$\frac{d\mathbb{P}_2|\mathcal{F}_{\underline{u}(t)} \cap \{T^u > t\}}{d\mathbb{P}_4|\mathcal{F}_{\underline{u}(t)} \cap \{T^u > t\}} = \frac{M^u_{4\to 2}(t)}{M^u_{4\to 2}(0)}, \quad t \geq 0.$$

We may define G^u such that $M^u_{4\to 2}(t) = e^{-\alpha_0 t} G^u(Z_1^u(t), Z_2^u(t))^{-1}$. So we obtain the transition density $\widetilde{p}_t^Z$ for (Z_1^u, Z_2^u) under $\mathbb{P}_2$:

$$\widetilde{p}_t^Z((z_1, z_2), (z_1^*, z_2^*)) = e^{-\alpha_0 t} p_t^Z((z_1, z_2), (z_1^*, z_2^*)) \frac{G^u(z_1, z_2)}{G^u(z_1^*, z_2^*)}$$

This means that, under the law $\mathbb{P}_2$, if (Z_1^u, Z_2^u) starts from (z_1, z_2), then for any $t > 0$ and any measurable function f on $\mathbb{D}$,

$$\mathbb{E}_2[\mathbf{1}_{\{T^u > t\}} f(Z_1^u, Z_2^u)] = \int\int_{\mathbb{D}} f(z_1^*, z_2^*) \widetilde{p}_t^Z((z_1, z_2), (z_1^*, z_2^*)) dz_1^* dz_2^*.$$

In particular, $\mathbb{P}_2[T^u > t] = \int\int_{\mathbb{D}} \widetilde{p}_t^Z((z_1, z_2), (z_1^*, z_2^*)) dz_1^* dz_2^*$.

Quasi-invariant Density

Using $p^Z_\infty(z_1^*, z_2^*)$ we then derive a quasi-invariant density $\widetilde{p}^Z_\infty(z_1^*, z_2^*) := \frac{1}{\mathcal{Z}} \frac{p^Z_\infty(z_1^*, z_2^*)}{G^u(z_1^*, z_2^*)}$ for (Z_1^u, Z_2^u) under $\mathbb{P}_2$: if (Z_1^u, Z_2^u) has initial density $p^Z_\infty(z_1^*, z_2^*)$, then for any $t > 0$, $\mathbb{P}_2[T^u > t] = e^{-\alpha_0 t}$, and the law of $(Z_1^u(t), Z_2^u(t))$ conditional on the event $T^u > t$ is still $p^Z_\infty(z_1^*, z_2^*)$. Using the convergence of $p^Z_t((z_1, z_2), (z_1^*, z_2^*))$ to $p^Z_\infty(z_1^*, z_2^*)$ we get the convergence of $e^{\alpha_0 t} \widetilde{p}^Z_t((z_1, z_2), (z_1^*, z_2^*))$ to $\mathcal{Z} G^u(z_1, z_2) \widetilde{p}^Z_\infty(z_1^*, z_2^*)$.

To complete the proof of the theorem, we use Koebe's distortion theorem and the technique in [Lawler-Rezaei], which was used to derive Green's function in Euclidean distance.

Sketch Proof of the Theorem

Suppose r is very small. Choose big $t_0 > 0$ such that $r \ll e^{-t_0} \ll 1$. Then $\operatorname{dist}(z_1, \eta_j) < r$, $j = 1, 2$, if and only if $T^u > t_0$ and the images of $\eta_1(u_1(t_0 + \cdot))$ and $\eta_2(u_2(t_0 + \cdot))$ under $g_{u_1(t_0),u_2(t_0)}$, denoted by $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$, both visit the region $\Omega := g_{u_1(t_0),u_2(t_0)}(\{|z| < r\})$.

By Koebe's distortion theorem, $\Omega \approx \{|z| < e^{t_0} r\}$. By DMP for 2-SLE, $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ form a 2-SLE$_\kappa$ in $\mathbb{D}$ from $\widehat{a}^u_j(t_0)$ to $\widehat{b}^u_j(t_0)$, $j = 1, 2$. From the assumption on (u_1, u_2), we know that $\widehat{b}^u_1(t_0)$ and $\widehat{b}^u_2(t_0)$ are opposite points on $\partial\mathbb{D}$. Because t_0 is big, $\mathbb{P}[T^u > t_0] \approx e^{-\alpha_0 t_0} * G^u(z_1, z_2)$ and the joint law of the arguments of $\widehat{a}^u_j(t_0)/\widehat{b}^u_j(t_0)$, $j = 1, 2$, conditional on the event $T^u > t_0$ is close to the quasi-invariant density. Putting these ingredients together, we then finish the proof of the theorem.

Two-curve two-point Green's Function

We expect some subsequent works after this. Some of them will be joint with Xin Sun (Columbia University).

One project is to derive the two-curve two-point Green's function for 2-SLE, i.e., the limit of the rescaled probability that two curves of a 2-SLE both pass through two small discs centered at two different points. We will follow the approach of [Lawler-Werness] and expect that the two-point Green's function can be written as the product of a one-point Green's function and the expectation of another one-point Green's function in a random domain.

Minkowski Content

After that we plan to derive the existence of $(2-\alpha_0)$-dimensional Minkowski content of the intersection of two curves of a 2-SLE following the approach of [Lawler-Rezaei], which may be further used to prove the following decomposition statement for 2-SLE: the following two procedure gives the same measure on the triple (η_1, η_2, z):

(I) first sample a 2-SLE (η_1, η_2) and then sample a point z on $\eta_1 \cap \eta_2$ according to the Minkowski content;

(II) first sample z according to the two-curve Green's function, then sample a 4-SLE connecting z with the marked boundary points, and join two pairs of them at z to get (η_1, η_2).

The decomposition may be further used to derive 2-SLE loop measure. It is expected to be an infinite measure on a pair of loops, which touch but not cross each other, such that conditional on any loop, the other loop is a single SLE loop in a complement domain of the first loop.

Intersection of Flow Lines

Another long term plan is to derive the one-point, two-point Green's function and Minkowski content of the intersection of two flow lines η_1, η_2. The two flow lines commute with each other in the sense that conditional on any one curve, the other is an $\mathrm{SLE}_\kappa(\rho)$. The force value ρ can vary in an interval. When $\rho = 0$, we get the 2-SLE_κ as a special case. Another special case gives the cut-point set of a single chordal SLE_κ, $\kappa \in (4, 8)$. The Hausdorff dimension of $\eta_1 \cap \eta_2$ was derived in [Miller-Wu]. The exponent α will be $2 - \dim_H(\eta_1 \cap \eta_2)$.

The existence of Green's function will improve this estimate in [Miller-Wu], which has the form of

$$\mathbb{P}[\mathrm{dist}(z, \eta_1 \cap \eta_2) < \varepsilon] \approx \varepsilon^{\alpha + o(1)}.$$

Thank you!