Connectivity properties of the adjacency graph of SLE_κ bubbles for $\kappa \in (4,8)$

Random Conformal Geometry and Related Fields, KIAS, Seoul

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Introduction

Consider a chordal SLE_κ curve η from 0 to ∞ in $\mathbb{H}$ for $\kappa \in (4, 8)$.
A *bubble* of η is a connected component of $\mathbb{H} \backslash \eta$.

Introduction

Today's talk is about the following question, originally posed by Duplantier, Miller and Sheffield (2014):

Question

Is the adjacency graph of bubbles connected? I.e., is there a finite path in the adjacency graph between any pair of bubbles?

The analogous question for Brownian motion is a well-known open problem.

Introduction

Is the adjacency graph of bubbles even connected for *any* κ? The answer is not obvious.

- The set of points on the curve which do not lie on the boundary of any bubble has full Hausdorff dimension.
- There could exist pairs of macroscopic bubbles separated by an infinite "cloud" of small bubbles.

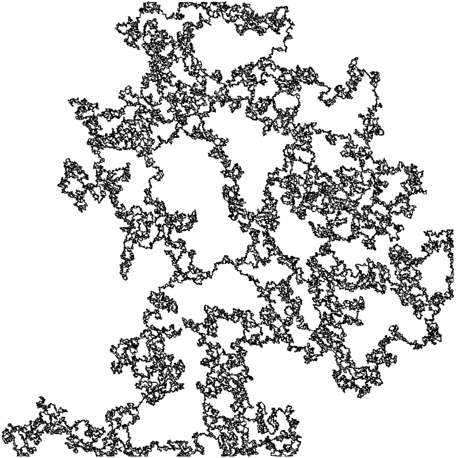


Figure: An SLE_6 in a square domain. Simulation by Jason Miller.

Main Result

Theorem (Gwynne, P. (2018))

For each fixed $\kappa \in (4, \kappa_0]$*, the adjacency graph of bubbles of a chordal* SLE_κ *curve is almost surely connected, where* $\kappa_0 \approx 5.6158$ *is the unique solution of the equation* $\pi \cot(\pi\kappa/4) + \psi(2 - \kappa/4) - \psi(1) = 0$ *on the interval* $(4, 8)$*.*

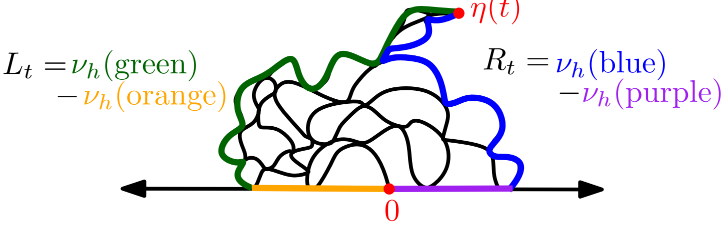
Here, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denotes the digamma function.

Corollary (Gwynne, P. (2018))

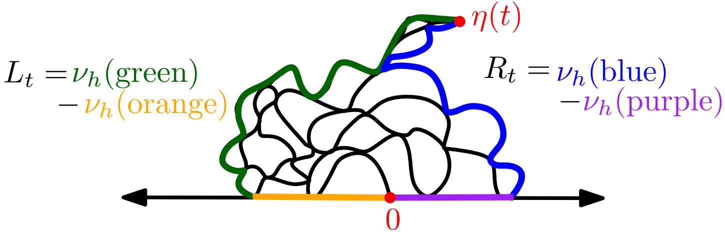
For $\kappa \in (4, \kappa_0]$*, the set of points on a chordal* SLE_κ *curve which do not lie on the boundary of any bubble is almost surely totally disconnected.*

Our Approach

- By theory of LQG, the left and right boundaries of an SLE_κ curve (with a particular parametrization) are a pair (L, R) of independent $\kappa/4$-stable Levy processes.



Our Approach



- We use (L, R) to define a stronger "Markovian" connectivity condition for the graph of bubbles.
- We will show that this condition holds for $\kappa \in (4, \kappa_0]$ and fails for κ sufficiently close to 8. (Reminder: $\kappa_0 \approx 5.6158$ is the unique solution of the equation $\pi \cot(\pi\kappa/4) + \psi(2 - \kappa/4) - \psi(1) = 0$ on the interval $(4, 8)$.)

Review of LQG

To define (L, R) precisely, we need some definitions from LQG.

Review of LQG

- $D \subset \mathbb{C}$ open
- h a GFF-type distribution on D

"Definition"

The γ-**LQG surface associated with** h is the random Riemannian surface with Riemann metric tensor $e^{\gamma h(z)}(dx^2 + dy^2)$, where $dx^2 + dy^2$ is the Euclidean metric tensor.

- This definition does not make literal sense since h is a distribution, not a pointwise-defined function.

Review of LQG

- However, certain objects associated with γ-LQG surfaces can be defined rigorously using regularization procedures:
  - a γ-LQG area measure on D (defined as limit of regularized versions of $e^{\gamma h(z)}\,dz$)
  - a γ-**LQG length measure** on certain curves in $\overline{D}$, such as ∂D and SLE_κ-type curves for $\kappa = \gamma^2$ (or, equivalently, the outer boundaries of $\mathrm{SLE}_{16/\kappa}$ curves by SLE duality)

Review of LQG

- We parametrize our SLE_κ curve by quantum natural time.
- Roughly speaking, this is the same as parametrizing by "quantum Minkowski content"
- It is the quantum analogue of the so-called natural parametrization of SLE.

Definition of (L, R)

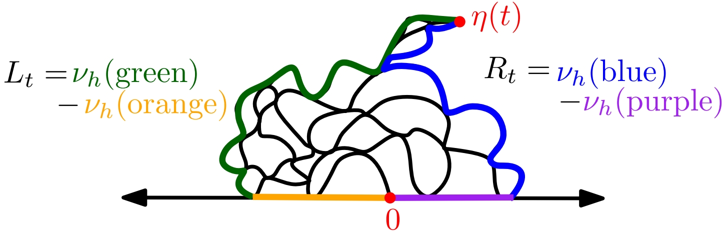
Sample an SLE_κ curve independently on a $\frac{4}{\gamma} - \frac{\gamma}{2}$ - quantum wedge with $\gamma = 4/\sqrt{\kappa}$.

To construct the $\frac{4}{\gamma} - \frac{\gamma}{2}$ - quantum wedge:

- Take the distribution $\tilde{h} - \left(\frac{4}{\gamma} - \frac{\gamma}{2}\right) \log|\cdot|$ on $\mathbb{H}$, where $\tilde{h}$ a free boundary GFF on $\mathbb{H}$.
- "Zoom in near the origin"
- Rescale so the γ-LQG mass of $\mathbb{D} \cap \mathbb{H}$ remains of constant order.

Definition of (L, R)

Sample an SLE_κ curve independently on a $\frac{4}{\gamma} - \frac{\gamma}{2}$ - quantum wedge with $\gamma = 4/\sqrt{\kappa}$.



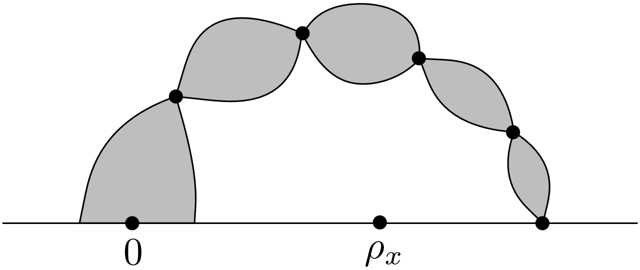
Theorem (Duplantier, Miller and Sheffield (2014))

The processes L_t and R_t are i.i.d. totally asymmetric $\frac{\kappa}{4}$-stable Levy processes with only negative jumps.

SLE-Levy process dictionary

We can use (L, R) to describe some geometric features of an SLE_κ curve.

- Points on boundary of bubble
- Points of the boundary of two different bubbles
  $\Leftrightarrow$ (local) cut points $\Leftrightarrow$ **edges of adjacency graph of bubbles**



Proof of our main result

1. We define an "(L, R)-Markovian path to infinity"
   - If it exists, the graph must be connected.
2. We reduce the task of proving existence of this path to an estimate for a single bubble.
3. We outline the proof of the estimate for a single bubble.

After the proof, we will remark on our converse result for large κ and discuss open problems.

Table of Contents

Step 1: (L, R)-Markovian path to infinity

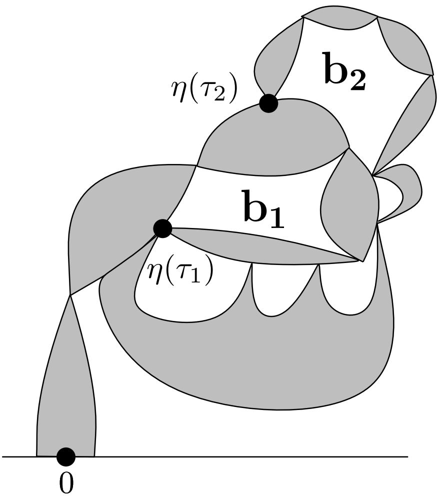
Definition

For $\kappa \in (4, 8)$, an (L, R)-*Markovian path to infinity* in the adjacency graph of bubbles of η is an infinite increasing sequence of stopping times $\tau_1 < \tau_2 < \tau_3 < \cdots$ for (L, R) such that almost surely

- $\tau_k \to \infty$,
- η forms a bubble b_k at each time τ_k (equivalently, either L or R has a downward jump at time τ_k), and
- b_k and b_{k+1} are connected in the adjacency graph (i.e., $\partial b_k \cap \partial b_{k+1} \neq \emptyset$) for each k.

Note: this is a *random* path defined for almost every realization of the SLE_κ curve.

Step 1: (L, R)-Markovian path to infinity



There is an (L, R)-Markovian path to infinity started at every stopping time at which η forms a bubble almost surely.

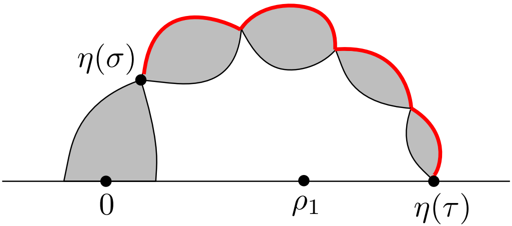
$\Downarrow$

Adjacency graph of bubbles is connected almost surely.

Table of Contents

Step 2: Reducing to a single bubble

$$\mathbb{E} \log(L_\tau - L_\sigma) \geq 0 \quad \Rightarrow$$



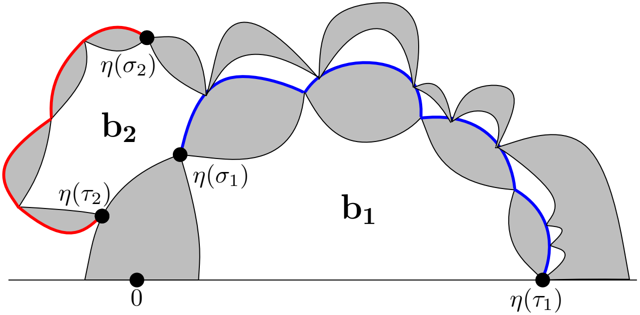
There is an (L, R)-Markovian path to infinity started at every stopping time at which η forms a bubble almost surely.

$$\Downarrow$$

Adjacency graph of bubbles is connected almost surely.

Step 2: Reducing to a single bubble

Using $\mathbb{E}\log(L_\tau - L_\sigma) \geq 0$ to construct (L, R) Markovian path to infinity:



$X_1 = L_{\tau_1} - L_{\sigma_1}$

$X_2 = R_{\tau_2} - R_{\sigma_2}$

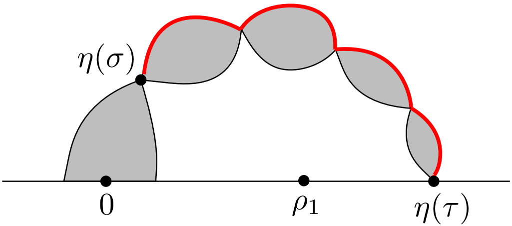
etc.

$$X_{k+1}/X_k \overset{i.i.d.}{\equiv} L_\tau - L_\sigma$$

Step 2: Reducing to a single bubble

$\mathbb{E}\log(L_\tau - L_\sigma) \geq 0 \quad \Rightarrow$

There is an (L, R)-Markovian path to infinity started at every stopping time at which η forms a bubble almost surely.



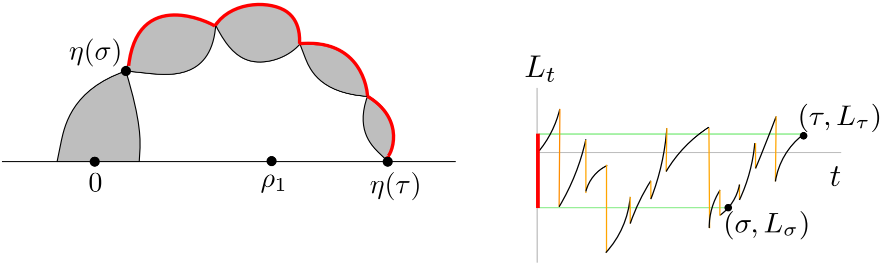
$\Downarrow$

Adjacency graph of bubbles is connected almost surely.

Table of Contents

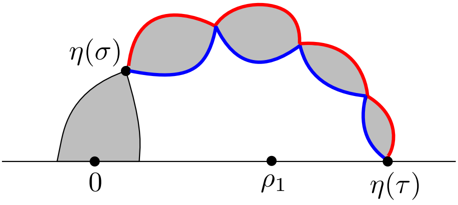
Step 3: Estimate for a single bubble

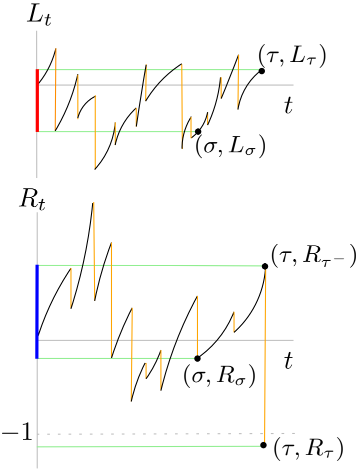
We have reduced the theorem to proving $\mathbb{E} \log(L_\tau - L_\sigma) \geq 0$.



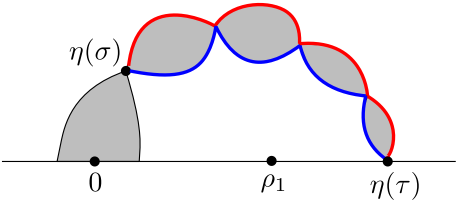
Why is this estimate tricky?
The laws of τ and σ are not known explicitly!

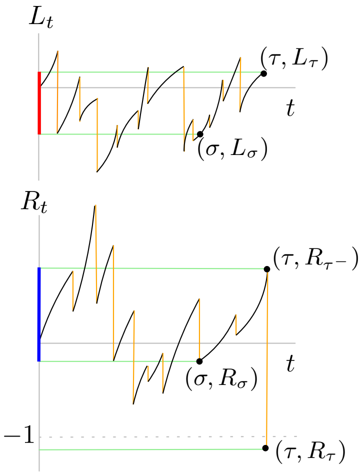
Step 3: Estimate for a single bubble





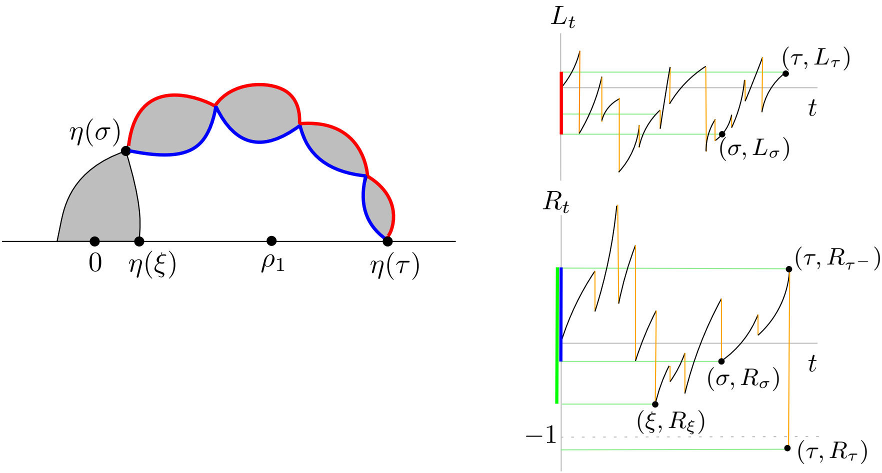
Step 3: Estimate for a single bubble





$L_\tau - L_\sigma$ stochastically dominates $R_{\tau^-} - R_\sigma$.

Step 3: Estimate for a single bubble



Step 3: Estimate for a single bubble

(L, R) run backwards from τ to ξ conditional on $\{R_{\tau^-} - R_\xi = r\}$	$\overset{\mathcal{L}}{=}$	(L, R) run backward until R hits $-r$

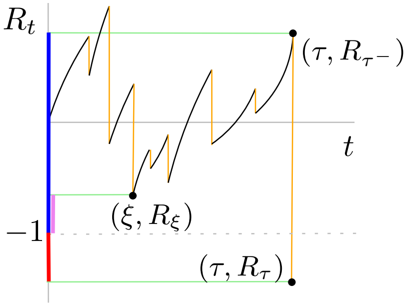
- For fixed r, we know the law of the last simultaneous running infimum of (L, R) run backward until R hits $-r$.
  - The set of simultaneous running infima is a subordinator with known index (by result of Brownian motions), so can use arcsine law for subordinators.
- Now the only thing we need to know is the law of $R_{\tau^-} - R_\xi$.
- Fortunately, this *is* known!

Step 3: Estimate for a single bubble

Theorem (Doney and Kyprianou (2006))

$$\mathbb{P}(-1 - R_\tau \in du, R_{\tau^-} + 1 \in dv, R_\xi + 1 \in dy)$$
$$= \frac{\kappa}{4}\left(1 - \frac{\kappa}{4}\right)\frac{\sin(\pi\kappa/4)}{\pi}\frac{(1-y)^{\kappa/4-2}}{(v+u)^{\kappa/4+1}}\, du\, dv\, dy$$

for $u > 0$, $y \in [0,1]$, *and* $v \geq y$.



Converse for large κ

- We can prove that, for large κ, there does not exist an (L, R)-Markovian path to infinity.
- The proof is similar to the small κ case; this time we wish to show the expected log of some random variable is negative.
- Our proof is based on the fact that a $\kappa/4$-stable process converges in law to Brownian motion as κ increases to 8.
  - We do not get an explicit range of large κ for which there is no (L, R)-Markovian path to infinity.
- Our result does *not* imply that the graph of bubbles of SLE_κ is not connected a.s. for large κ.

Open Problems

- Three versions of connectivity for the graph of bubbles:
  1. The graph is a.s. connected
  2. There a.s. exists a path to infinity from any fixed bubble.
  3. There exists an (L, R)-Markovian path started at any stopping time ζ when bubble is formed a.s.
- We have $3 \Rightarrow 2 \Rightarrow 1$.
- We show 3 holds for explicit small range of κ, and 3 fails for κ sufficiently large.
- There should be a phase transition for each of these properties. Do these phase transitions coincide?

Thank you for your attention!