The Fyodorov-Bouchaud formula and Liouville conformal field theory

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Introduction

Two fields of physics:

- Log-correlated fields, Gaussian multiplicative chaos (GMC)
- Liouville conformal field theory (LCFT)

DKRV 2014: link between GMC and LCFT

Why is this link interesting ?

- GMC theory $\Rightarrow$ Rigorous definition of Liouville CFT
- CFT techniques $\Rightarrow$ Exact formulas on GMC
  DOZZ formula / Fyodorov-Bouchaud formula

Gaussian Free Field (GFF)

Gaussian free field X on the unit circle $\partial\mathbb{D}$

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}$$

- $X(e^{i\theta})$ has an infinite variance
- X lives in the space of distributions
- Cut-off approximation X_ϵ

  Ex: $X_\epsilon = \rho_\epsilon * X, \quad \rho_\epsilon = \frac{1}{\epsilon}\rho(\frac{\cdot}{\epsilon}),$ with smooth ρ.

Gaussian multiplicative chaos (GMC)

For $\gamma \in (0,2)$, define on $\partial \mathbb{D}$ the measure $e^{\frac{\gamma}{2}X} d\theta$

- Cut-off approximation $e^{\frac{\gamma}{2}X_\epsilon} d\theta$
- $\mathbb{E}[e^{\frac{\gamma}{2}X_\epsilon}] = e^{\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon^2]}$
- Renormalized measure: $e^{\frac{\gamma}{2}X_\epsilon - \frac{\gamma^2}{8}\mathbb{E}[X_\epsilon^2]} d\theta$

Proposition

The following limit holds in probability, for any continuous test function f, $\forall \gamma \in (0,2)$:

$$\int_0^{2\pi} e^{\frac{\gamma}{2}X(e^{i\theta})} f(\theta) d\theta = \lim_{\epsilon \to 0} \int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta}) - \frac{\gamma^2}{8}\mathbb{E}[X_\epsilon^2(e^{i\theta})]} f(\theta) d\theta$$

Moments of the GMC

We introduce:

$$\forall \gamma \in (0,2), \; Y_\gamma := \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{\gamma}{2} X(e^{i\theta})} d\theta$$

Existence of the moments of Y_γ:

$$\mathbb{E}[Y_\gamma^p] < +\infty \iff p < \frac{4}{\gamma^2}.$$

The Fyodorov-Bouchaud formula

Theorem (R. 2017)

Let $\gamma \in (0,2)$ and $p \in (-\infty, \frac{4}{\gamma^2})$, then:

$$\mathbb{E}[Y_\gamma^p] = \frac{\Gamma(1 - p\frac{\gamma^2}{4})}{\Gamma(1 - \frac{\gamma^2}{4})^p}$$

We also have a density for Y_γ,

$$f_{Y_\gamma}(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1} e^{-(\beta y)^{-\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y),$$

where $\beta = \Gamma(1 - \frac{\gamma^2}{4})$. Equivalently $Y_\gamma \overset{\text{law}}{=} \frac{1}{\beta}\text{Exp}(1)^{-\frac{\gamma^2}{4}}$.

Application 1: maximum of the GFF

Derivative martingale: work by Duplantier, Rhodes, Sheffield, Vargas.

$\gamma \to 2$ in our GMC measure (Aru, Powell, Sepúlveda):

$$Y' := \lim_{\gamma \to 2} \frac{1}{2-\gamma} Y_\gamma.$$

$\ln Y'$ has the following density:

$$f_{\ln Y'}(y) = e^{-y} e^{-e^{-y}}$$

$\ln Y' \sim \mathcal{G}$ where $\mathcal{G}$ follows a standard Gumbel law

Application 1: maximum of the GFF

Following an impressive series of works (2016):

Theorem (Ding, Madaule, Roy, Zeitouni)

For a reasonable cut-off X_ϵ of the GFF:

$$\max_{\theta \in [0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon \to 0}{\to} \mathcal{G} + \ln Y' + C$$

where $\mathcal{G}$ is a standard Gumbel law and $C \in \mathbb{R}$.

Application 1: maximum of the GFF

The Fyodorov-Bouchaud formula implies:

Corollary (R 2017)

For a reasonable cut-off X_ϵ of the GFF:

$$\max_{\theta \in [0,2\pi]} X_\epsilon(e^{i\theta}) - 2 \ln \frac{1}{\epsilon} + \frac{3}{2} \ln \ln \frac{1}{\epsilon} \underset{\epsilon \to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C$$

where $\mathcal{G}_1, \mathcal{G}_2$ are independent Gumbel laws and $C \in \mathbb{R}$.

Application 2: random unitary matrices

$U_N := N \times N$ random unitary matrix

Its eigenvalues $(e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the distribution:

$$\frac{1}{n!} \prod_{k<j} |e^{i\theta_k} - e^{i\theta_j}|^2 \prod_{k=1}^{n} \frac{d\theta_k}{2\pi}$$

Let $p_N(\theta) = \det(1 - e^{-i\theta} U_N) = \prod_{k=1}^{N} (1 - e^{i(\theta_k - \theta)})$

Webb (2015): $\forall \alpha \in (-\frac{1}{2}, \sqrt{2})$,

$$\frac{|p_N(\theta)|^\alpha}{\mathbb{E}[|p_N(\theta)|^\alpha]} d\theta \underset{N\to\infty}{\to} e^{\frac{\alpha}{2} X(e^{i\theta})} d\theta$$

Application 2: random unitary matrices

Conjecture by Fyodorov, Hiary, Keating (2012):

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\rightarrow} \mathcal{G}_1 + \mathcal{G}_2 + C.$$

Chhaibi, Madaule, Najnudel (2016), tightness of:

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N.$$

With our result it is sufficient to show:

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\rightarrow} \mathcal{G}_1 + \ln Y' + C.$$

Integer moments of the GMC

The computation of Fyodorov and Bouchaud

Fyodorov Y.V., Bouchaud J.P.: Freezing and extreme value statistics in a Random Energy Model with logarithmically correlated potential, *Journal of Physics A: Mathematical and Theoretical*, Volume 41, Number 37, (2008).

Integer moments of the GMC

For $n \in \mathbb{N}^*$, $n < \frac{4}{\gamma^2}$:

$$\mathbb{E}[(\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta})^2]}d\theta)^n]$$

$$= \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} \mathbb{E}[\prod_{i=1}^n e^{\frac{\gamma}{2}X_\epsilon(e^{i\theta_i})-\frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(e^{i\theta_i})^2]}]d\theta_1 \dots d\theta_n$$

$$= \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} e^{\frac{\gamma^2}{4}\sum_{i<j}\mathbb{E}[X_\epsilon(e^{i\theta_i})X_\epsilon(e^{i\theta_j})]}d\theta_1 \dots d\theta_n$$

Integer moments of the GMC

For $n \in \mathbb{N}^*$, $n < \frac{4}{\gamma^2}$:

$$\mathbb{E}[Y_\gamma^n] = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} e^{\frac{\gamma^2}{4} \sum_{i<j} \mathbb{E}[X(e^{i\theta_i})X(e^{i\theta_j})]} d\theta_1 \dots d\theta_n$$

$$= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \prod_{i<j} \frac{1}{|e^{i\theta_i} - e^{i\theta_j}|^{\frac{\gamma^2}{2}}} d\theta_1 \dots d\theta_n$$

$$= \frac{\Gamma(1 - n\frac{\gamma^2}{4})}{\Gamma(1 - \frac{\gamma^2}{4})^n}$$

- Question: can we replace $n \in \mathbb{N}^*$ by a real $p < \frac{4}{\gamma^2}$?

Proof of the Fyodorov-Bouchaud formula

Framework of conformal field theory

Belavin A.A., Polyakov A.M., Zamolodchikov A.B.: Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear. Physics.*, B241, 333-380, (1984).

The BPZ differential equation

We introduce the following observable for $t \in [0,1]$:

$$G(\gamma, p, t) = \mathbb{E}[(\int_0^{2\pi} |t - e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2} X(e^{i\theta})} d\theta)^p]$$

BPZ equation:

$$(t(1-t^2)\frac{\partial^2}{\partial t^2} + (t^2-1)\frac{\partial}{\partial t} + 2(C-(A+B+1)t^2)\frac{\partial}{\partial t} - 4ABt)G(\gamma, p, t) = 0$$

where:

$$A = -\frac{\gamma^2 p}{4}, \; B = -\frac{\gamma^2}{4}, \; C = \frac{\gamma^2}{4}(1-p) + 1.$$

Solutions of the BPZ equation

BPZ equation in t $\rightarrow$ hypergeometric equation in t^2

Two bases of solutions:

- $G(\gamma, p, t) = C_1 F_1(t^2) + C_2 t^{\frac{\gamma^2}{2}(p-1)} F_2(t^2)$
- $G(\gamma, p, t) = B_1 \tilde{F}_1(1-t^2) + B_2 (1-t^2)^{1+\frac{\gamma^2}{2}} \tilde{F}_2(1-t^2)$

where:

- $C_1, C_2, B_1, B_2 \in \mathbb{R}$
- $F_1, F_2, \tilde{F}_1, \tilde{F}_2 :=$ hypergeometric series depending on γ and p.

Change of basis: $(C_1, C_2) \leftrightarrow (B_1, B_2)$.

The shift relation

By direct asymptotic expansion:

- $C_1 = (2\pi)^p \mathbb{E}[Y_\gamma^p]$
- $C_2 = 0$
- $B_1 = \mathbb{E}[(\int_0^{2\pi} |1 - e^{i\theta}|^{\frac{\gamma^2}{2}} e^{\frac{\gamma}{2} X(\theta)} d\theta)^p]$
- $B_2 = (2\pi)^p p \frac{\Gamma(-\frac{\gamma^2}{2} - 1)}{\Gamma(-\frac{\gamma^2}{4})} \mathbb{E}[Y_\gamma^{p-1}]$

The change of basis implies:

$$\mathbb{E}[Y_\gamma^p] = \frac{\Gamma(1 - p\frac{\gamma^2}{4})}{\Gamma(1 - \frac{\gamma^2}{4})\Gamma(1 - (p-1)\frac{\gamma^2}{4})} \mathbb{E}[Y_\gamma^{p-1}].$$

Negative moments of GMC

The shift relation gives all the negative moments:

$$\mathbb{E}[Y_\gamma^{-n}] = \Gamma(1 + \frac{n\gamma^2}{4})\Gamma(1 - \frac{\gamma^2}{4})^n, \quad \forall n \in \mathbb{N}.$$

We check:

$$\forall \lambda \in \mathbb{R}, \ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\Gamma(1 + \frac{n\gamma^2}{4})\Gamma(1 - \frac{\gamma^2}{4})^n < +\infty$$

Negative moments $\Rightarrow$ determine the law of Y_γ !

Explicit probability densities

Probability densities for Y_γ^{-1} and Y_γ

$$f_{\frac{1}{Y_\gamma}}(y) = \frac{4}{\beta\gamma^2}(\frac{y}{\beta})^{\frac{4}{\gamma^2}-1} e^{-(\frac{y}{\beta})^{\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y)$$

$$f_{Y_\gamma}(y) = \frac{4\beta}{\gamma^2}(\beta y)^{-\frac{4}{\gamma^2}-1} e^{-(\beta y)^{-\frac{4}{\gamma^2}}} \mathbf{1}_{[0,\infty[}(y)$$

where $\gamma \in (0,2)$ and $\beta = \Gamma(1 - \frac{\gamma^2}{4})$.

What is Liouville field theory?

Path integral formalism

$\Sigma = \{X : \mathbb{D} \to \mathbb{R}\}$

For $X \in \Sigma$, energy of $X := \frac{1}{4\pi} \int_{\mathbb{D}} |\partial X|^2 dx^2 + \int_{\partial \mathbb{D}} e^{\frac{\gamma}{2} X} ds$

Random field ϕ_L:

$$\mathbb{E}[F(\phi_L)] = \int_{\Sigma} F(X) e^{-\frac{1}{4\pi} \int_{\mathbb{D}} |\partial X|^2 dx^2 - \int_{\partial \mathbb{D}} e^{\frac{\gamma}{2} X} ds} DX$$

with $\gamma \in (0, 2)$.

$\Rightarrow \phi_L$ is the Liouville field

Correlations of Liouville theory

Correlation function of $z_i \in \mathbb{D}$, $\alpha_i \in \mathbb{R}$:

$$\langle \prod_{i=1}^{N} e^{\alpha_i \phi_L(z_i)} \rangle_{\mathbb{D}} = \int_{X:\mathbb{D}\mapsto\mathbb{R}} DX \prod_{i=1}^{N} e^{\alpha_i X(z_i)} e^{-\frac{1}{4\pi}\int_{\mathbb{D}} |\partial X|^2 dx^2 - \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X} ds}$$

Expressed in terms of Gaussian multiplicative chaos

- $\langle e^{\alpha \phi_L(0)} \rangle_{\mathbb{D}} = \tilde{C}_1 \mathbb{E}[Y_\gamma^{p-1}]$
- $\langle e^{\alpha \phi_L(0)} e^{-\frac{\gamma}{2}\phi_L(t)} \rangle_{\mathbb{D}} = \tilde{C}_2 t^{\frac{\alpha\gamma}{2}} (1-t^2)^{-\frac{\gamma^2}{8}} G(\gamma, p, t)$

  with $p = 2 - 2\alpha - \frac{4}{\gamma^2}$.

Correlations of Liouville theory

Liouville theory is a conformal field theory

- Degenerate fields: $e^{-\frac{\gamma}{2}\phi_L(z)}$ and $e^{-\frac{2}{\gamma}\phi_L(z)}$.
- BPZ equation, for $z_1, z \in \mathbb{D}$, $\alpha \in \mathbb{R}$, $\gamma \in (0,2)$:

$$(z_1, z) \mapsto \langle e^{\alpha \phi_L(z_1)} e^{-\frac{\gamma}{2}\phi_L(z)} \rangle_{\mathbb{D}}$$

  is solution of a differential equation.
- Use conformal map ψ, $\psi(z_1) = 0$, $\psi(z) = t$.

BPZ equation on the upper half plane $\mathbb{H}$

Proposition (R. 2017)

Let $\gamma \in (0,2)$ and $\alpha > Q + \frac{\gamma}{2}$. Then:

$$\Big(\frac{4}{\gamma^2}\partial_{zz} + \frac{\Delta_{-\frac{\gamma}{2}}}{(z-\overline{z})^2} + \frac{\Delta_\alpha}{(z-z_1)^2} + \frac{\Delta_\alpha}{(z-\overline{z_1})^2} + \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \frac{1}{z-z_1}\partial_{z_1} + \frac{1}{z-\overline{z_1}}\partial_{\overline{z_1}}\Big)\langle e^{-\frac{\gamma}{2}\phi_L(z)} e^{\alpha\phi_L(z_1)}\rangle_{\mathbb{H}} = 0$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$, $\Delta_{-\frac{\gamma}{2}} = -\frac{\gamma}{4}(Q + \frac{\gamma}{4})$.

$\Rightarrow$ differential equation for $G(p,\gamma,t)$.

Analogue for the unit interval [0, 1]

Log-correlated field X on $[0,1]$: $\mathbb{E}[X(x)X(y)] = -2\ln|x-y|$

For $\gamma \in (0,2)$, and suitable a, b and p, define:

$$M(\gamma, p, a, b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p].$$

Theorem (R., Zhu 2018)

$M(\gamma, p, a, b)$ has the following expression,

$$\frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})},$$

$$\ln\Gamma_\gamma(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2}-x)^2}{2}e^{-t} + \frac{x-\frac{Q}{2}}{t}\right], \quad Q = \frac{\gamma}{2}+\frac{2}{\gamma}.$$

Analogue for the unit interval $[0,1]$

Log-correlated field X on $[0,1]$: $\mathbb{E}[X(x)X(y)] = -2\ln|x-y|$

Equivalent statement:

$$\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx \overset{\text{law}}{=} c e^{\mathcal{N}(0,\gamma^2\ln 2)} Y_\gamma X_1^{-1} X_2^{-1} X_3^{-1}$$

- $c :=$ deterministic constant
- $Y_\gamma \overset{\text{law}}{=} \frac{1}{\beta}\text{Exp}(1)^{-\frac{\gamma^2}{4}}$
- $X_i :=$ generalized beta law

Analogue for the bulk meaure on $\mathbb{D}$

Work in progress, for $\gamma \in (0,2)$, $\alpha \in (\frac{\gamma}{2}, Q)$:

$$\mathbb{E}[(\int_{\mathbb{D}} \frac{1}{|x|^{\gamma\alpha}} e^{\gamma X(x)} dx^2)^{\frac{Q-\alpha}{\gamma}}] =$$

$$\gamma^2 (\pi \frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})})^{\frac{Q-\alpha}{\gamma}} \cos(\frac{\alpha - Q}{\gamma}\pi) \frac{\Gamma(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2})\Gamma(\frac{\gamma}{2}(\alpha - Q))}{\Gamma(\frac{\alpha-Q}{\gamma})}$$

Liouville theory with action: $\int_{\mathbb{D}} (|\partial X|^2 + e^{\gamma X}) dx^2$

Outlook and perspectives

Integrability program for GMC and Liouville theory

- More general Liouville correlations on $\mathbb{D}$
- Work in progress to recover the law of the quantum disk of the Duplantier-Miller-Sheffield approach to Liouville Quantum Gravity
- Other geometries, higher genus
- Conformal bootstrap