Towards $2D$ random Kähler geometry

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Plan of the talk

Which random geometries for $2d$ quantum gravity?

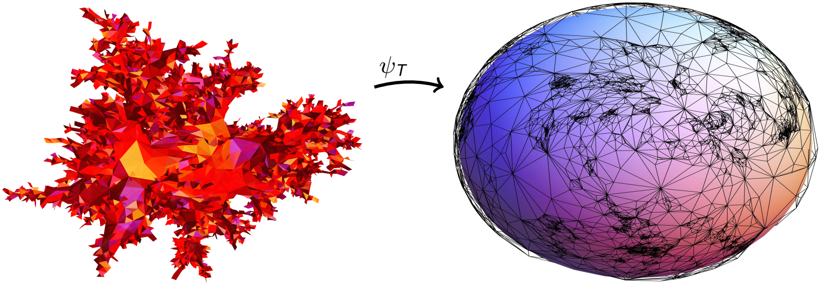
Overview of the model

Mathematical content

2d quantum gravity recast in terms of random triangulations

- glue N flat equilateral triangles together along their edges so as to get a (complex) manifold T with spherical topology.
- Push the flat metric on the faces of T to the sphere via a conformal map $\psi_T : T \to \mathbb{S}$ to get a metric g_T on $\mathbb{S}$

Goal: Sample T at random among $\mathcal{T}_N := \{T \,:\, T \text{ with } N \text{ faces}\}$. What is the law of g_T in the scaling limit $N \to \infty$?



Coupling $2d$ quantum gravity to conformal matter

- **Pure gravity:** sample T uniformly at random among $\mathcal{T}_N$.
  $\hookrightarrow$ Scaling limit of g_T described by **Liouville CFT** with $\gamma = \sqrt{8/3}$ (D-M-S '14-'18)

- **Conformal matter:** sample T according to the partition function of a CFT on T. E.g. partition function for GFF defined for g metric on $\mathbb{S}$ by

$$Z(g) := \int \exp\Big(-\frac{1}{4\pi}\int_{\mathbb{S}} |dX|_g^2 \, \mathrm{d}v_g\Big) DX = \Big(\frac{\det'(-\triangle_g)}{v_g(\mathbb{S})}\Big)^{-\mathbf{c}_{\mathrm{mat}}/2},$$

  which makes sense in terms of regularized determinants (Ray-Singer '71). Here $\mathbf{c}_{\mathrm{mat}} = 1$. Then pick T according to

$$\mathbb{P}_N(T) = \frac{Z(g_T)}{Z_N} \quad \text{with} \quad Z_N := \sum_{T \text{ with } N \text{ faces}} Z(g_T)$$

  $\hookrightarrow$ Scaling limit of g_T described by **Liouville CFT** with $\gamma = \text{function}(\mathbf{c}_{\mathrm{mat}})$ (to be proven)

The heuristics (DDK '80s) rely on **Polyakov's anomaly formula:** for $\omega : \mathbb{S} \to \mathbb{R}$ smooth

$$\ln \frac{Z(e^{\omega} g)}{Z(g)} = \frac{\mathbf{c}_{\mathrm{mat}}}{96\pi} S_L(g_0, \omega)$$

where S_L is the Liouville functional (more later).

Coupling 2d quantum gravity to massive matter?

- **Massive GFF:** partition function ($q \in \mathbb{R}$ and mass $m > 0$)

$$Z(g,q,m) = \int \exp\Big(-\frac{1}{4\pi} \int_M \big(|dX|_g^2 + iq\, R_g X + m^2 X^2\big)\, \mathrm{dv}_g \Big) DX$$

Question: what if we sample triangulations according to this model?

- $m = 0$ **model:** remove divergencies to compute the $m \to 0$ partition function

$$Z_0(g,q) := \lim_{m \to 0} Z(g,q,m)$$

Anomaly formula: if $\hat{g} = e^{\omega} g$ for some $\omega : \mathbb{S} \to \mathbb{R}$ smooth

$$\ln \frac{Z_0(e^{\omega} g, q)}{Z_0(g,q)} = \frac{1-6q^2}{96\pi} S_L(g,\omega) + \frac{q^2}{4\pi} S_M(g,\hat{g})$$

where S_L is the **Liouville functional** and S_M the **Mabuchi K-energy**.
→ advocated by Bilal/Ferrari/Klevtsov/Zelditch in a series of papers 2011-2014.

Upshot: scaling limit of g_T when sampling triangulations w.r.t. this model conjecturally described by path integral involving Liouville+K-energy, which we see as a natural version of "quantum Kähler geometry"

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Which random geometries for $2d$ quantum gravity?

Overview of the model

Mathematical content

What is a (natural) random geometry?

- In geometry, basic objects in view of classification are manifolds with uniformized curvature
- such manifolds can generally be found by solving variational problems: one looks for the minimizer of some functional

$$\varphi \in \Sigma \mapsto S(\varphi).$$

- Corresponding random geometry is a functional measure (path integral) on Σ

$$e^{-S(\varphi)} D\varphi$$

  where $D\varphi$ is the "Lebesgue measure" on Σ.

  Approach inherited from Feynmann's view on quantum mechanics.

Riemmanian geometry

Consider a $2d$ manifold M equipped with a Riemannian metric g_0.

- Ricci scalar curvature R_{g_0} (in $2d$)

$$\frac{\text{volume of } B_{g_0}(x,\epsilon) \text{ in } M}{\text{Euclidean vol. of } B(0,\epsilon)} = 1 - \frac{R_{g_0}(x)}{24}\epsilon^2 + O(\epsilon^4)$$

- **Uniformization:** a story that goes back to E. Picard and H. Poincaré is to find a metric g of the form

$$g = e^{\omega} g_0 \quad \text{for some smooth map } \omega : M \to \mathbb{R}$$

  with constant Ricci scalar curvature

$$R_g = -\mu$$

Classical Liouville functional

Let $\gamma, \mu > 0$ be some parameters.

The map $\omega : M \to \mathbb{R}$ such that the metric $g = e^{\gamma\omega} g_0$ has uniformized curvature

$$R_g = -2\pi\mu\gamma^2$$

is a critical point of the **Liouville functional**

$$\omega \mapsto S_L(g_0, \omega) = \frac{1}{4\pi} \int_M \left(|d\omega|^2_{g_0} + Q_c R_{g_0}\omega + 4\pi\mu e^{\gamma\omega} \right) \mathrm{dv}_{g_0}$$

with

$$Q_c = \frac{2}{\gamma}$$

Notations: $\triangle_g =$ *Laplacian*, R_g=Ricci curvature, v_g=volume form

Kähler geometry

Consider a $2d$ manifold M equipped with a Riemannian metric g_0.

- Kähler potential ϕ of the metric $g = e^{\omega} g_0$ w.r.t. g_0 defined by

$$\frac{e^{\omega}}{\mathrm{v}_g(M)} - \frac{1}{\mathrm{v}_{g_0}(M)} = \frac{1}{2} \triangle_{g_0} \phi$$

  Another parametrization of the set of metrics that allows one to translate the search for constant curvature metrics in terms of complex Monge-Ampère equation.

- This has led to a classification of Kähler manifolds (any dimension) with successive works by Aubin, Yau, Tian , Donaldson (1978-2015).

Mabuchi K-energy

- Let ϕ be the Kähler potential of the metric $g = e^{\omega} g_0$ w.r.t. g_0

$$\frac{e^{\omega}}{\mathrm{v}_g(M)} - \frac{1}{\mathrm{v}_{g_0}(M)} = \triangle_{g_0}\phi$$

- Definition of **Mabuchi K-energy**

$$S_{\mathrm{M}}(g_0, g) = \int_M \Big(2\pi(1-\mathbf{h})\phi\triangle_{g_0}\phi + \big(\frac{8\pi(1-\mathbf{h})}{\mathrm{v}_{g_0}(M)} - R_{g_0}\big)\phi + \frac{2}{\mathrm{v}_g(M)}\omega e^{\omega}\Big)\mathrm{dv}_{g_0}$$

with $\mathbf{h}$:= genus of M.

- Critical points give metrics $g := e^{\omega} g_0$ with uniformized curvature

Notations: $\triangle_g$ = *Laplacian*, R_g=Ricci curvature, v_g=volume form

Quantum Riemannian geometry/Liouville theory

Consider a Riemann surface M equipped with a metric g_0, and parameters $\mu > 0$, $\gamma \in (0, 2)$.

Quantum Liouville theory is a measure formally defined by

$$\langle F \rangle_{L,g_0} := \int F(\varphi) e^{-\mathcal{S}_L(g_0,\varphi)} D\varphi$$

where

- $\mathcal{S}_L$ is the **quantum Liouville functional**

$$\mathcal{S}_L(g_0, \varphi) = \frac{1}{4\pi} \int_M \Big(|d\varphi|_g^2 + QR_g\varphi + 4\pi\mu e^{\gamma\varphi} \Big) d\mathrm{v}_g$$

- $D\varphi$ is the "Lebesgue measure" on the space of maps $\varphi : M \to \mathbb{R}$.
- Q is a parameter tuned at its quantum value

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

David, Guillarmou, Kupiainen, R., Vargas (2014-2016): *Construction on Riemann surfaces*

Quantum Riemannian geometry/Liouville theory

- Random geometry is then understood as associated to the random metric tensor

$$e^{\gamma\varphi} g_0$$

where the random "function" φ has probability law characterized by functional expectations

$$\mathbb{E}[F(\varphi)] = \frac{1}{Z} \langle F \rangle_{L,g_0}$$

with

$$\langle F \rangle_{L,g_0} := \int F(\varphi) e^{-S_L(g_0,\varphi)} D\varphi$$

and $Z = \langle 1 \rangle_{L,g_0}$ is the normalizing constant to have mass 1.

- As it turns out, φ is not a fairly defined function a.s. $\Rightarrow$ rich multifractal geometry
  - **Volume form**: uses Gaussian multiplicative chaos (GMC) theory

    KAHANE (1985)
  - **Distance**: understood for $\gamma = \sqrt{8/3}$.

    DUPLANTIER, MILLER, SHEFFIELD,... (2014-2018)

Symmetries of CFTs are encoded in the way they react to changes of background metrics

Conformal anomaly (David-Kupiainen-Rhodes-Vargas 14')

Consider a conformal metric $g = e^{\omega} g_0$ then

$$\langle F \rangle_{\mathrm{L},g} = \langle F(\cdot - \tfrac{Q}{2}\omega) \rangle_{\mathrm{L},g_0} \exp\Big(\frac{\mathbf{c}_{\mathrm{L}}}{96\pi} S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) \Big) \qquad (1)$$

where $S_{\mathrm{L}}^{\mathrm{cl},0}$ is the classical Liouville functional (with $\mu = 0$)

$$S_{\mathrm{L}}^{\mathrm{cl},0}(g_0, \omega) := \int_M \big(|d\omega|_g^2 + 2R_{g_0}\omega \big) \mathrm{dv}_{g_0}, \qquad (2)$$

and $\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2$ is the *central charge* of the Liouville theory.

Contains a great deal of information about the theory

- leads to PDEs (Ward/BPZ equations) that can be used to solve the theory
  - KUPIAINEN-RHODES-VARGAS (2016-2017)
    Local conformal structure and DOZZ formula
- connection with quantum gravity models (Polyakov, David-Distler-Kawai,...)
- **Question:** can we come up with a path integral producing a further Mabuchi term?

Quantum Kähler geometry

Consider a Riemann surface M with genus $\mathbf{h}$ equipped with a metric g_0, and parameters $\mu, \beta > 0$, $\gamma \in (0, 2)$. Set $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$

Quantum Kähler theory is a measure formally defined by

$$\langle F \rangle_{\mathrm{ML}, g_0} := \int F(\varphi) e^{-\beta^2 \mathcal{S}_{\mathrm{M}}(g_0, e^{\gamma\varphi} g_0) - \mathcal{S}_L(g_0, \varphi)} D\varphi$$

where

- $\mathcal{S}_M$ is the **quantum Mabuchi K-energy:** if $g = e^{\gamma\varphi} g_0$

$$\mathcal{S}_{\mathrm{M}}(g_0, g) = \int_M \Big(2\pi(1-\mathbf{h})\phi \triangle_{g_0} \phi + \big(\frac{8\pi(1-\mathbf{h})}{\mathrm{v}_{g_0}(M)} - R_{g_0}\big)\phi + \frac{2}{1-\frac{\gamma^2}{4}} \frac{1}{V_g(M)} \gamma\varphi e^{\gamma\varphi} \Big) \mathrm{dv}_{g_0}$$

and ϕ is the Kähler potential of the metric $e^{\gamma\varphi} g_0$ w.r.t. g_0

LACOIN, RHODES, VARGAS (2018 to appear)

Existence: main results

Assume M has genus $\mathbf{h} \geqslant 2$ and $\gamma \in (0,1)$ and $\beta > 0$.

Theorem (Lacoin, Rhodes, Vargas 2018)

One can make sense probabilistically to the path integral

$$\langle F \rangle_{\mathrm{ML},g_0} := \int F(\varphi) e^{-\beta^2 \mathcal{S}_{\mathrm{M}}(g_0, e^{\gamma\varphi} g_0) - \mathcal{S}_{\mathrm{L}}(g_0,\varphi)} D\varphi$$

This path integral has finite mass, i.e. $\langle 1 \rangle_{\mathrm{ML},g_0} < +\infty$ provided that the Mabuchi coupling constant is small enough

$$\beta \in \Big(0, \frac{\mathbf{h}-1}{2}\big(\tfrac{4}{\gamma^2} - \tfrac{\gamma^2}{4}\big)\Big)$$

Remark:

- the constraint on β is not a technical restriction, it is a **topological obstruction**!

Conformal anomaly (LRV 18')

Consider a conformal metric $g = e^{\omega} g_0$ then

$$\langle F \rangle_{\mathrm{ML},g} = \langle F(\cdot - \tfrac{Q}{2}\omega, \phi) \rangle_{\mathrm{ML},g_0} \exp\Big(\frac{\mathbf{c}_{\mathrm{L}}}{96\pi} S^{\mathrm{cl},0}_{\mathrm{L}}(g_0, \omega) + \beta S_{\mathrm{M}}(g_0, g) \Big) \tag{3}$$

where $S^{\mathrm{cl},0}_{\mathrm{L}}$, S_{M} are respectively classical Liouville functional (with $\mu = 0$) and classical Mabuchi K-energy.

String susceptibility (LRV 18')

The path integral $\langle \cdot \rangle_{\mathrm{ML},g_0}$ defines a random geometry under which the "volume of the manifold" has Gamma law $\Gamma(s, \mu)$.
The area scaling exponent s, called string susceptibility, has the expression

$$s := \frac{2Q}{\gamma}(\mathbf{h} - 1) - \frac{2\beta}{1 - \frac{\gamma^2}{4}}$$

Remarks:

- our formula agrees with the asymptotic expansion $\gamma \to 0$ given in physics (exact expression was open question)
- these properties make the connection with quantum gravity models (Rand. Triang.)
- QFT with global conformal invariance...not a CFT!

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Overview of the model

Mathematical content

Gaussian Free Field (GFF)

- The GFF X_{g_0} with vanishing g_0-mean is a centered Gaussian Schwartz distribution
- for f, f' test functions

$$\mathbb{E}\Big[\Big(\int_M f\, X_{g_0}\, \mathrm{dv}_{g_0}\Big)\Big(\int_M f' X_{g_0}\, \mathrm{dv}_{g_0}\Big)\Big] = \iint_{M^2} f(x) f'(y) G_{g_0}(x,y)\, \mathrm{v}_{g_0}(dx)\, \mathrm{dv}_{g_0}(dy)$$

  with G_{g_0} the integral kernel for the Poisson equation

$$-\triangle_{g_0} u = 2\pi f, \quad \text{with b.c.} \quad \int_M u\, \mathrm{dv}_{g_0} = 0$$

- Almost surely X_{g_0} lives in a Sobolev space $H^{-\delta}(M)$ with $\delta > 0$
- **Interpretation:** for positive continuous functionals F on $H^{-\delta}(M)$

$$\int F(\varphi) e^{-\frac{1}{4\pi}\int_M |d\varphi|^2_{g_0}\, \mathrm{dv}_{g_0}} D\varphi = \int_{\mathbb{R}} \mathbb{E}[F(c + X_{g_0})]\, dc$$

Gaussian Multiplicative chaos (GMC)

- **Goal**: construct a random measure formally given by

$$e^{\gamma X_{g_0}}\,\mathrm{dv}_{g_0}$$

Ill-defined as X_{g_0} is not a fairly defined function. At short scale

$$\mathbb{E}[X_{g_0}(z)X_{g_0}(z')] \approx \ln \frac{1}{d_{g_0}(z,z')}$$

- Call X_ϵ a mollification of the field X_{g_0} at scale ϵ

$$\mathbb{E}[X_\epsilon(z)X_\epsilon(z')] \approx \ln \frac{1}{d_{g_0}(z,z')+\epsilon}.$$

Theorem (Kahane 1985)

For $\gamma \in (0,2)$ there exists a random measure $\mathcal{G}_{g_0}^{\gamma}$ such that, almost surely, the limit

$$\lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\epsilon(z)}\,\mathrm{v}_{g_0}(dz) = \mathcal{G}_{g_0}^{\gamma}(dz)$$

holds in the space of Radon measure. $\mathcal{G}_{g_0}^{\gamma}$ does not depend on the regularization.

Some properties of GMC measures

- Non triviality

Theorem (Kahane 1985)

The random measure $\mathcal{G}_{g_0}^{\gamma}$ is non trivial if and only if $\gamma \in (0, 2)$.

- Multifractal behaviour

Theorem (Kahane 1985)

For $\gamma \in (0, 2)$ almost surely, the measure $\mathcal{G}_{g_0}^{\gamma}(d^2 z)$ lives on a set of Hausdorff dimension $2 - \frac{\gamma^2}{2}$.

- the total mass $\mathcal{G}_{g_0}^{\gamma}(M)$ is finite almost surely.

Quantum Liouville: definition

Rigorously the measure is defined by (assuming g_0 is uniformized)

$$\langle F \rangle_{L,g_0} := \int_{\mathbb{R}} e^{-2Q(1-\mathbf{h})c} \mathbb{E}\Big[F(c + X_{g_0}) \exp\Big(-\mu e^{\gamma c} \mathcal{G}^{\gamma}_{g_0}(M) \Big) \Big] \, dc$$

where

- $\mathbf{h}$ is the genus of M
- X_{g_0} is a Gaussian Free Field under $\mathbb{E}$ with vanishing g_0-mean
- $\mathcal{G}^{\gamma}_{g}(M)$ is a Gaussian multiplicative chaos (GMC) formally understood as

$$\mathcal{G}^{\gamma}_{g_0}(M) = \int_M e^{\gamma X_{g_0}} \, \mathrm{dv}_{g_0}$$

Quantum Liouville-Mabuchi: construction

Quantum Mabuchi theory (assuming g_0 is uniformized) defined by

$$\langle F\rangle_{\mathrm{L},\mathrm{g}_0} := \int_{\mathbb{R}} e^{-2Q(1-\mathbf{h})c}\mathbb{E}\Big[e^{-\beta\mathcal{S}_{\mathrm{M}}}F(c+X_{g_0})\exp\Big(-\mu e^{\gamma c}\mathcal{G}^{\gamma}_{g_0}(M)\Big)\Big]\,dc$$

where $\mathcal{S}_{\mathrm{M}}$ is the quantum Mabuchi K-energy of the "Liouville random metric" $e^{\gamma(c+X_{g_0})}g_0$ w.r.t. g_0. It can be given sense and involves:

- Kähler potential of the "Liouville random metric"

$$\Phi(z) := -\frac{2}{\mathcal{G}^{\gamma}_{g}(M)}\int G_{g_0}(z,w)\mathcal{G}^{\gamma}_{g_0}(dw)$$

- Entropy term $\frac{\mathcal{D}^{\gamma}_{g_0}(M)}{\mathcal{G}^{\gamma}_{g_0}(M)}$ where $\mathcal{D}^{\gamma}_{g_0}(M)$ is a variant of GMC that we call **derivative GMC**

$$\mathcal{D}^{\gamma}_{g_0}(M) := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}(\gamma X_\epsilon(z)-\gamma^2\ln\epsilon)e^{\gamma X_\epsilon(z)}\,\mathrm{v}_{g_0}(dz)$$

Technical backbone

Establish negative exponential moments for the entropy

$$\forall \beta > 0, \quad \mathbb{E}\Big[\exp\Big(-\beta \frac{\mathcal{D}^{\gamma}_{g_0}(M)}{\mathcal{G}^{\gamma}_{g_0}(M)}\Big)\Big] < +\infty$$

Simple consequence of

- left **Gaussian concentration** for derivative GMC

$$\forall x > 0 \text{ large}, \quad \mathbb{P}(\mathcal{D}^{\gamma}_{g_0}(M) < -x) \leqslant C \exp\big(-C^{-1} x^2\big)$$

the field $X_{g_0} e^{\gamma X_{g_0}}$ is not bounded from below

- sharp **small deviation** result for GMC (for some κ)

$$\forall x > 0 \text{ small}, \quad \mathbb{P}(\mathcal{G}^{\gamma}_{g_0}(M) < x) \leqslant C \exp\big(-C^{-1} |\ln x|^{\kappa} x^{4/\gamma^2}\big)$$

compare with results by Duplantier, Nikulae, Sheffield (2008) or Garban, Holden, Sepúlveda, Xin Sun (on fractal sets 2018)

$$\mathbb{P}(\mathcal{G}^{\gamma}_{g_0}(M) > x) \sim C \exp(-C(\ln x)^2)$$

Why?

Thanks!