Hypergeometric SLE and Convergence of Multiple Interfaces in Lattice Models

Hao Wu

Yau Mathematical Sciences Center, Tsinghua University, China

Table of contents

Ising Model

Curie temperature [Pierre Curie, 1895]

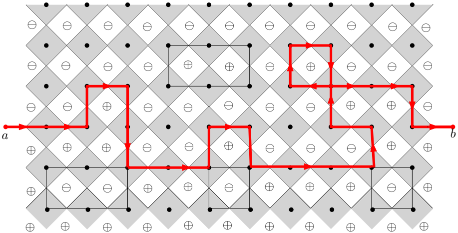
Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.

Ising Model [Lenz, 1920]

A model for ferromagnet, to understand the critical temperature

- $G = (V, E)$ is a finite graph
- $\sigma \in \{\oplus, \ominus\}^V$
- The Hamiltonian

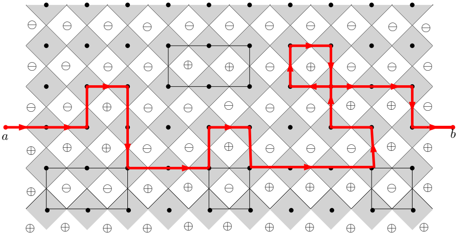
$$H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$$

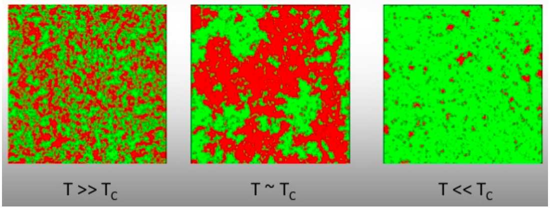


Ising Model

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$





Kramers-Wannier, Onsager-Kaufman, 1940

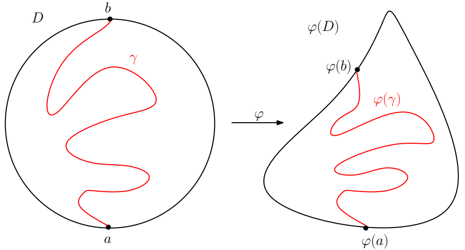
Ising model on $\mathbb{Z}^2$:
$\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

Interface

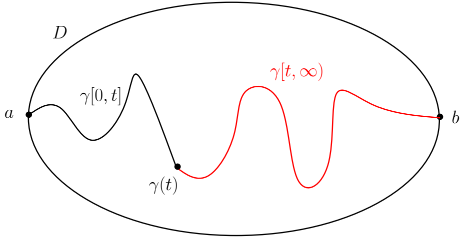
Conformal invariance + Domain Markov Property

SLE (Schramm Loewner Evolution)

Random fractal curves in $D \subset \mathbb{C}$ from *a* to *b*. Candidates for the scaling limit of discrete Statistical Physics models.



Conformal invariance :
If γ is in *D* from *a* to *b*,
and $\varphi : D \to \varphi(D)$ conformal map,
then $\varphi(\gamma) \overset{d}{\sim}$ the one in $\varphi(D)$ from $\varphi(a)$ to $\varphi(b)$.



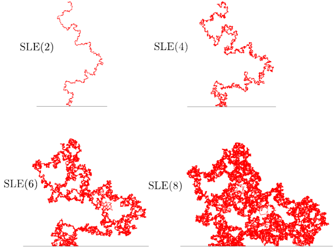
Domain Markov Property :
the conditional law of
$\gamma[t, \infty)$ given $\gamma[0, t] \overset{d}{\sim}$
the one in $D \setminus \gamma[0, t]$ from $\gamma(t)$ to *b*.

Examples of SLE

Lemma [Schramm 1999]

There exists a one-parameter family of random curves that satisfies Conformal Invariance and Domain Markov Property : SLE_κ for $\kappa \geq 0$.

Simple, $\kappa \in [0, 4]$; Self-touching, $\kappa \in (4, 8)$; Space-filling, $\kappa \geq 8$.



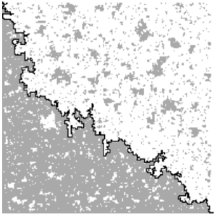
Courtesy to Tom Kennedy.

- $\kappa = 2$: LERW
- $\kappa = 8$: UST (Lawler, Schramm, Werner)
- $\kappa = 3$: Critical Ising
- $\kappa = 16/3$: FK-Ising (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov)
- $\kappa = 6$: Percolation (Camia, Newman, Smirnov)

Critical Ising

Thm [Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov 2010]

The interface of critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary condition converges to SLE(3).



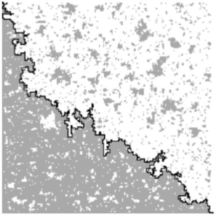
Their Strategy

- Tightness : RSW
- Identify the scaling limit : Holomorphic observable

Other results on the convergence ?

Thm [Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov 2010]

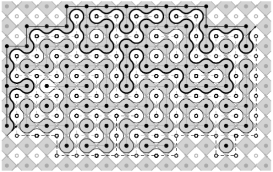
The interface of critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary condition converges to SLE(3).



- Different Models ?
- Different lattices ?
- Different Boundary Conditions ?

- Many conjectures.
- Universality : open.
- Some results.

Open Question : Other Models



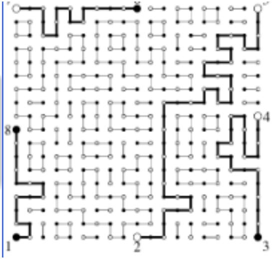
Conjecture

For $q \leq 4$, the interface of critical Random Cluster Model converges to SLE(κ) where

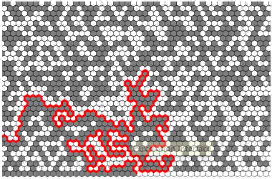
$$\kappa = 4\pi / arccos(-\sqrt{q}/2).$$

Conjecture

The interface of Double Dimer Model converges to SLE(4).



Open Question : Universality

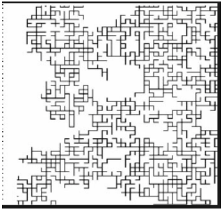


Thm [Smirnov 2000]

The interface of critical site percolation on triangular lattice converges to SLE(6).

Conjecture

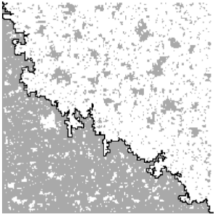
The interface of critical bond percolation on square lattice converges to SLE(6).



Other results on the convergence ?

Thm [Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov 2010]

The interface of critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary condition converges to SLE(3).

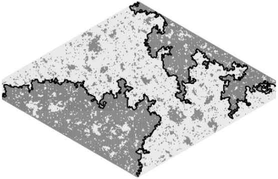


- Different Models ?
- Different lattices ?
- Different Boundary Conditions ?

- Many conjectures.
- Universality : open.
- Some results.

Table of contents

Critical Ising in Quad



Thm [Izyurov 2014, W. 2017]

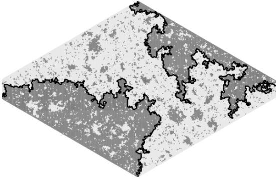
The interface of critical Ising model on $\mathbb{Z}^2$ with alternating boundary condition converges to Hypergeometric SLE_3, denoted by hSLE_3.

- Q1 : What is Hypergeometric SLE ?
- Q2 : Why are they the limit ?
- Q3 : How do we prove the convergence ?

- Answer to Q1 :
  - random fractal curves in quad $q = (\Omega; x_1, x_2, x_3, x_4)$
  - $\mathrm{hSLE}_\kappa(\nu)$ for $\kappa \in (0, 8)$ and $\nu \in \mathbb{R}$.
  - driving function :

$$dW_t = \sqrt{\kappa} dB_t + \kappa \partial_{x_1} \log \mathcal{Z}_{\kappa,\nu}(W_t, V_t^2, V_t^3, V_t^4) dt.$$

General Boundary Conditions



Thm [Izyurov 2014, W. 2017]

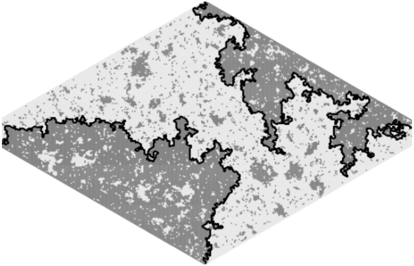
The interface of critical Ising model on $\mathbb{Z}^2$ with alternating boundary condition converges to Hypergeometric SLE_3, denoted by hSLE_3.

- Q1 : What is Hypergeometric SLE ?
- Q2 : Why they are the limit ?
- Q3 : How to prove the convergence ?

- Answer to Q1 :
  - when $\nu = -2$, it equals SLE_κ
  - when $\kappa \in (4, 8)$, SLE_κ in Ω from x_1 to x_4 conditioned to avoid (x_2, x_3) is $\mathrm{hSLE}_\kappa(\kappa - 6)$
  - reversibility : the time-reversal has the same law.
    ✓ proved for $\nu \geq \kappa/2 - 4$; ? should be true for $\nu > -4 \vee (\kappa/2 - 6)$.

Q2 : Why they are the limit ?

Recall : Conformal Invariance + Domain Markov Property $\to$ SLE(κ).



Assume the scaling limit exists, then the limit should satisfy

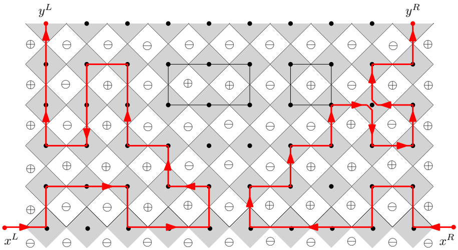
- (CI) Conformal Invariance
- (DMP) Domain Markov Property
- (SYM) Symmetry

Thm [W.2017]

Suppose $(\mathbb{P}_q, q \in \mathcal{Q})$ is a collection of proba, measures on pairs of simple curves that satisfies CI, DMP, and SYM. Then there exist $\kappa \in (0, 4]$ and $\nu < \kappa - 6$ such that $\mathbb{P}_q \sim \text{hSLE}_\kappa(\nu)$.

Key in the proof : J. Dubédat's commutation relation.

Q3 : How to prove the convergence ?



$(\eta^L; \eta^R)$: any subseq. limit

- $\mathcal{L}(\eta^L \,|\, \eta^R) = \mathrm{SLE}_3$
- $\mathcal{L}(\eta^R \,|\, \eta^L) = \mathrm{SLE}_3$

Proposition

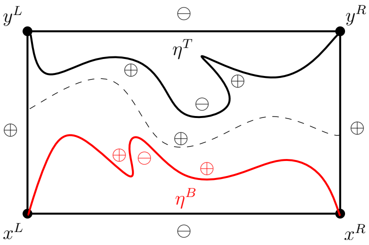
Fix $\kappa \in (0, 4]$. There exists a unique probability measure on $(\eta^L; \eta^R)$ such that

- $\mathcal{L}(\eta^L \,|\, \eta^R) = \mathrm{SLE}_\kappa$
- $\mathcal{L}(\eta^R \,|\, \eta^L) = \mathrm{SLE}_\kappa$

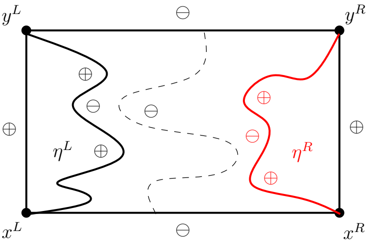
The marginal of η^R is hSLE_κ from x^R to y^R.

Conclusion

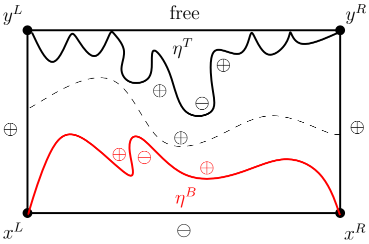
η^R : hSLE_3 from x^R to y^R.



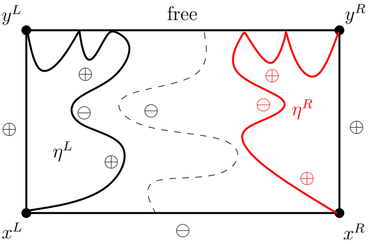
hSLE$_3$



hSLE$_3(-5)$

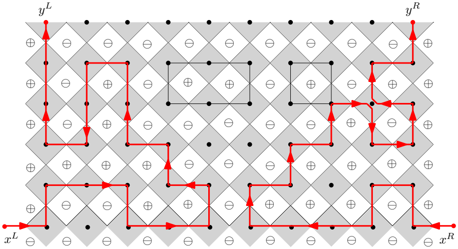


hSLE$_3(-3/2)$



hSLE$_3(-7/2)$

Convergence of Ising Interface to hSLE_3



Dobrushin b.c. : Interface $\to \mathrm{SLE}_3$

- RSW $\Longrightarrow$ tightness
- Holomorphic observable

Alternating b.c. : Interface $\to \mathrm{hSLE}_3$

First approach [Izyurov]

- RSW $\Longrightarrow$ tightness
- New holomorphic observable

Advantage :
more general b.c.

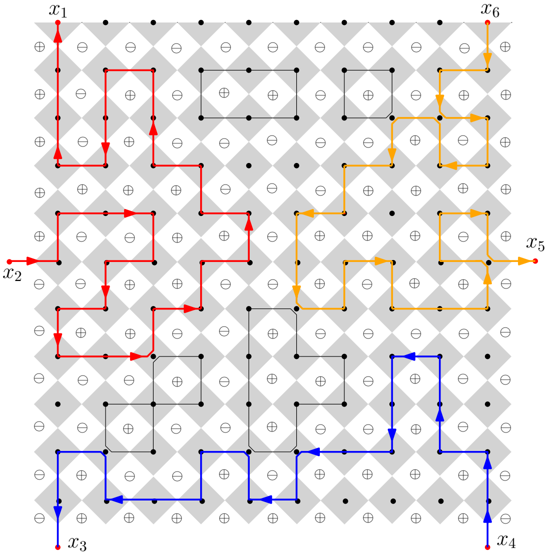
Second approach [W.]

- RSW $\Longrightarrow$ tightness
- Cvg with Dobrushin b.c.

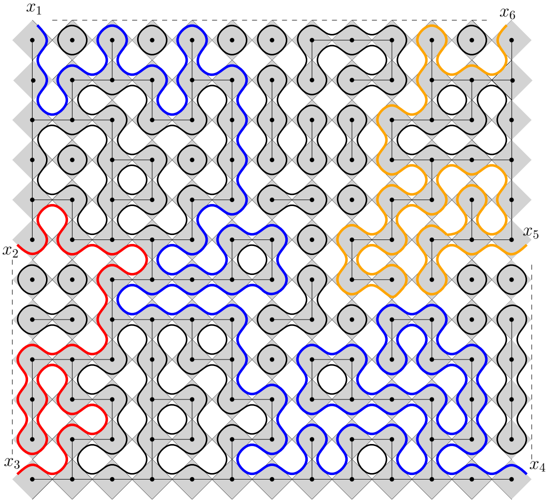
Advantage :
more general b.c.
and other lattice models.

Table of contents

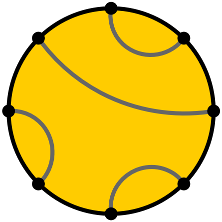
What about more complicated b.c. ?

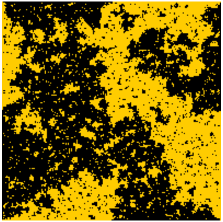


What about more complicated b.c. ?



What about more complicate b.c. ?





courtesy to E. Peltola

Global Multiple SLEs

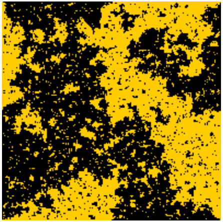
A collection of N disjoint simple curves $(\eta_1, \ldots, \eta_N) \in X^\alpha(\Omega; x_1, \ldots, x_{2N})$ such that

$$\forall j, \quad \mathcal{L}(\eta_j \,|\, \eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_N) = \mathrm{SLE}_\kappa$$

Thm [Korzdon & Lawler, Beffara & Peltola & W.]

Fix $\kappa \in (0, 4] \cup \{16/3, 6\}$ and link pattern $\alpha \in \mathrm{LP}_N$. There exists a unique global multiple SLE_κ associated to α.

- Existence and uniqueness : see E. Peltola's talk



Thm [Korzdon & Lawler, Beffara & Peltola & W.]

Fix $\kappa \in (0,4] \cup \{16/3, 6\}$ and link pattern $\alpha \in \mathrm{LP}_N$. There exists a unique global multiple SLE_κ associated to α.

Corollary

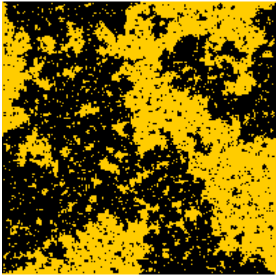
- *Multiple LERWs in UST* $\to$ *Multiple SLE*(2)*s*
- *Multiple Interfaces in Ising* $\to$ *Multiple SLE*(3)*s*
- *Multiple Interfaces in FK-Ising* $\to$ *Multiple SLE*$(16/3)$*s*
- *Multiple Interfaces in Percolation* $\to$ *Multiple SLE*(6)*s*

Summary : RSW+ Cvg with Dobrushin b.c.+ Uniqueness.

Question : What is the marginal law ?

Table of contents

What is the marginal law ?



For $\Omega = \mathbb{H}$ and $x_1 < \cdots < x_{2N}$,
$$dW_t = \sqrt{\kappa} dB_t + \kappa \partial_{x_1} \log \mathcal{Z}_\alpha(W_t, V_t^2, \ldots, V_t^{2N}) dt,$$

Pure Partition Functions

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}\}$ is a collection of smooth functions satisfying PDE, COV, ASY.

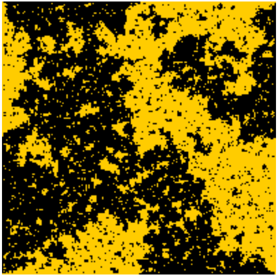
PDE : $\left[\frac{\kappa}{2} \partial_i^2 + \sum_{j \neq i} \left(\frac{2}{x_j - x_i} \partial_j - \frac{(6-\kappa)/\kappa}{(x_j - x_i)^2} \right) \right] \mathcal{Z}(x_1, \ldots, x_{2N}) = 0.$

COV : $\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})).$

ASY : $\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \mathcal{Z}_{\hat{\alpha}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N})$

- Q1 : Do they exist ?
- Q2 : Are they unique ?

Pure Partition Functions



Uniqueness [Flores & Kleban 2015]

If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

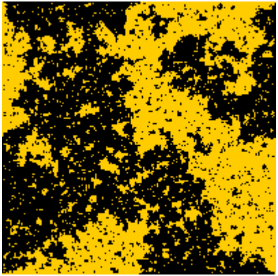
Existence

- Kytölä & Peltola 2016 : $\kappa \in (0,8) \setminus \mathbb{Q}$
- Peltola & W. 2017 : $\kappa \in (0,4]$
- W. 2018 : $\kappa \in (0,6]$

- Coulomb gas technique
- Global Multiple SLEs
- Hypergeometric SLE

The 2nd construction : see E. Peltola's talk.

Pure Partition Functions



Existence [W. 2018]

- $\kappa \in (0, 6]$: Hypergeometric SLE

Proof

Cascade relation + Induction.

- COV, ASY : by construction
- PDE : Hypergeometric SLE
- Smoothness : Hypoellipticity [Dubédat 2015] (see also Lawler & Jahangoshahi [arXiv : 1710.00854])
- Cascade relation : by construction
- Positivity : by construction
- Optimal power law bound : $h = (6-\kappa)/(2\kappa)$,

$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \leq \prod |x_{b_i} - x_{a_i}|^{-2h}, \quad \alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}.$$

Multiple SLEs vs. Pure Partition Functions

Global Multiple SLEs

Fix $\kappa \in (0,4] \cup \{16/3, 6\}$, there exists a unique global multiple SLE.

Global Multiple SLEs : conjecture

True for $\kappa \in (0,8)$.

- Proved for $\kappa \in (4,6]$ using the convergence of RCM.
- Wrong for $\kappa \geq 8$.

Pure Partition Functions

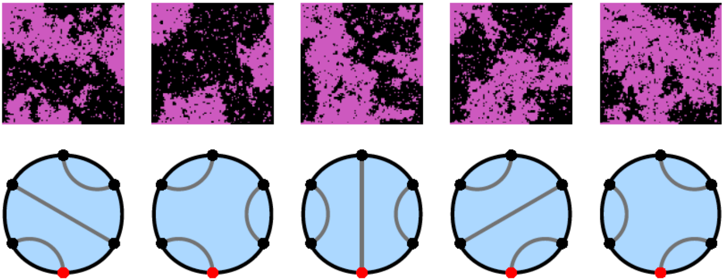
Fix $\kappa \in (0,6]$, there exists a unique collection of pure partition functions.

Pure Partition Functions : conjecture

True for $\kappa \in (0,8)$.

- The optimal power law bound might fail for $\kappa \in (6,8)$.
- Might be true for $\kappa \geq 8$.

Crossing Probabilities of Ising Interfaces



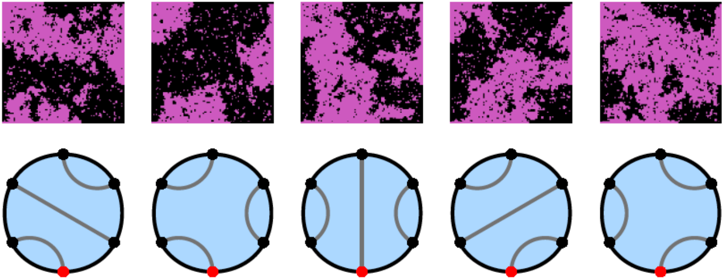
Courtesy to E. Peltola

Conjecture : In progress

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{A}_\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \ldots, x_{2N})}{\mathcal{Z}_{Ising}(\Omega; x_1, \ldots, x_{2N})}.$$

Crossing Probabilities



- Connection probabilities in LERWs (SLE(2)) :
  Kenyon & Wilson 2011, Karrila & Kytölä & Peltola 2017
- Crossing probabilities in Ising (SLE(3)) : in progress
- Connection probabilities for level lines of GFF (SLE(4)) :
  Kenyon & Wilson 2011, Peltola & W. 2017
- Connection probabilities in percolation (SLE(6)) : OK.

Thanks ! References

- E. Peltola, H. Wu
  Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities of Level Lines of GFF (arXiv : 1703.00898)
- H. Wu
  Hypergeometric SLE : Conformal Markov Characterization and Applications (arXiv : 1703.02022)
- V. Beffara, E. Peltola, H. Wu
  On the Uniqueness of Global Multiple SLEs (arXiv :1801.07699)