Multiple SLEs, pure partition functions, and connection probabilities

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Based on joint works with

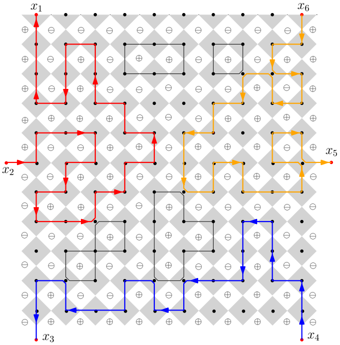
Vincent Beffara, **Alex Karrila**, **Kalle Kytölä**, and **Hao Wu**

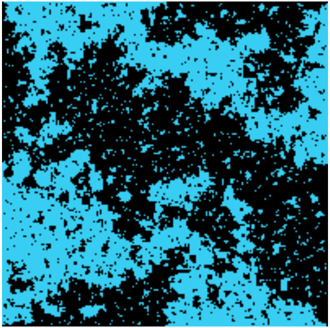
Random Conformal Geometry and Related Fields, KIAS

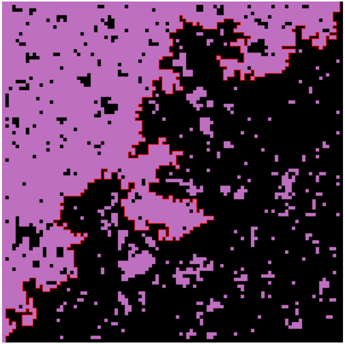
Plan

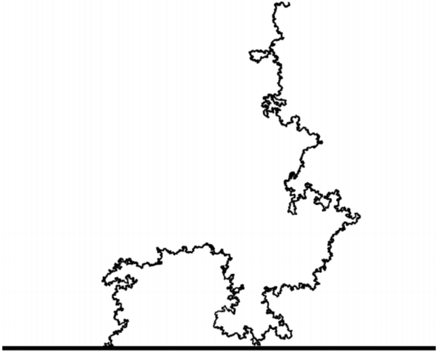
1. Motivation
    - critical models in statistical physics, scaling limits
    - conformal invariance of interfaces & correlations
2. Multiple SLEs: classification
    - global multiple SLEs
    - local multiple SLEs (i.e., commuting SLEs)
    - partition functions of multiple SLEs
    - local $\Leftrightarrow$ global ???
3. Relation to connection probabilities
    - multichordal loop-erased random walks
    - level lines of the Gaussian free field
    - double-dimer pairings
    - Ising model crossing probabilities
4. More on partition functions?

Motivation









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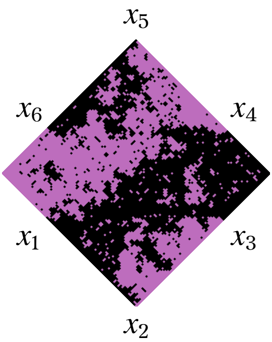
interface of Ising model $\xrightarrow{\delta \to 0}$ Schramm-Loewner evolution, SLE_3

[Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov (2014)]



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider the critical Ising model in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating b.c.
- **Izyurov (2015):** interfaces $\xrightarrow{\delta\to 0}$ (local) multiple SLE_3
  Proof: multi-point holomorphic observable
- If we condition on the event that the interfaces connect the boundary points according to a given connectivity ...

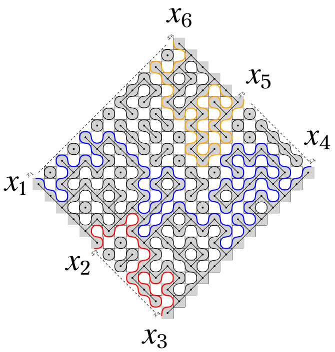
... then we have:

Theorem

The law of the N macroscopic interfaces of the critical Ising model **converges in the scaling limit $\delta \to 0$ to the N-SLE_κ with $\kappa = 3$.**

Wu **[arXiv:1703.02022]**
Beffara, P. & Wu **[arXiv:1801.07699]**

Proof: convergence for $N = 1$ and classification of global multiple SLE_κ



- fix discrete domain data $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$
- consider the critical FK-Ising model in $\Omega^\delta \subset \delta\mathbb{Z}^2$ with alternating b.c.
- **Kemppainen & Smirnov (2018):** 2 interfaces $\xrightarrow{\delta\to0}$ $\mathrm{hSLE}_{16/3}$

  Proof: holomorphic observable
- If we condition on the event that the interfaces connect the boundary points according to a given connectivity ...

... then we have:

Theorem

The law of the N macroscopic interfaces of the critical FK-Ising model **converges in the scaling limit $\delta \to 0$ to the N-SLE$_\kappa$ with $\kappa = 16/3$.**

Beffara, P. & Wu **[arXiv:1801.07699]**

Proof: convergence for $N = 1$ and classification of global multiple SLE$_\kappa$

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✓

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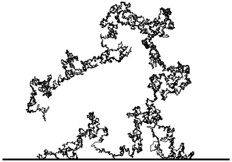
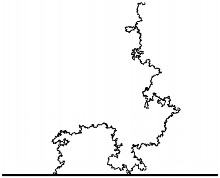
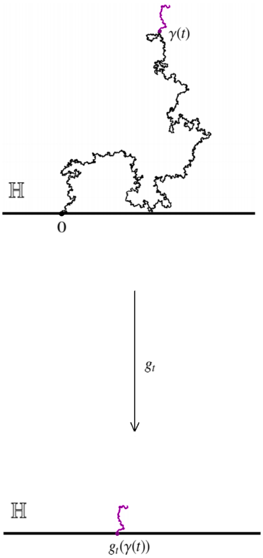
 

SCHRAMM-LOEWNER

EVOLUTIONS

SLE_κ



Theorem [Schramm ~2000]

$\exists!$ one-parameter family $(\mathrm{SLE}_\kappa)_{\kappa\geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**

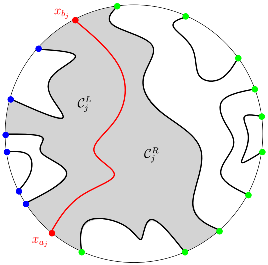
- encode SLE$_\kappa$ random curves in **random conformal maps** $(g_t)_{t\geq 0}$
- driving process = **image of the tip**:

$$X_t := \lim_{z\to\gamma(t)} g_t(z) = \sqrt{\kappa} B_t$$

- $g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ solutions to **Loewner equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z$$

- family of **random curves** in $(\Omega; x_1, \ldots, x_{2N})$
- various connectivities encoded in **planar pair partitions** $\alpha \in \mathrm{LP}_N$



Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists a unique probability measure on N curves** such that

conditionally on $N-1$ of the curves, the remaining one is the chordal SLE$_\kappa$ *in the random domain where it can live.*

Kozdron & Lawler (2007); Lawler (2009); Miller & Sheffield (2016);

P. & Wu **[arXiv:1703.00898]** and Beffara, P. & Wu **[arXiv:1801.07699]**

Uniqueness: $\exists$ at most one global N-SLE$_\kappa$

Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points,

there exists at most one probability measure on N curves

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conditionally on $N-1$ of the curves, the remaining one is the chordal SLE_κ *in the random domain where it can live.*

Idea of proof:

- sample curves according to conditional law
  $\Rightarrow$ **Markov chain** on space of curves
- prove that there is a coupling of two such Markov chains, started from *any* two initial configurations, such that they have a *uniformly positive chance to agree after a few steps*
  $\Rightarrow$ **there is *at most one* stationary measure**

Uniqueness: $\exists$ at most one global N-SLE$_\kappa$

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Remarks:

- the case of $N = 2$ was proved in
  Miller & Sheffield's Imaginary geometry II
  using the coupling of SLE with the Gaussian free field (GFF)
- BUT that proof does not generalize to $N \geq 3$ curves
  since those *cannot* be coupled with the GFF
- **our proof does not use the GFF in any way**

Construction of global N-SLE_κ

Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points,

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conditionally on $N-1$ of the curves, the remaining one is the chordal SLE_κ in the random domain where it can live.

How to find them?

Construction of global N-SLE$_\kappa$

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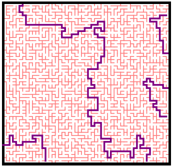
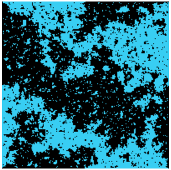
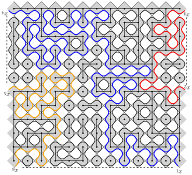
conditionally on $N-1$ of the curves, the remaining one is the chordal SLE$_\kappa$ *in the random domain where it can live.*

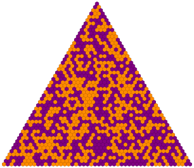
1. From scaling limits of **multiple interfaces** in critical models: use *convergence of one curve & RSW* (works for $\kappa \in \{2, 3, 4, 16/3, 6\}$)

[Schramm & Sheffield (2013); Izyurov (2015); Beffara, P. & Wu (2018)]









Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points,

there exists a unique probability measure on N curves

such that

conditionally on $N-1$ of the curves, the remaining one is the chordal SLE_κ in the random domain where it can live.

2. **Global construction** by Brownian loop measure

   (works for $\kappa \in (0,4]$) [Kozdron & Lawler (2007–2009); P. & Wu (2017)]

- density w.r.t product measure of independent SLEs:

$$\frac{\mathrm{d}(N\text{-}\mathrm{SLE}_\kappa)}{\mathrm{d}(\otimes_{j=1}^{N}\mathrm{SLE}_\kappa)} := \mathbb{1}_{\{\gamma_j \cap \gamma_k = \emptyset \ \forall\ j \neq k\}} \exp\left(\frac{(3\kappa-8)(6-\kappa)}{2\kappa}\, m_\alpha(\Omega; \gamma_1, \ldots, \gamma_N) \right)$$

m_α = combinatorial expression of Brownian loop measures

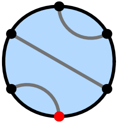
- normalize to get a probability measure

Example

Density w.r.t product measure of independent SLEs:

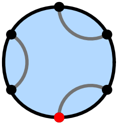
$$\frac{\mathrm{d}(N\text{-SLE}_\kappa)}{\mathrm{d}(\otimes_{j=1}^{N}\mathrm{SLE}_\kappa)} := \mathbf{1}_{\{\gamma_j\cap\gamma_k=\emptyset\ \forall\ j\neq k\}} \exp\left(\frac{(3\kappa-8)(6-\kappa)}{2\kappa}\ m_\alpha(\Omega;\gamma_1,\ldots,\gamma_N)\right)$$

- for $\alpha = \{\{1,6\},\{2,5\},\{3,4\}\}$ we have:



$$m_\alpha(\Omega;\gamma_1,\gamma_2,\gamma_3) = \mu(\Omega;\gamma_1,\gamma_2) + \mu(\Omega;\gamma_2,\gamma_3)$$

- for $\alpha = \{\{1,2\},\{3,4\},\{5,6\}\}$ we have:



$$m_\alpha(\Omega;\gamma_1,\gamma_2,\gamma_3) = \mu(\Omega;\gamma_1,\gamma_2) + \mu(\Omega;\gamma_1,\gamma_3) + \mu(\Omega;\gamma_2,\gamma_3) - \mu(\Omega;\gamma_1,\gamma_2,\gamma_3)$$

where $\mu(\Omega;\gamma_1,\gamma_2,\ldots)$ is the measure of Brownian loops in Ω that intersect all curves $\gamma_1,\gamma_2,\ldots$

Theorem

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points,

there exists a unique probability measure on N curves

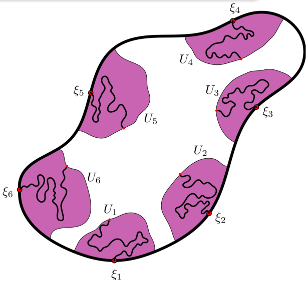
such that

conditionally on $N-1$ of the curves, the remaining one is the chordal SLE$_\kappa$ *in the random domain where it can live.*

3?. **Local construction** by Loewner evolutions (*should* work $\forall\, \kappa \in (0,8)$) [Dubédat (2006)]

$$dW_t = \sqrt{\kappa}\, dB_t + \kappa\, \partial_1 \log \mathcal{Z}\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right) dt$$

$$dV_t^i = \frac{2dt}{V_t^i - W_t}, \quad W_0 = x_1; \quad V_0^i = x_i \text{ for } i \neq 1.$$



- **Problems:**
  - How to find correct partition function $\mathcal{Z}$?
  - Makes sense *locally*, but if curves touch...?

Theorem:

Let $\kappa \in (0,4] \cup \{16/3, 6\}$. For any fixed connectivity α of $2N$ points, **there exists a unique probability measure on N curves** s.t. *conditionally on $N-1$ of the curves, the remaining one is the chordal* SLE_κ *in the random domain where it can live.*

Proposition

Let $\kappa \in (0,4]$. The marginal law of the curve starting from x_1 is given by the Loewner chain with driving process

$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}_\alpha\left(W_t, V_t^2, V_t^3, \ldots, V_t^{2N}\right)\mathrm{d}t, \qquad W_0 = x_1$$

$$\mathrm{d}V_t^i = \frac{2\mathrm{d}t}{V_t^i - W_t}, \qquad V_0^i = x_i, \quad \text{for } i \neq 1$$

Therefore, **local** N-SLE_κ with partition function $\mathcal{Z}_\alpha$

= **global** N-SLE_κ associated to connectivity α

P. & Wu **[arXiv:1703.00898]**

What is $\mathcal{Z}_\alpha$? Recall global construction...

Global construction of multiple SLE by Brownian loop measure (works for $\kappa \in (0,4]$) [Kozdron & Lawler (2007–2009); P. & Wu (2017)]

- fix connectivity pattern $\alpha \in \mathrm{LP}_N$
- density w.r.t product measure of independent SLEs:

$$\frac{\mathrm{d}(N\text{-}\mathrm{SLE}_\kappa)}{\mathrm{d}(\otimes_{j=1}^N \mathrm{SLE}_\kappa)} := \mathbf{1}_{\{\gamma_j \cap \gamma_k = \emptyset \;\forall\; j \neq k\}} \exp\left(\frac{(3\kappa-8)(6-\kappa)}{2\kappa} \, m_\alpha(\Omega; \gamma_1, \ldots, \gamma_N) \right) =: R_\alpha(\Omega)$$

  m_α = combinatorial expression of Brownian loop measures

- normalize to get a probability measure
- total mass defines (pure) *partition function*

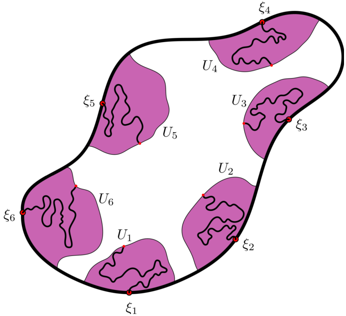
$$\mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) := \mathbb{E}\left[R_\alpha(\mathbb{H})(x_1, \ldots, x_{2N})\right] \times \prod_{(a,b)\in\alpha} \left((x_b - x_a)^{-2}\right)^{\frac{6-\kappa}{2\kappa}}$$

- **Remark:** smoothness of $\mathcal{Z}_\alpha$ problematic
  [Dubédat (2013); Lawler & Jahangoshahi (2017)]

Dubédat's commuting SLEs

and

pure partition functions



Dubédat's commuting SLEs

A multiple SLE_κ ***partition function*** is a **smooth positive** function $\mathcal{Z}(x_1, \ldots, x_{2N})$ of $2N$ real variables $x_1 < \cdots < x_{2N}$ that satisfies

(PDE): system of $2N$ partial differential equations

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{Z}(x_1, \ldots, x_{2N}) = 0 \qquad \forall\, 1 \leq j \leq 2N$$

(COV): conformal covariance

$$\mathcal{Z}(f(x_1), \ldots, f(x_{2N})) = \prod_{j=1}^{2N} \left| f'(x_j) \right|^{\frac{\kappa-6}{2\kappa}} \times \mathcal{Z}(x_1, \ldots, x_{2N})$$

Theorem **[Dubédat (2006)]**

- Any partition function generates a **local** N-SLE_κ via "multiple Loewner evolution" $\mathrm{d}W_t^{(j)} = \sqrt{\kappa}\, \mathrm{d}B_t + \kappa\, \partial_j \log \mathcal{Z}\, \mathrm{d}t$
- Conversely, any **local** N-SLE_κ has a partition function.

local N-SLE_κ **processes in** $(\mathbb{H}; x_1, \ldots, x_{2N})$ $\Leftrightarrow$ **partition functions** $\mathcal{Z}(x_1, \ldots, x_{2N})$

(PDE): system of $2N$ PDEs
(COV): conformal covariance
(POS): positivity $\mathcal{Z}(x_1, \ldots, x_{2N}) > 0$
(C^∞): smoothness

Theorem [Dubédat (2006)]

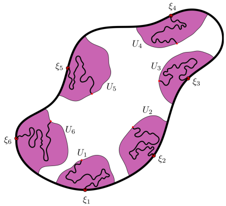
Any partition function $\mathcal{Z}$ generates a **local** N-SLE_κ
(with $\mathrm{d}W_t^{(j)} = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_j \log \mathcal{Z}\,\mathrm{d}t$)

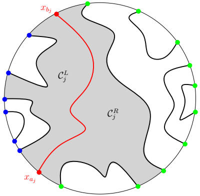
Pure partition functions are certain basis $\{\mathcal{Z}_\alpha\}_{\alpha \in \mathrm{LP}_N}$.

Proposition

P. & Wu **[arXiv:1703.00898]** Let $\kappa \in (0, 4]$.

- The local N-SLE_κ's with partition functions $\{\mathcal{Z}_\alpha\}_{\alpha \in \mathrm{LP}_N}$ are the **extremal measures** in a convex set of local N-SLE_κ's.
- These extremal measures are "localizations of" global N-SLE_κ's with connectivities $\alpha \in \mathrm{LP}_N$.





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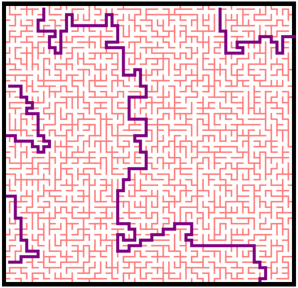
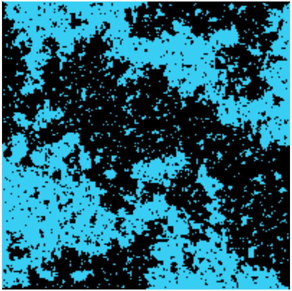
Connection probabilities





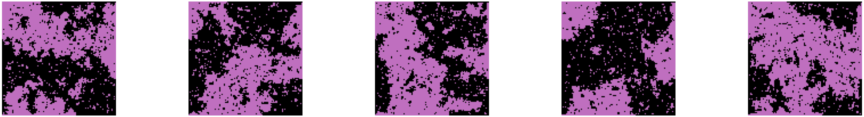
Application: connection probabilities

Idea: discrete connection probabilities $\xrightarrow{\delta \to 0}$ partition functions:

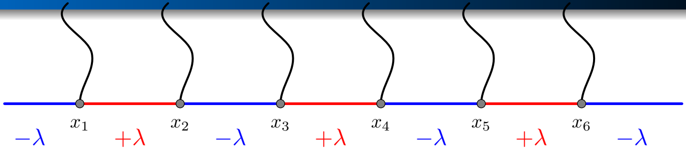
$$\lim_{\delta \to 0} \mathbb{P}\,[\text{interfaces form connectivity } \alpha] = \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{\sum_\beta \mathcal{Z}_\beta(x_1, \ldots, x_{2N})}$$

- **Gaussian free field** ($\kappa = 4$) with alternating boundary data $+\lambda, -\lambda, +\lambda, -\lambda, \ldots$ P. & Wu **[arXiv:1703.00898]**
- boundary touching branches in the **uniform spanning tree** ($\kappa = 2$) with wired boundary Karrila, Kytölä, P. **[arXiv:1702.03261]**

Todo: **critical Ising model** with alternating boundary conditions



Multiple level lines of GFF: multiple SLE_4



- level lines form some connectivity $\vartheta = \vartheta^{\mathrm{GFF}} \in \mathrm{LP}_N$
- exact solvability of $\mathcal{Z}_\alpha^{(\kappa=4)}$ and $\mathbb{P}[\vartheta = \alpha]$ (neat combinatorics)
- Proof: the ratio is bounded martingale, use optional stopping

Theorem

$$\mathbb{P}\,[\vartheta^{\mathrm{GFF}} = \alpha] \;=\; \frac{\mathcal{Z}_\alpha^{(\kappa=4)}(x_1, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta^{(\kappa=4)}(x_1, \ldots, x_{2N})} \;>0 \qquad \text{for all } \alpha \in \mathrm{LP}_N$$

For the double-dimer model (with certain b.c.), the connection probabilities converge in the scaling limit to the same expressions:

$$\mathbb{P}\,[\vartheta_\delta^{\mathrm{dd}} = \alpha] \quad \xrightarrow{\delta\to 0} \quad \mathbb{P}\,[\vartheta^{\mathrm{GFF}} = \alpha] \qquad \text{for all } \alpha \in \mathrm{LP}_N$$

Kenyon & Wilson (2011) and P. & Wu **[arXiv:1703.00898]**

Multichordal LERW: multiple SLE_2

Theorem

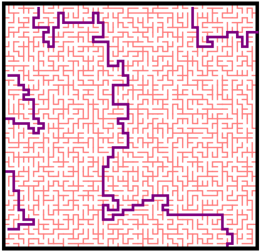
$$\frac{1}{\delta^{2N}} \mathbb{P}[\text{ UST connectivity } \alpha \text{ between edges } e_1, \ldots, e_{2N} \text{]}$$

$$\underset{\delta \to 0}{\longrightarrow} \quad \mathcal{Z}_\alpha^{(\kappa=2)}(x_1, \ldots, x_{2N})$$

Kenyon & Wilson (2011) and Karrila, Kytölä & P. **[arXiv:1702.03261]**

<u>Proof:</u> use explicit formula & conv. of discrete harmonic functions

- boundary branches in uniform spanning tree = loop-erased random walks
- *condition on the event that branches connect given 2N points pairwise*
- exact solvability of $\mathcal{Z}_\alpha^{(\kappa=2)}$ (neat combinatorics again)



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Construction of pure partition functions $\mathcal{Z}_\alpha$

(PDE): system of $2N$ PDEs
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(POS): positivity $\mathcal{Z}(x_1, \ldots, x_{2N}) > 0$
(C^∞): smoothness

Theorem [Dubédat (2006)]

Any partition function $\mathcal{Z}$ generates a **local** N-SLE$_\kappa$.

- Flores & Kleban (2015): construction for $\kappa \in (0, 8)$
  - algorithm to find formulas via Coulomb gas integrals
- Kytölä & P. (2016): construction for $\kappa \in (0, 8) \setminus \mathbb{Q}$
  - algorithm to find formulas via Coulomb gas integrals
  - representation theory of quantum group $U_q(\mathfrak{sl}_2)$
  - natural fusion structure and asymptotic properties

Formulas look like:

$$\mathcal{Z}_\alpha(\boldsymbol{x}) = \prod_{i<j}(x_i - x_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_r \prod_j (w_r - x_j)^{-4/\kappa} \prod_{r<s}(w_r - w_s)^{8/\kappa} \mathrm{d}\boldsymbol{w},$$

where Γ_α are certain integration surfaces (complicated)

In this approach:

- (PDE), (COV), (C^∞) readily follow; *fail to check (POS)*

Construction of pure partition functions $\mathcal{Z}_\alpha$

(PDE): system of $2N$ PDEs
(COV): conformal covariance
(POS): positivity $\mathcal{Z}(x_1, \ldots, x_{2N}) > 0$
(C^∞): smoothness

Theorem [Dubédat (2006)]

Any partition function $\mathcal{Z}$ generates a **local** N-SLE$_\kappa$.

- P. & Wu (2017): construction for $\kappa \in (0, 4]$
  via Brownian loop measure á la Kozdron & Lawler (2007-2009)
  - (COV), (POS) immediate; (PDE) from Itô calculus
  - (C^∞) problematic:
    use hypoelliptic PDE theory [Dubédat (2013); P. & Wu (2017)]
    or SLE estimates [Lawler & Jahangoshahi (2017)]
- Wu (2018): construction for $\kappa \in (0, 6]$
  - (COV), (POS) immediate
  - (PDE) using properties of hSLE
  - (C^∞) problematic as before
- **Remark:** By an uniqueness result [from Flores & Kleban (2015)],
  all these constructions give the *same* functions!


Thanks !