Geometric descriptions of the Loewner energy

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Outline

Introduction

- Loewner introduced in 1923 a way to encode the uniformizing conformal map of a simply connected domain $D \subset \mathbb{C}$ via continuous iterations of conformal distortions to "straighten" its boundary,

$$\text{non self-intersecting curve } \gamma \Leftrightarrow \text{ real-valued driving function } W.$$

- Main tool to solve Bieberbach's conjecture by De Branges in 1985 (using W smooth).
- Random fractal non self-intersecting curves: Schramm-Loewner Evolution introduced by Oded Schramm in 1999
(SLE$_\kappa$ when $W = \sqrt{k}B$, where B is the standard 1-d Brownian Motion).

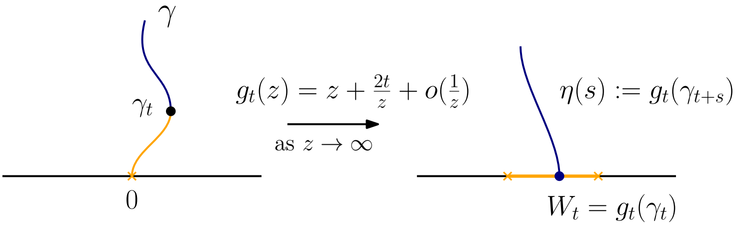
Introduction

- In this talk, $W \in$ Cameron-Martin space of the Brownian motion (having finite Dirichlet energy: $I(W) < \infty$)
  $\implies W \in Lip_{loc}^{1/2}(\mathbb{R}_+)$
  $\implies$ the chordal Loewner chain generated by W is a transient simple curve.
- We call $I(W)$ the *Loewner energy* of γ.
- Connection to zeta-regularized determinants of Laplacians.
- Weil-Petersson class in the universal Teichmüller space.

Outline

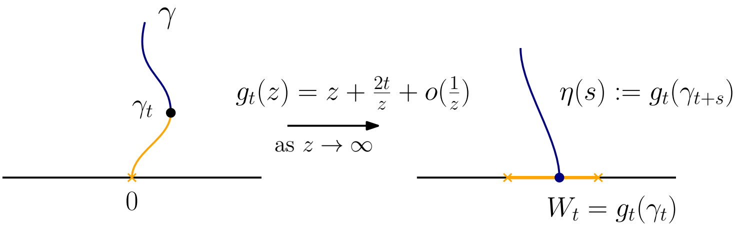
Background on the Loewner chains

Let γ be a simple chord in $\mathbb{H}$ from 0 to ∞.



- γ is *capacity-parametrized* by $t \in [0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the *driving function* of γ.
- One can recover γ from W using Loewner's differential equation, satisfied by $(g_t)_{t \geq 0}$.

Properties of the driving function



- $W_0 = 0$;
- W is continuous;
- *Scaling property:* let $c > 0$, the driving function of $c\gamma$ is given by a Brownian-scaling of W:

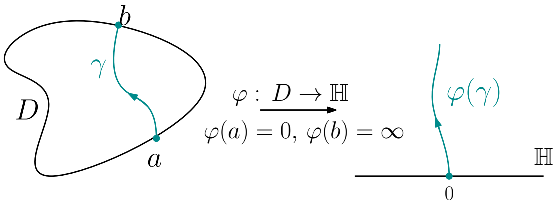
$$\tilde{W}_t = cW_{c^{-2}t};$$

- *Additivity:* for a fixed $t \geq 0$, the chord $s \mapsto \eta(s) - W_t$ from 0 to ∞ is capacity parametrized, and has the driving function:

$$\tilde{W}_s = W_{t+s} - W_t.$$

(satisfied by $t \mapsto g_t(z)$).

The chordal Loewner energy



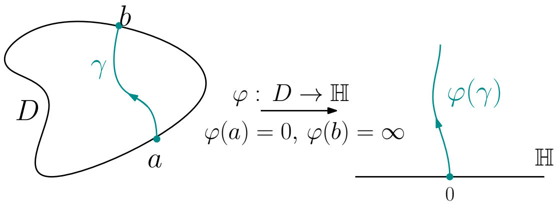
Definition: Loewner energy (Friz & Shekhar 2015, W. 2016)

We define the *Loewner energy of a simple chord γ in* (D, a, b) to be

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W)$$
$$:= \begin{cases} \frac{1}{2}\int_0^\infty W'(t)^2 \, \mathrm{d}t, & \text{if } W \text{ is absolutely continuous;} \\ \infty, & \text{otherwise,} \end{cases}$$

where W is the driving function of $\varphi(\gamma)$.

The chordal Loewner energy



- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(c\gamma).$$

- $I_{D,a,b}(\gamma) = 0$ *iff* γ is the hyperbolic geodesic connecting a and b.
- Additivity: if $[0, 1] \to D$ is a continuous parametrization of γ, with $\gamma(0) = a$, $\gamma(1) = b$,

$$I_{D,a,b}(\gamma) = I_{D,a,b}(\gamma[0, t]) + I_{D\setminus\gamma[0,t],\gamma_t,b}(\gamma[t, 1]).$$

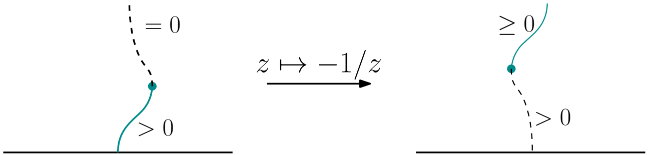
- $I_{D,a,b}(\gamma) < \infty$, then γ is rectifiable. (Friz & Shekhar)

Reversibility of chordal Loewner energy

Theorem (W. 2016)

Let γ be a simple chord in D connecting two boundary points a and b, we have

$$I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma).$$



The deterministic result is based on a probabilistic interpretation of the Loewner energy as a large deviation rate function of SLE_κ as $\kappa \to 0$. Loosely speaking,

$$\text{“}\mathrm{P}(\text{SLE}_\kappa \text{ stays close to } \gamma) \approx \exp\left(-\frac{I(\gamma)}{\kappa}\right).\text{”}$$

Reversibility of Loewner energy

Theorem (Reversibility of SLE, Zhan 2008, Miller-Sheffield 2012)

For $\kappa \leq 4$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

- $\mathrm{SLE}_2 \leftrightarrow$ Loop-erased random walk;
- $\mathrm{SLE}_3 \leftrightarrow$ Critical Ising model interface;
- $\mathrm{SLE}_4 \leftrightarrow$ Level line of the Gaussian free field;
- $\mathrm{SLE}_6 \leftrightarrow$ Critical independent percolation interface;
- $\mathrm{SLE}_{8/3} \leftrightarrow$ Self-avoiding random walk (conjecture).

Goal:

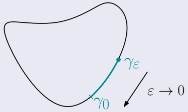
Understand the geometric meaning of the Loewner energy.

Loewner loop energy

Definition

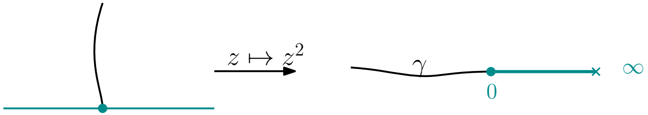
We define the *Loewner energy of a simple loop* $\gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\gamma_0 = \gamma_1$ to be

$$I^L(\gamma, \gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon], \gamma_\varepsilon, \gamma_0}(\gamma[\varepsilon, 1]).$$



The loop energy generalizes the chordal energy:

$$I_{\mathbb{C} \setminus \mathbb{R}_+, 0, \infty}(\gamma) = I^L(\gamma \cup \mathbb{R}_+, \infty).$$



Loewner loop energy

Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the parametrization of the loop.

$\Longrightarrow$ I^L is Möbius-invariant on the set of free loops vanishing only on circles.

Moreover,

- $I^L(\gamma) < \infty$, then γ has no corner, is rectifiable and is a K-quasicircle with K depending only on $I^L(\gamma)$ (i.e. γ is the image of S^1 by a K-quasiconformal homeomorphism of $\hat{\mathbb{C}}$);
- $\gamma \in C^{3/2+\varepsilon} \implies I^L(\gamma) < \infty$, where $\varepsilon > 0$.

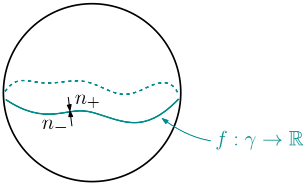
- The proof is based on the reversibility of the chordal energy.
- It shows that the Loewner energy has even more symmetries in the loop setting.

Outline

We will consider only *smooth* C^∞ loops γ in $\hat{\mathbb{C}} \simeq S^2$. Let

$$g_0(z) = \frac{4}{(1+|z|^2)^2}\,\mathrm{d}z^2$$

denote the spherical metric.



Definition

For any Riemannian metric g on S^2, we define

$$\mathscr{H}(\gamma, g) := \log \det{}'_\zeta N(\gamma, g) - \log \mathrm{Length}_g(\gamma),$$

where $N(\gamma, g)$ is the *Neumann Jump operator*: $\forall f \in C^\infty(\gamma, \mathbb{R})$, $x \in \gamma$,

$$N(\gamma, g) f(x) := \partial_{n^+} \mathscr{P}_+[f](x) + \partial_{n^-} \mathscr{P}_-[f](x).$$

Zeta-regularizated determinant of N

- $N(\gamma, g)$ is non-negative, essentially self-adjoint for the L^2 product.
- The spectrum $N(\gamma, g)$ is
$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$$
- Define the Zeta-function
$$\zeta_N(s) := \sum_{i \geq 1} \lambda_i^{-s},$$
it can be analytically continued to a neighborhood of 0.
- Define (following Ray & Singer 1976)
$$\log \det{}'_{\zeta} N := -\zeta'_N(0)$$
$$\text{``} = \sum_{i \geq 1} \log(\lambda_i) \lambda_i^{-s}|_{s=0} = \log(\prod_{i \geq 1} \lambda_i).\text{''}$$
- The Zeta-regularization of determinants has been used by physicists to perform Feynman path integrals, and is also important in Polyakov's quantum string theory.

Loewner Energy vs. Determinants

Recall $\mathscr{H}(\gamma, g) = \log \det'_\zeta N(\gamma, g) - \log \mathrm{Length}_g(\gamma)$.

Theorem (W. 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $\mathscr{H}(\cdot, g) = \mathscr{H}(\cdot, g_0)$
2. Circles minimize $\mathscr{H}(\cdot, g)$ among all smooth Jordan curves.
3. Let γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\gamma, \gamma(0)) &= 12\mathscr{H}(\gamma, g) - 12\mathscr{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1,g})\det_\zeta(-\Delta_{\mathbb{D}_2,g})}{\det_\zeta(-\Delta_{D_1,g})\det_\zeta(-\Delta_{D_2,g})}, \end{aligned}$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

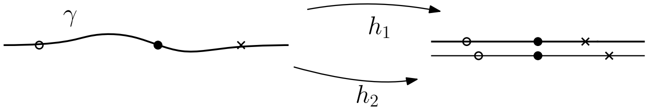
In particular, the above identity gives already the parametrization independence of the Loewner loop energy for smooth loops.

Proof sketch

- First prove the identity: when γ passes through ∞,

$$I^L(\gamma,\infty) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\nabla(\operatorname{Re}\log h'(z))\right|^2 \,\mathrm{d}z^2\right) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\frac{h''}{h'}\right|^2 \,\mathrm{d}z^2\right),$$

  where h maps conformally the complement of γ to two half-planes and fixes ∞.



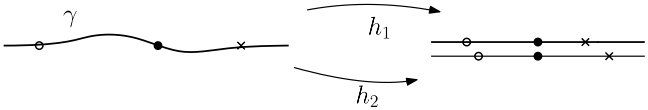
  - Prove the additivity of the right-hand side.
  - Prove the identity for γ driven by a linear function.
  - Prove the identity by approximating the driving function by piecewise linear ones.

Proof (less than sketch)

- First prove the identity: when γ passes through ∞,

$$I^L(\gamma,\infty) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\nabla(\operatorname{Re}\log h'(z))\right|^2 \mathrm{d}z^2\right) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\frac{h''}{h'}\right|^2 \mathrm{d}z^2\right),$$

where h maps conformally the complement of γ to two half-planes and fixes ∞.

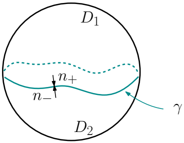


- Write the functional $\mathscr{H}$ in terms of zeta-determinants of Laplacians. [Burghelea, Friedlander, Kappeler, 1993].

Theorem (Burghelea, Friedlander, Kappeler, 1993)

Let (M, g) be a Riemannian surface, $\gamma \subset M$ a smooth Jordan curve dividing M into two components D_1 and D_2. Denote by $\Delta_{D_i,g}$ the Laplace-Beltrami operator with Dirichlet boundary condition on (D_i, g), then we have

$$\begin{aligned}\mathscr{H}(\gamma, g) &= \log \det{}'_\zeta(N(\gamma, g)) - \log \mathrm{Length}_g(\gamma) \\ &= \log \det{}'_\zeta(-\Delta_{M,g}) - \log \mathrm{vol}_g(M) - \log \det{}_\zeta(-\Delta_{D_1,g}) - \log \det{}_\zeta(-\Delta_{D_2,g}).\end{aligned}$$



$$\begin{aligned}\Longrightarrow \mathscr{H}(\gamma, g) - \mathscr{H}(S^1, g) &= \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1,g})\det_\zeta(-\Delta_{\mathbb{D}_2,g})}{\det_\zeta(-\Delta_{D_1,g})\det_\zeta(-\Delta_{D_2,g})} \\ &= \text{“renormalized mass of Brownian loops attached to } \gamma\text{.”}\end{aligned}$$

$\Longrightarrow$ Use the Polyakov-Alvarez conformal anomaly formula to compare determinants of Laplacians.

Polyakov-Alvarez conformal anomaly formula

Take $g = e^{2\sigma} g_0$ a metric conformally equivalent to g_0. (Here think $\sigma = \operatorname{Re}\log h'$.)

Theorem ([Polyakov 1981], [Alvarez 1983], [Osgood, et al. 1988])

For a compact surface M without boundary,

$$\left(\log \det{}'_\zeta(-\Delta_g) - \log \mathrm{vol}_g(M)\right) - \left(\log \det{}'_\zeta(-\Delta_0) - \log \mathrm{vol}_0(M)\right)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_M K_0 \sigma \,\mathrm{dvol}_0.\right]$$

The analogue for a compact surface D with smooth boundary is:

$$\log \det{}_\zeta(-\Delta_g) - \log \det{}_\zeta(-\Delta_0)$$
$$= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_D K_0 \sigma \,\mathrm{dvol}_0 + \int_{\partial D} k_0 \sigma \,\mathrm{dl}_0\right] - \frac{1}{4\pi}\int_{\partial D} \partial_n \sigma \,\mathrm{dl}_0.$$

We get:

$$I^L(\gamma,\gamma(0)) = 12\mathscr{H}(\gamma,g) - 12\mathscr{H}(S^1,g).$$

$\square$

But only for γ smooth!

Outline

Universal Teichmüller space

We write

- $QS(S^1)$ the group of quasisymmetric sense-preserving homeomorphism of S^1;

A sense-preserving homeomorphism $\varphi : S^1 \to S^1$ is *quasisymmetric* if there exists $M \geq 1$ such that for all $\theta \in \mathbb{R}$ and $t \in (0, \pi)$,

$$\frac{1}{M} \leq \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \leq M.$$

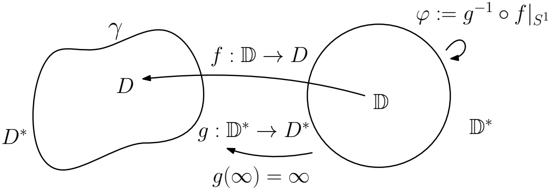
- $\text{Möb}(S^1) \simeq \text{PSL}(2, \mathbb{R})$ the subgroup of Möbius function of S^1.

The *universal Teichmüller space* is

$$T(1) := QS(S^1)/\text{Möb}(S^1) \simeq \{\varphi \in QS(S^1),\ \varphi \text{ fixes } -1, -i \text{ and } 1\}.$$

Welding function

- We know already that $I^L(\gamma) < \infty \implies \gamma$ is a quasicircle and the Loewner energy is Möbius-invariant (Rohde, W. 2017).
- Consider γ as a point $[\varphi] \in T(1)$ via its welding function $\varphi \in QS(S^1)$.



- Quasicircles with corners has ∞ Loewner energy.

Question

What is the class of finite energy loops in $T(1)$?

Weil-Petersson Class

- It is well-known in the literature that the homogeneous space of C^∞-smooth diffeomorphisms
$$M := \mathrm{Diff}(S^1)/\mathrm{Möb}(S^1) \subset T(1)$$
has a Kähler structure on it, studied by many physicists in string theory: Bowick, Rajeev, Kirillov, Yur'ev, Witten, etc.
- There is a unique homogeneous Kähler metric (up to constant factor): the *Weil-Petersson metric*.

Weil-Petersson metric

The Lie algebra of M consists of C^∞ vector fields on S^1:

$$v = v(\theta)\frac{\partial}{\partial\theta} = \sum_{m\in\mathbb{Z}\setminus\{-1,0,1\}} v_m e^{im\theta}\frac{\partial}{\partial\theta}, \text{ where } v_{-m} = \overline{v_m}.$$

The complex structure $J^2 = -Id$ is given by the Hilbert transform:

$$J(v)_m = -i\,\mathrm{sgn}(m) v_m, \text{ for } m \in \mathbb{Z}\setminus\{-1,0,1\}.$$

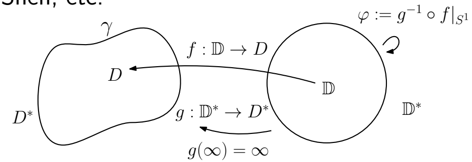
The Weil-Petersson form $\omega(\cdot,\cdot)$ and the metric $\langle\cdot,\cdot\rangle_{WP}$ is given at the origin by

$$\omega(v,w) = ib\sum_{m\in\mathbb{Z}\setminus\{-1,0,1\}} (m^3-m)v_m w_{-m},$$

$$\langle v,w\rangle_{WP} = \omega(J(v),w) = b\sum_{m=2}^{\infty}(m^3-m)\,\mathrm{Re}(v_m w_{-m})$$

for some $b>0$.

- The *Weil-Petersson class* $T_0(1)$ is the closure of $\mathrm{Diff}(S^1)/\mathrm{Möb}(S^1) \subset T(1)$ under the WP-metric.
- The above description and many other characterizations are provided by Nag, Cui, Takhtajan, Teo, Shen, etc.



Theorem (Takhtajan & Teo, 2006)

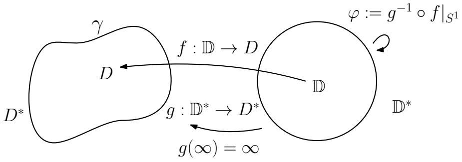
The *universal Liouville action* $\mathbf{S_1} : T_0(1) \to \mathbb{R}$,

$$\mathbf{S_1}([\varphi]) := \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 + 4\pi \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

is a Kähler potential of the Weil-Petersson metric, where

$$g'(\infty) = \lim_{z \to \infty} g'(z) = \tilde{g}'(0)^{-1} \text{ and } \tilde{g}(z) = 1/g(1/z).$$

Loewner Energy vs. Weil-Petersson Class



Theorem (W. 2018)

A bounded simple loop γ in $\hat{\mathbb{C}}$ has finite Loewner energy if and only if $[\varphi] \in T_0(1)$. Moreover,

$$I^L(\gamma) = \mathbf{S_1}(\gamma)/\pi.$$

- This gives a new characterization of the WP-Class, and a new viewpoint on the Kähler potential on $T_0(1)$ (or alternatively a way to look at the Loewner energy).
- Again the root-invariance (and also the reversibility) of the loop energy follows immediately.

Proof

- When γ is smooth, choose cleverly a metric g on S^2 to apply the identity I^L with $\log \det_\zeta$.
- Approximate a general loop by a well-chosen family of smooth curves.

□

Characterizations of the WP-Class (an incomplete list)

The welding function $[\varphi]$ is in $T_0(1)$ if one of the following equivalent conditions holds:

- $\int_{\mathbb{D}} |\nabla \operatorname{Re}(\log f'(z))|^2 \, \mathrm{d}z^2 = \int_{\mathbb{D}} |f''(z)/f'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}} |S(f)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |S(g)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- φ has quasiconformal extension to $\mathbb{D}$, whose complex dilation $\mu = \partial_{\bar{z}}\varphi / \partial_z \varphi$ satisfies

$$\int_{\mathbb{D}} |\mu(z)|^2 \rho(z) \, \mathrm{d}z^2 < \infty;$$

- φ is absolutely continuous with respect to arc-length measure, such that $\log \varphi'$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- the Grunsky operator associated to f or g is Hilbert-Schmidt,

where $\rho(z) \, \mathrm{d}z^2 = 1/(1 - |z|^2)^2 \, \mathrm{d}z^2$ is the hyperbolic metric on $\mathbb{D}$ or $\mathbb{D}^*$ and

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of f.

Outline

We have see‌n:

Loewner Energy $\int W'(t)^2/2dt$ $\xleftarrow[\text{??}]{\text{Large deviation}}$ Schramm Loewner Evolution

This talk | Quantum zipper

$\int_{\mathbb{C}\setminus\gamma} \nabla|\mathrm{Re}(\log h')(z)|^2/\pi dz^2$ $\xleftarrow[\text{??}]{\text{“Large deviation”}}$ Quantum Surfaces $e^{\sqrt{\kappa}\mathrm{GFF}}dz^2$ Liouville quantum gravity

This talk

$\mathscr{H}(\gamma, g)$, ζ-regularized determinants of Δ $\xleftrightarrow{\text{(Dubédat 2008)}}$ Gaussian free field partition function Brownian loop soups

This talk | ??

Weil-Petersson Class $\subset T(1)$ $\xleftrightarrow{\text{??}}$ What is the random object?

What's next?

- What is the random model naturally associated to the WP-class?
- In which space does the random welding belong to?
- Understand the Kähler structure on the WP-class in the probabilistic language.
- Inspired by the welding of two quantum disks, we show that the isometric welding of two finite energy domains still has finite Loewner energy (in progress with Viklund).
- Understand the gradient flow of the Loewner energy on loops (studied by Burghelea et al. 1993) and the meaning under Loewner's framework.

Thanks for your attention!