A multifractal $\mathrm{SLE}_\kappa(\rho)$ spectrum

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Random Conformal Geometry and Related Fields

Outline

1. Preliminaries
   - Multifractal spectrum
   - Imaginary geometry
2. Martingales, one-point estimate and concentration
   - Martingales
3. Two-point estimate
   - Frostman's lemma
   - Perfect points
   - Two-point estimate

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Multifractal spectrum

Consider $\text{SLE}_\kappa(\rho)$ with force point $x_R = 0^+$. Find dimension of set of points x such that

- Curve hits x with certain "angle".
- $g'_{\tau_s}(x) \approx e^{-\beta s}$ as $s \to \infty$,
  where $\tau_s = \inf\{t \geq 0 : \text{dist}(\eta([0,t]), x) \leq e^{-s}\}$.
- $\omega_\infty((O_{\tau_s}, x], \mathbb{H} \setminus K_{\tau_s}) \approx e^{-\alpha s}$ as $s \to \infty$,
  where $\omega_\infty(A, \mathbb{H} \setminus K) = \lim_{y \to \infty} y\omega(iy, A, \mathbb{H} \setminus K)$, $O_t = \max K_t \cap \mathbb{R}$.

Multifractal spectrum

- SLE curves very rough $\Rightarrow$ no up to constants estimates
- Formal set:

$$V_\beta = \left\{ x > 0 : \lim_{s\to\infty} \frac{1}{s} \log g'_{\tau_s}(x) = -\beta(1+\rho/2), \right.$$
$$\left. \tau_s = \tau_s(x) < \infty \ \forall s > 0 \right\}.$$

Main result

Theorem

Let $\kappa > 0$, $\rho \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$, $x_R = 0^+$ *and write* $a = 2/\kappa$. *Define*

$$d(\beta) := 1 - \frac{a\beta}{2}\left(\frac{(1 - a\rho)}{2a} - \frac{1 + 2\beta}{\beta}\right)^2 \left(1 + \frac{\rho}{2}\right)$$

and let $\beta_0 = \frac{2a}{|4a - 1 + a\rho|}$, $\beta_- = \inf\{\beta : d(\beta) > 0\}$ *and* $\beta_+ = \sup\{\beta : d(\beta) > 0\}$. *Then, if* $\kappa \in (0, 4]$

$$\dim_H V_\beta = d(\beta) \text{ for } \beta \in [\beta_-, \beta_+],$$

and if $\kappa \in (4, 8)$,

$$\dim_H V_\beta = d(\beta) \text{ for } \beta \in [\beta_-, \beta_0].$$

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Imaginary geometry

We can couple a GFF h and SLE such that $\mathrm{SLE}_\kappa(\underline{\rho})$ arise as the flow lines of the "vector field" $e^{ih/\chi}$ (but we will call them flow lines of h), where χ is a constant, depending on κ.

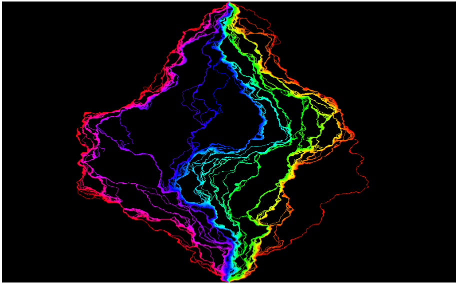


Figure: Flow lines of a GFF on the square $[-1, 1]^2$. (Simulation by Jason Miller.)

Imaginary geometry

Fix $0 < \kappa < 4$ and let $\chi = \chi(\kappa) = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ and $\kappa' = \frac{16}{\kappa} \in (4, \infty)$. Let h be a GFF in $\mathbb{H}$ with piecewise constant boundary data. We can couple h with SLE such that:

- The flow line η of h (of $e^{ih/\chi}$) is an $\mathrm{SLE}_\kappa(\underline{\rho})$ curve from 0 to ∞ (the locations and weights of the force points depend on the boundary data of h).
- A flow line of angle θ, denoted η_θ, is a flow line of $h + \theta\chi$.
- The counterflow line of h is an $\mathrm{SLE}_{\kappa'}(\underline{\rho})$ curve from ∞ to 0 coupled with $-h$. We denote the counterflow line by η'.
- A counterflow line is the "light cone" of flow lines. The outer boundaries of η' are given by $\eta_{-\frac{\pi}{2}}$ and $\eta_{\frac{\pi}{2}}$.
- In this coupling, η, η_θ and η' are almost surely determined by the GFF.

The same holds in other simply connected domains than $\mathbb{H}$ analogously.

Imaginary geometry

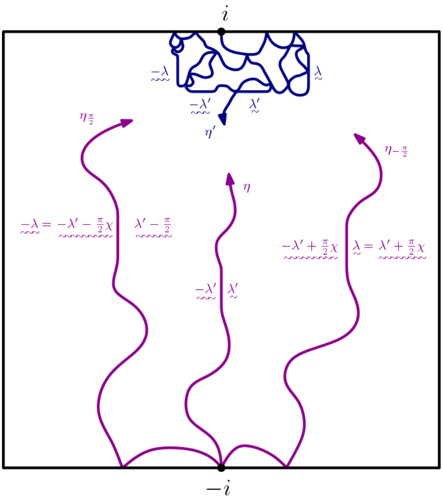


Figure: Flow lines and a counterflow line coupled in the same imaginary geometry.

Imaginary geometry

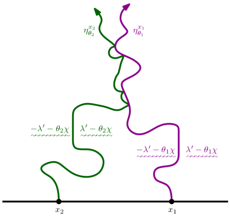
Let η_θ^x denote the flow line of angle θ from x to ∞. Fix $x_1, x_2 \in \mathbb{R}$ such that $x_1 \geq x_2$. Then,

(i) $\theta_1 < \theta_2 \Rightarrow \eta_{\theta_1}^{x_1}$ a.s. stays to the right of $\eta_{\theta_2}^{x_2}$. Can hit if $\theta_2 - \theta_1 < \frac{\pi\kappa}{4-\kappa}$.

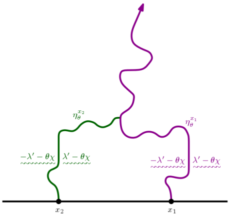
(ii) $\theta_1 = \theta_2 \Rightarrow \eta_{\theta_1}^{x_1}, \eta_{\theta_2}^{x_2}$ can intersect and if they do, they merge and never separate.

(iii) $\theta_2 < \theta_1 < \theta_2 + \pi \Rightarrow \eta_{\theta_1}^{x_1}$ and $\eta_{\theta_2}^{x_2}$ can intersect and:
intersecting $\Rightarrow$ cross and never cross back.
Can hit after crossing if $\theta_1 - \theta_2 < \frac{\pi\kappa}{4-\kappa}$.

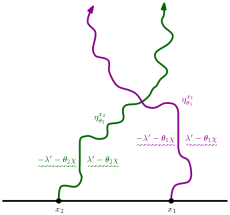
Imaginary geometry



(a) $\theta_1 < \theta_2$.



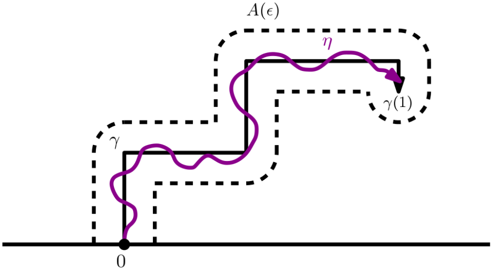
(b) $\theta = \theta_1 = \theta_2$.



(c) $\theta_2 < \theta_1 < \theta_2 + \pi$.

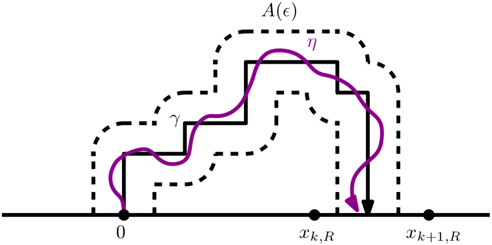
Imaginary geometry

Let $\epsilon > 0$, $\eta \sim \text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$, $x_{1,L} = 0^-$, $x_{1,R} = 0^+$, $\rho_{1,L}, \rho_{1,R} > -2$ and $\gamma : [0,1] \to \overline{\mathbb{H}}$, $\gamma(0) = 0$, $\gamma((0,1]) \subset \mathbb{H}$, then with positive probability, η does not leave the ϵ-neighborhood, $A(\epsilon)$, of γ before coming within distance ϵ from the tip $\gamma(1)$.



$\eta \sim \mathrm{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$, $x_{1,L} = 0^-$, $x_{1,R} = 0^+$, $\rho_{1,L}, \rho_{1,R} > -2$ that can hit $[x_{k,R}, x_{k+1,R}]$ and $\epsilon > 0$ such that $|x_{2,q}| > \epsilon$ for $q \in \{L, R\}$ and $x_{k+1,R} - x_{k,R} \geq \epsilon$ and $x_{k,R} \leq \epsilon^{-1}$. γ curve in $\mathbb{H}$, from 0 to $[x_{k,R}, x_{k+1,R}]$.

$$\mathbb{P}(\eta \text{ hits } [x_{k,R}, x_{k+1,R}] \text{ before leaving } A(\epsilon)) \geq p_0(\kappa, \max_{j,q} |\rho_{j,q}|, \overline{\rho}_{k,R}, \epsilon) > 0$$



Imaginary geometry

Absolute continuity: Let $c = (D, z_0, \underline{x}_L, \underline{x}_R, z_\infty)$ be a configuration and U a bounded open neighborhood of z_0. Let μ_c^U denote the law of an $\text{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process with configuration c, stopped upon exiting U. Let $\tilde{c} = (\widetilde{D}, z_0, \underline{\tilde{x}}_L, \underline{\tilde{x}}_R, \tilde{z}_\infty)$ be another configuration.

- If the force points of c and $\tilde{c}$ in U agree, and the distance from U to the force points that differ is positive, then μ_c^U and $\mu_{\tilde{c}}^U$ are mutually absolutely continuous.
- If $D = \mathbb{H}$, $\widetilde{D} \subseteq \mathbb{H}$, $z_0 = 0$, the force points agree in U and $\varsigma > 0$ such that $\text{dist}(U, \mathbb{H} \setminus \widetilde{D}) > \varsigma$ and the force points of c and $\tilde{c}$ which disagree are at distance at least ς from U, then there exists a constant $C = C(U, \varsigma, \kappa, \{\rho_{j,q}\}_{j,q}) \geq 1$ such that

$$\frac{1}{C} \leq \frac{d\mu_{\tilde{c}}^U}{d\mu_c^U} \leq C.$$

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$\text{SLE}_\kappa(\rho)$

We fix $\kappa \in (0,8)$ and let $a = 2/\kappa$ and parametrize the $\text{SLE}_\kappa(\rho)$ as the solution to

$$\partial_t g_t(z) = \frac{a}{g_t(z) - W_t}, \qquad g_0(z) = z,$$

with

$$\begin{aligned} dW_t &= dB_t + \frac{a\rho/2}{W_t - V_t} dt, & W_0 &= 0; \\ dV_t &= \frac{a}{V_t - W_t} dt, & V_0 &= x_R, \end{aligned}$$

where B_t is a one-dimensional standard Brownian motion with $B_0 = 0$. From now on, assume that $\rho \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$.

Martingale

- We let $\mu_c = 2a - \frac{1}{2} + \frac{a\rho}{2}$ and fix $-\frac{\mu_c^2}{2a} < \zeta < \infty$, and

$$\mu = \mu_c + \sqrt{\mu_c^2 + 2a\zeta}, \quad \beta = \frac{a}{\sqrt{\mu_c^2 + 2a\zeta}}.$$

- We write

$$\delta_t(x) = \frac{g_t(x) - V_t}{g_t'(x)}, \quad Q_t(x) = \frac{g_t(x) - V_t}{g_t(x) - W_t}.$$

- Then

$$M_t^\zeta(x) = g_t'(x)^\zeta Q_t^\mu \delta_t^{-\mu(1+\frac{\rho}{2})},$$

is a local martingale and

$$\frac{dM_t^\zeta(x)}{M_t^\zeta(x)} = \frac{\mu}{f_t(x)} dB_t,$$

where $f_t(x) = g_t(x) - W_t$.

Martingale

- Weight by M_t^ζ to get the measure $\mathbb{P}^*$.
- This measure change is practical, since

$$\delta_t(x) \asymp \mathrm{dist}(x, \eta([0,t]))$$

and since under $\mathbb{P}^*$, Q_t has an invariant distribution, and hence we have good control.

- Furthermore, Q_t is the quotient of two harmonic measures.
- Gives one-point estimate sufficient for upper bound on dimension.
- $\tilde{I}_t$ good event until the time $\delta = (x - x_R)e^{-at}$. Concentration estimates $\Rightarrow \mathbb{P}^*(\tilde{I}_t)$ arbitrarily close to 1.

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Frostman's lemma

Let $A \subset \mathbb{R}^n$ be a set and let ν be a measure with support contained in A and let

$$J_s(\nu) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{|x-y|^s} d\nu(x) d\nu(y).$$

If $J_s(\nu) < \infty$, then $\dim_H A \geq s$.

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Perfect points

- Fix $0 < \kappa < 4$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$, h GFF in $\mathbb{H}$ with boundary data $-\lambda$ in $\mathbb{R}_-$ and $\lambda(1+\rho)$ on $\mathbb{R}_+$. Let η be the zero-angle flow line emanating from 0, i.e., an $\text{SLE}_\kappa(\rho)$ curve with force point 0^+.
- Denote the (zero-angle) flow line from x by η^x. $\eta^x \sim \text{SLE}_\kappa(2+\rho, -2-\rho; \rho)$ with configuration $(\mathbb{H}, x, (0, x^-), x^+, \infty)$.
- Fix $\delta \in (0, \frac{1}{2})$ and $\epsilon = e^{-\alpha}$, $\alpha > 10$ (to be determined later).
- For $x \geq 1$ and $k \in \mathbb{N}$, we write

$$x_k = \begin{cases} x - \frac{1}{4}\epsilon^k & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases}$$

- For $U \subset \mathbb{H}$,

$$\sigma^x(U) = \inf\{t \geq 0 : \eta^x(t) \in \overline{U}\}.$$

  and $\sigma_k^x = \sigma^{x_k}(B(x, \epsilon^{k+1}))$.
- $\tilde{I}_t^k = \tilde{I}_t^k(x)$ is the good event for η^{x_k} regarding concentration of measure (and its indicator), and

$$I_k^k = \mathbb{E}\left[\tilde{I}^k_{k/a+G(x,x_k)}\middle|\mathscr{F}_{\sigma_k^x}\right],$$

  where $G(x, x_k)$ is a function such that $\tilde{t}(\frac{k}{a} + G(x, x_k)) \geq \sigma_k^x$.
- Let $\eta^{x_k,R}$ denote the right side of η^{x_k} and $V_t^k = \max\{y \in \eta^{x_k}([0,t]) \cap \mathbb{R}\}$ and

$$Q_t^k = \frac{\omega_\infty([V_t^k, x], \mathbb{H} \setminus \eta^{x_k}([0,t]))}{\omega_\infty(\eta^{x_k,R}([0,t]) \cup [V_t^k, x], \mathbb{H} \setminus \eta^{x_k}([0,t]))}.$$

Perfect points

- $A_k^1(x)$ is the event that
  (i) $\sigma_k^x < \infty$,
  (ii) $Q^k_{\sigma_k^x} \in [\delta, 1-\delta]$ and
  (iii) $\sigma_k^x < \sigma^{x_k}(\mathbb{H} \setminus B(x, \frac{1}{2}\epsilon^k))$,

  and $E_k^1(x) = 1_{A_k(x)} I_k^{u,k}$.
- We let $A_k^2(x)$ be the event that on $A_k^1(x)$
  (i) $\eta^{x_{k-1}}|_{[\sigma_{k-1}^x, \infty)}$ merges with $\eta^{x_k}|_{[0,\sigma_k^x)}$ before exiting $B(x, \frac{1}{2}\epsilon^k) \setminus B(x, \epsilon^{k+1})$
  (ii) $\arg(\eta^{x_k}(t) - x) \geq \frac{2}{3}\min(\arg(\eta^{x_{k+1}}(\sigma_{k+1}^x) - x), \arg(\eta^{x_k}(\sigma_k^x) - x))$ for $t > \sigma_k^x$ but before merging with $\eta^{x_{k+1}}$,

  and $E_k^2(x) = 1_{A_k^2(x)}$.

Perfect points

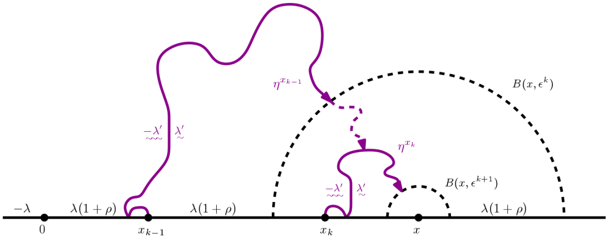

Figure: If $E_k^1(x) = 1$, then $\eta^{x_{k-1}}$ hits $B(x, \epsilon^k)$, $Q^k_{\sigma^x_k} \in [\delta, 1-\delta]$ and the derivatives of the Loewner chain for $\eta^{x_{k-1}}$ behave as we want. Furthermore, given that $E_k^1(x) = 1$, we have that if $E_k^2(x) = 1$, then $\eta^{x_{k-1}}$ merges with η^{x_k} before exiting $B(x, \frac{\epsilon^k}{2}) \setminus B(x, \epsilon^{k+1})$ and does not go too close to $\{s > x\}$ before doing so.

- Let $E_k(x) = E_k^1(x)E_k^2(x)$, $E^{m,n}(x) = E_{m+1}^1(x)\prod_{k=m+2}^n E_k(x)$, and $E^n(x) = E^{0,n}(x)$.
- Is this the right event to consider? Yes, because by Koebe 1/4 theorem,

$$\begin{aligned} g'_{\sigma(B(x,\epsilon))}(x) &\asymp \epsilon^{-1}(g_{\sigma(B(x,\epsilon))}(x) - V_{\sigma(B(x,\epsilon))}) \\ &= \epsilon^{-1}\omega_\infty((O_{\sigma(B(x,\epsilon))}, x], \mathbb{H} \setminus K_{\sigma(B(x,\epsilon))}) \end{aligned}$$

where O_t is the rightmost point of $K_t \cap \mathbb{R}$, and the harmonic measure from ∞ of the "inner" parts of $\eta^{x_k}([0,\sigma_k^x])$ and $\eta([0,\sigma(B(x,\epsilon^{k+1}))])$ are comparable.

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Two-point estimate

The sequence of measures that we will consider is $\{\nu_n\}$ where

$$\nu_n(A) = \int_A \sum_{x \in \mathscr{D}_n} \frac{E^n(x)}{\mathbb{E}[E^n(x)]} 1_{J_n(x)}(t)dt,$$

$\mathscr{D}_n = \{1 + (j - \frac{1}{2})\epsilon^n : j = 1, \ldots, \epsilon^{-n}\}$ and $J_n(x) = [x - \frac{\epsilon^n}{2}, x + \frac{\epsilon^n}{2}]$.
The aim is to prove the following.

Proposition

For each sufficiently small $\delta \in (0, \frac{1}{2})$ there exist a constant $c(\delta) > 0$ and a subpower function ψ such that for all $x, y \in [1, 2]$ and $m \in \mathbb{N}$ such that $2\epsilon^{m+2} \leq |x - y| \leq \frac{1}{2}\epsilon^m$, we have

$$\begin{aligned}\mathbb{E}[E^n(x)E^n(y)] \leq & c(\delta)^{2m+2}\psi(-\log \epsilon)^{(3m+2)|\zeta|} \\ & \times \epsilon^{(m+2)(\zeta\beta-\mu)(1+\rho/2)}\mathbb{E}[E^n(x)]\mathbb{E}[E^n(y)].\end{aligned}$$

Two-point estimate

Strategy: want to mimic the strategy of Miller and Wu [2017] to separate points, view them as "almost independent". Do this via the result on the boundedness of Radon-Nikodym derivatives. Need the following.

Lemma

For every $x \geq 1$ and $m, n \in \mathbb{N}$ such that $m \leq n$, it holds that

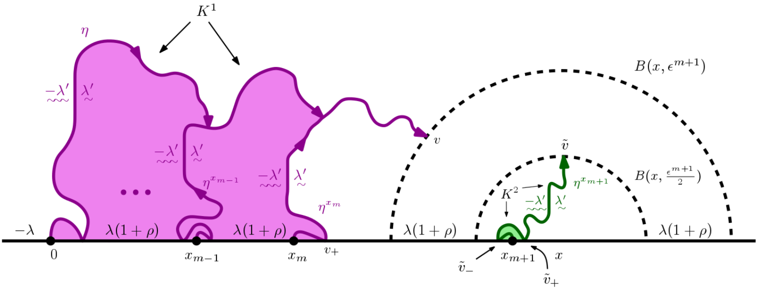
$$\mathbb{E}[E^m(x)E^{m,n}(x)] \asymp \mathbb{E}[E^m(x)]\mathbb{E}[E^{m,n}(x)].$$

Furthermore, if y is such that $2\epsilon^{m+2} \leq |x-y| \leq \frac{1}{2}\epsilon^m$, then

$$\mathbb{E}[E^{m-1}(x)E^{m+1,n}(x)E^{m+1,n}(y)] \asymp \mathbb{E}[E^{m-1}(x)]\mathbb{E}[E^{m+1,n}(x)]\mathbb{E}[E^{m+1,n}(y)].$$

The constants in $\asymp$ depend only on κ and ρ.

Two-point estimate



Proof idea: the R-N derivative between the laws of the green part with and without the purple, is bounded above and below by a constant, which is independent of m, since

$$\frac{\text{dist}(K^1, K^2)}{\text{diam}(U)} \gtrsim 1.$$

.

Two-point estimate

The next result we need is the following.

Lemma

For each $x \geq 1$ and $m, n \in \mathbb{N}$ such that $m \leq n$, it holds that

$$\mathbb{E}[E^n(x)] \asymp \mathbb{E}[E^m(x)]\mathbb{E}[E^{m,n}(x)],$$

where the constants depend only on κ, ρ and δ.

Proof idea: the condition on $Q^k_{\sigma^x_m}$ makes sure that the harmonic measure (from ∞) of each side of the curve, and $[V_{\sigma^x_m}, x]$ and hence $\eta^{x_{m+1}}([0, \sigma^x_{m+1}])$ are comparable. Hence, using the mapping out function, each of them will have a positive length, and using that the curve then will follow any curve we want with positive probability gives the result.

Two-point estimate

The last lemma we need is:

Lemma

For each $\delta \in (0, \frac{1}{2})$, sufficiently small, there exist a constant $c(\delta) > 0$ and a subpower function ψ such that the for each $x \geq 1$,

$$\mathbb{E}[E^m(x)] \geq c(\delta)^m \psi(-\log \epsilon)^{-m|\zeta|} \epsilon^{m(\zeta\beta-\mu)(1+\rho/2)}.$$

Proof idea: by previous lemmas, we need only check that there exist a constant $c(\delta)$ and a subpower function ψ such that

$$\mathbb{E}[E_k^1(x)] \geq c(\delta)\psi(-\log \epsilon)^{-|\zeta|} \epsilon^{(\zeta\beta-\mu)(1+\rho/2)}$$
$$\mathbb{E}[E_k^2(x)|E^{k-1}(x) = 1, E_k^1(x) = 1] \asymp 1.$$

The latter follows by the same idea as the previous lemma.

- $\mathbb{E}[E_k^1(x)] = \mathbb{P}\left(A_k^1 \cap \tilde{I}^k_{k/a+G(x,x_k)}\right)$
- We can consider an $\text{SLE}_\kappa(-2-\rho;\rho)$ curve with configuration $(\mathbb{H}, x_k, x_k^-, x_k^+, \infty)$ instead of η^{x_k}.
- Translating and rescaling, the event $\{\sigma_k^x < \sigma^{x_k}(\mathbb{H} \setminus B(x, \frac{1}{2}\epsilon^k))\}$ turns into the event $\{\hat{\eta}$ hits $B(1,\epsilon)$ before leaving $B(1,2)\}$, where $\hat{\eta}$ is the rescaled curve. (The condition on Q remains roughly the same.)
- Denote by (g_t) the Loewner chain corresponding to $\hat{\eta}$ and weigh the probability measure $\mathbb{P}$ with the local martingale

$$M_t^\zeta(1) = g_t'(1)^\zeta Q_t^\mu \delta_t^{-\mu(1+\rho/2)}(g_t(1) - V_t^L)^{\mu(1+\rho/2)},$$

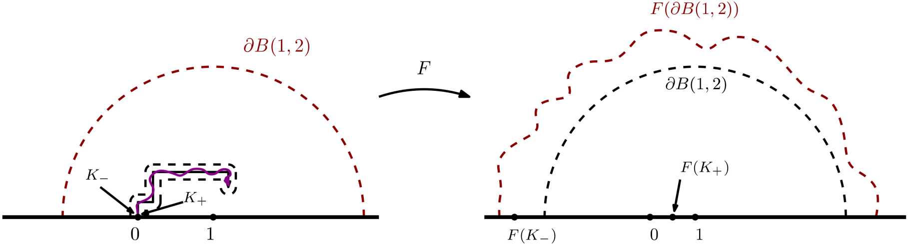
and denote the resulting measure by $\mathbb{P}^*$ (above quantities are the mentioned above, but for $\hat{\eta}$).

Two-point estimate

- Using estimates on g' and geometric estimates on the other quantities of M_t^ζ, we have

$$\psi(-\log\epsilon)^{-|\zeta|}\epsilon^{-(\zeta\beta-\mu)(1+\rho/2)}\mathbb{P}^*(A_k^1 \cap \tilde{I}^k_{k/a+G(x,x_k)})$$
$$\lesssim \mathbb{P}(A_k^1 \cap \tilde{I}^k_{k/a+G(x,x_k)})$$
$$\lesssim \psi(-\log\epsilon)^{|\zeta|}\epsilon^{-(\zeta\beta-\mu)(1+\rho/2)}\mathbb{P}^*(A_k^1 \cap \tilde{I}^k_{k/a+G(x,x_k)}).$$

Two-point estimate

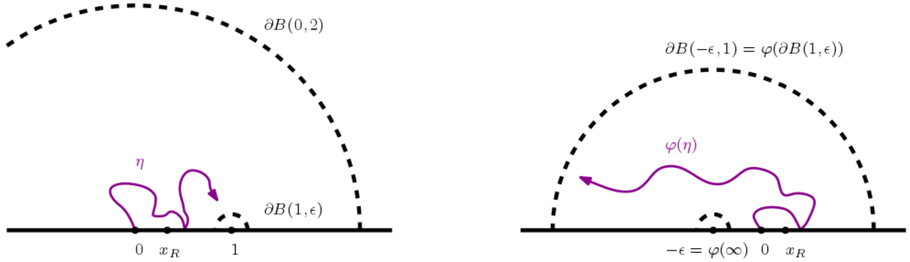


Let $\gamma : [0,1] \to \overline{\mathbb{H}}$, be a deterministic curve starting at 0 and remaining in $\mathbb{H}$ after that, and $\tilde{\epsilon} > 0$ be such that if $\hat{\eta}$ comes within distance $\tilde{\epsilon}$ of the tip $\gamma(1)$ before exiting the $\tilde{\epsilon}$-neighborhood of γ, then

$$\text{dist}(1, F(\partial B(1,2) \cap \mathbb{H})) \geq 2 \text{ and } F(\min\{\hat{K}_{\tilde{\sigma}_1} \cap \mathbb{R}\}) < -2,$$

and $\text{dist}(F(K^+), 1) \geq \tilde{\delta} > 0$. Now, we can consider a curve with only one force point, $F(K^+)$.

Two-point estimate



Let $\varphi(z) = \frac{\epsilon z}{1-z}$ and do a Schramm-Wilson coordinate change. $\varphi(B(1,\epsilon)) = B(-\epsilon, 1)$ and $\varphi(B(1,2)) = B(-\epsilon, \frac{\epsilon}{2})$. The event of hitting $B(1,\epsilon)$ before exiting $B(1,2)$ turns into hitting $\partial B(-\epsilon, 1)$ before hitting $B(-\epsilon, \frac{\epsilon}{2})$. Happens with probability $\geq p_0 > 0$. Thus,

$$\mathbb{P}^*(A_k \cap \tilde{I}^k_{k/a+G(x,x_k)}) \gtrsim 1.$$

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With these estimates at hand, separate as:

$$\mathbb{E}[E^n(x)E^n(y)] \leq \mathbb{E}[E^{m-1}(x)E^{m+2,n}(x)E^{m+2,n}(y)]$$
$$\lesssim \mathbb{E}[E^{m-1}(x)]\mathbb{E}[E^{m+2,n}(x)]\mathbb{E}[E^{m+2,n}(y)]$$

and then "patch up" with curves merging (without losing too much probability), and estimate with the last one-point estimate and we are done (after applying this together with Frostman's lemma).

Thanks for listening!