Characterising the Gaussian free field

Random conformal geometry (Seoul)

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Joint work with Nathanaël Berestycki and Gourab Ray

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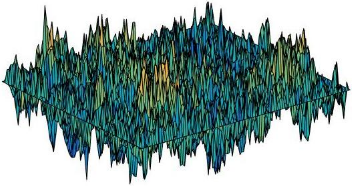
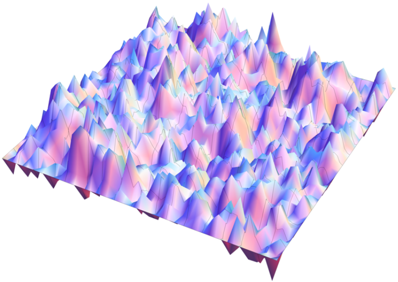


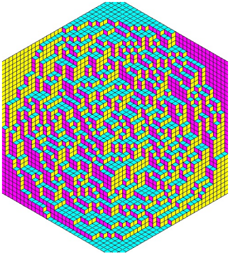
The Gaussian Free Field



@Scott Sheffield

Conjectured/proven scaling limit of many discrete models

The Gaussian Free Field



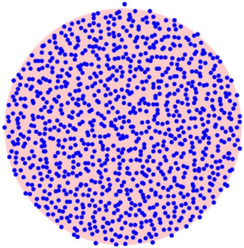
@Rick Kenyon

Conjectured/proven scaling limit of many discrete models

Height function of the dimer model

(Kenyon, Berestycki-Laslier-Ray, many others...)

The Gaussian Free Field

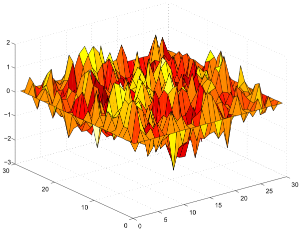


Conjectured/proven scaling limit of many discrete models

Log characteristic polynomial of random matrices (Ginibre Ensemble)

(Rider-Virag...)

The Gaussian Free Field



@Jason Miller

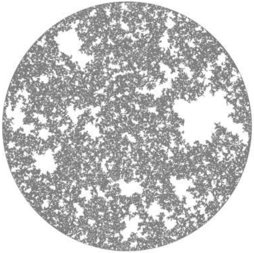
Conjectured/proven scaling limit of many discrete models

Ginsburg-Landau $\nabla\varphi$ interface model

$$\mathbb{P}(dh) \propto \exp(-\sum_{x\sim y} V(h_x - h_y)) \prod_x dh_x$$

(Giacomin-Olla-Spohn, Miller)

The Gaussian Free Field



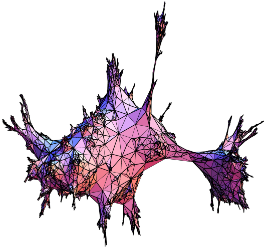
@David Wilson

Conjectured/proven scaling limit of many discrete models

Nesting field of CLE_4

(Miller-Sheffield)

The Gaussian Free Field



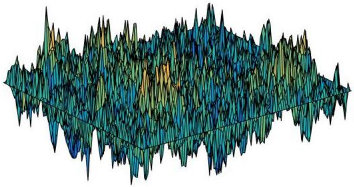
@Nicolas Curien

Conjectured/proven scaling limit of many discrete models

"Exponential of GFF" = Liouville quantum gravity = Metric space limit of random planar maps

(Le Gall, Miermont, Miller-Sheffield...)

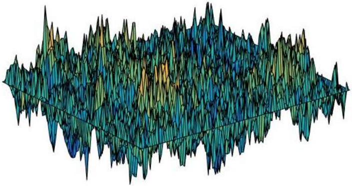
The Gaussian Free Field



Definition (zero-boundary condition GFF on $D \subset \mathbb{C}$)

- Gaussian "function" on D.
- Mean zero and covariance given by the Dirichlet Green function $G_D : D \times D \to \mathbb{R}$ on D.

The Gaussian Free Field



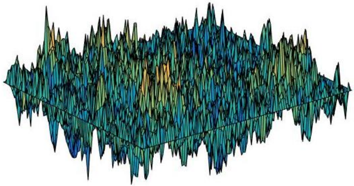
Definition (zero-boundary condition GFF on $D \subset \mathbb{C}$)

- Gaussian generalised function h^D on D.
- Mean zero and covariance given by

$$\mathbb{E}[(h^D, f)(h^D, g)] = \iint_{D \times D} f(x) G_D(x, y) g(y) \, dxdy$$

for f, g test functions.

The Gaussian Free Field



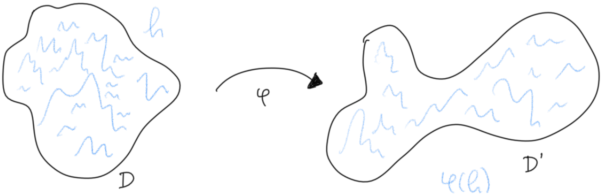
Definition (zero-boundary condition GFF on $D \subset \mathbb{C}$)

- Random Gaussian generalised function h^D on D.
- Mean zero, $\mathbb{E}[(h^D, f)(h^D, g)] = \iint f(x) G_D(x, y) g(y)$

G_D is the unique function on $D \times D$ such that $G_D(\cdot, y)$ is harmonic on $D \setminus \{y\}$ and blows up logarithmically on the diagonal.

Key properties

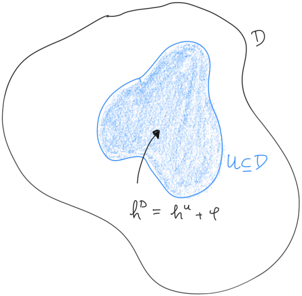
Conformal invariance



h a GFF on D, $\varphi : D \to D'$ conformal $\Rightarrow \varphi(h)$ a GFF on D'

Key properties

Domain Markov Property



$U \subset D$ open: $h^D = h^U + \varphi$ where h^U is a GFF on U, φ is independent of h^U and is harmonic when restricted to U.

Axiomatic characterisation

Theorem (Main result; Berestycki-P.-Ray)

Domain Markov property & Conformal invariance $\Rightarrow h = GFF$.

More precisely:

Setup

We assume for every D, we are given the law of h^D a linear stochastic process indexed by $f \in C_c^\infty(D)$.

We assume the following:

Assumptions

1. Moments

$\mathbb{E}(h^D, f)^4 < \infty$ for any $f \in C_c^\infty(D)$. Furthermore,

$$(f, g) \mapsto \mathbb{E}((h^D, f)(h^D, g)) =: K_D(f, g)$$

is continuous on $C_c^\infty(D) \times C_c^\infty(D)$

2. Dirichlet boundary condition

$\mathbb{E}(h^D, f) = 0$. Moreover, for any $(f_n)_{n\geq 1} \in C_c^\infty(D)$ radially symmetric with bounded mass, and with support$(f_n) \cap K = \emptyset$ eventually for any $K \subset D$,

$$\mathrm{Var}(h^D, f_n) \to 0$$

Assumptions

3. Conformal invariance

if $f : D \to D'$, then $h^{D'}$ has the law of $h^D \circ f^{-1}$.

4. Domain Markov property

If $U \subset D$ simply connected, we can write

$$h^D = h_D^U + \varphi_D^U,$$

where

- h_D^U has the given law in U, and zero outside;
- φ_D^U is harmonic in U;
- h_D^U, φ_D^U are independent.

Result and Remarks

Theorem: (1)-(4) Given family of laws corresponds to a multiple of the zero-boundary condition Gaussian free field.

Remarks

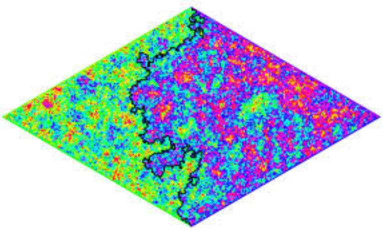
$d = 1$:

The theorem also works in 1d, giving us a new characterisation of Brownian bridge.

Role of assumptions

2, 3 and 4 seem indispensable. Moments assumptions, however...? Technically this is needed to extract some continuity.

Schramm's characterisation of SLE



- (Schramm)The only random curves satisfying conformal invariance and domain Markov property are SLE_κ for some value of κ
- (Schramm-Sheffield) $\mathrm{SLE}_4 =$ "level line" of the GFF

Open problems

- Moments?
- In $d \geq 3$ we have no analogous characterisation?
- What about free boundary / Neumann boundary conditions?
- What about Riemann surfaces? (see Berestycki-Laslier-Ray for dimer model, where limit known to exist and conformally invariant).
- What about a "stable" free field?

Thanks for listening!