Local spin correlations in the critical and near-critical Ising model

Random Conformal Geometry and Related Fields, KIAS, Seoul

S. C. Park

EPFL

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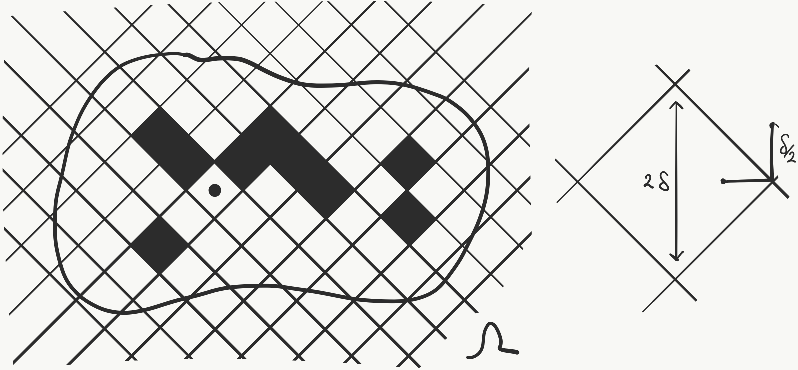
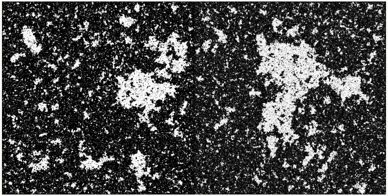
Ising model



- Simply connected $0 \in \Omega \Subset \mathbb{C}$, discretise $\Omega_\delta := \Omega \cap \delta(1+i)\mathbb{Z}^2$
- Probability of $\sigma : \Omega_\delta \to \{\pm 1\}$: $\mathbb{P}^{\beta}_{\Omega_\delta}[\sigma] \propto \exp\left[\beta \sum_{i\sim j} \sigma_i \sigma_j\right]$
- Phase transition: for $\beta \leq \beta_c$, unique limiting measure as $\Omega \to \mathbb{C}$

Scaling limit: $\delta \to 0$

- Fix + boundary condition and $\beta = \beta_c = \frac{1}{2}\ln(1+\sqrt{2})$. There is a scaling regime, i.e. a continuous field theory, that emerges as $\delta \to 0$:



- Resulting limit shows conformal invariance:

BeHo16 Interfaces converge to CLE_3

HoSm13 Energy density scales $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_0\sigma_{(1+i)\delta}\right] = \frac{\sqrt{2}}{2} + \frac{r_\Omega^{-1}(0)}{\pi}\delta + o(\delta)$

CHI15 Spin scales $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_0\right] = C r_\Omega^{-1/8}(0)\delta^{1/8} + o(\delta^{1/8})$

where $r_\Omega(a)$ is the conformal radius of $a \in \Omega$.

Scaling limit: $\delta \to 0$

HoSm13 $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\epsilon_{\frac{1+i}{2}\delta} := \frac{\sqrt{2}}{2} - \sigma_0\sigma_{(1+i)\delta}\right] = -\frac{r_\Omega^{-1}(0)}{\pi}\delta + o(\delta)$

CHI15 $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}[\sigma_0] = Cr_\Omega^{-1/8}(0)\delta^{1/8} + o(\delta^{1/8})$

where $r_\Omega(a)$ is the conformal radius of $a \in \Omega$.

Theorem (Gheissari, Hongler, P., 2017)

For any finite collection of edges $B = \{e_1, e_2, \dots\}$ in $\mathbb{C}_1 := (1+i)\mathbb{Z}^2$,

- $\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\prod_i \epsilon_{e_i\delta}\right] = \mathbb{E}^{\beta_c}_{\mathbb{C}_1}\left[\prod_i \epsilon_{e_i}\right] + P(B)r_\Omega^{-1}(0)\delta + o(\delta)$
- $\frac{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_0\prod_i \epsilon_{e_i\delta}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}[\sigma_0]} = \text{“}\frac{\mathbb{E}^{\beta_c}_{\mathbb{C}_1}\left[\sigma_0\prod_i \epsilon_{e_i}\right]}{\mathbb{E}^{\beta_c}_{\mathbb{C}_1}[\sigma_0]}\text{”} + \mathrm{Re}\left[P'(B)\partial_z \log r_\Omega(0)\right]\delta + o(\delta)$

where the $\mathbb{C}_1$ limits and P, P' are explicit and independent of Ω.

Example

$$\frac{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}\left[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}\right]}{\mathbb{E}^{\beta_c,+}_{\Omega_\delta}[\sigma_0]} = 2(\sqrt{2}-1) + \frac{5\sqrt{2}-7}{2}\cdot\partial_x \log r_\Omega(0)\cdot\delta + o(\delta)$$

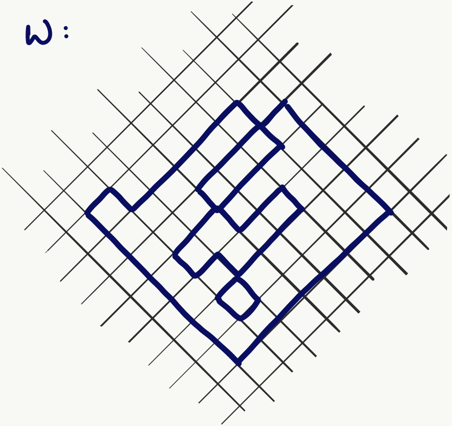
1. **Proof of the critical case**
   - Ising fermions
   - Discrete complex analysis
   - Proof ingredients

2. **Near-critical case**
   - What survives?
   - Current work

Combinatorial representations

Low-temperature expansion

- Trace edges between opposite spins
- $\mathcal{Z}^{\beta,+}_{\Omega_\delta} = \sum_\omega e^{-2\beta|\omega|}$
- $\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_0] = \frac{\sum_\omega e^{-2\beta|\omega|}(-1)^{\#loops_0(\omega)}}{\mathcal{Z}^{\beta,+}_{\Omega_\delta}}$
- $\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{(1+i)\delta}\sigma_{2\delta}\right]$ $= \mathbb{P}^{\beta,+}_{\Omega_\delta}[e\delta \notin \omega]$ $-\mathbb{P}^{\beta,+}_{\Omega_\delta}[e\delta \in \omega]$ where $e := \frac{3+i}{2}$.



Combinatorial representations

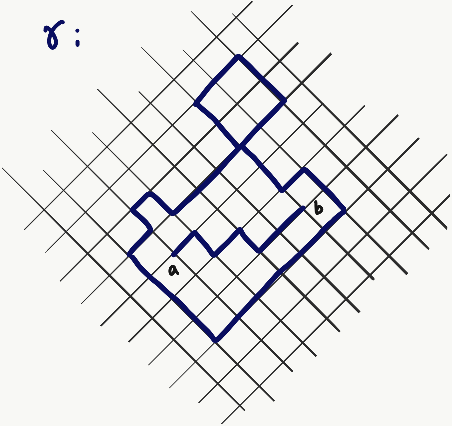
High-temperature expansion

- Dual model at $\tanh(\beta^*) := e^{-2\beta}$
- Krammers-Wannier duality

$$\mathcal{Z}^{\beta^*, free}_{\Omega^*_\delta} = \mathcal{Z}^{\beta,+}_{\Omega_\delta}$$

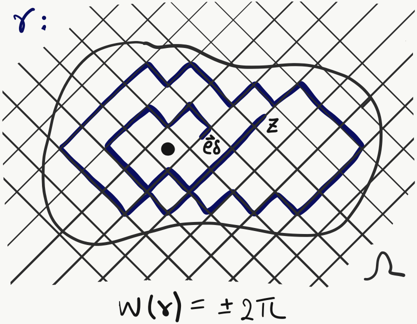
- $\mathbb{E}^{\beta,+}_{\Omega_\delta}[\sigma_a \sigma_b] =$

$$\frac{\sum_\gamma \tanh^{|\gamma|} \beta^*}{\mathcal{Z}^{\beta,+}_{\Omega_\delta}}$$

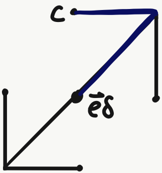


Fermion-fermions

- Define an observable $F_{\Omega_\delta}(\vec{a} = (a, \nu_{\vec{a}}), \cdot)$ on $\{\text{corners}\} \cup \{\text{edges}\}$



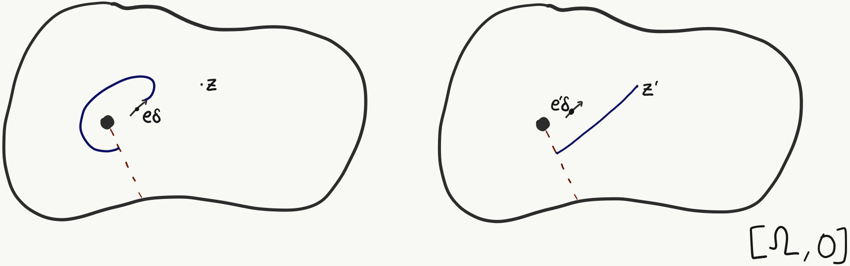
$$F_{\Omega_\delta}(\vec{e}\delta, z) \propto \sum_{\gamma} e^{-2\beta_c|\gamma|} e^{-\frac{i}{2}W(\gamma)}$$



- Spin correlation across $e\delta$: $\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_{(1+i)\delta}\sigma_{2\delta}\right] \propto \sum_{c\sim e\delta} F_{\Omega_\delta}(\vec{e}\delta, c)$

Spin-fermions

- Introduce *monodromy* at 0:



- $F_{[\Omega_\delta,0]}(\vec{e}\delta, z) \propto \sum_\gamma e^{-2\beta_c|\gamma|} e^{-\frac{i}{2}W(\gamma)} (-1)^{\mathbf{1}\{\vec{e}\delta \rightsquigarrow z'\}} (-1)^{\#loops_0(\gamma)}$
- $\dfrac{\mathbb{E}^{\beta,+}_{\Omega_\delta}\left[\sigma_0\sigma_{(1+i)\delta}\sigma_{2\delta}\right]}{E^{\beta,+}_{\Omega_\delta}[\sigma_0]} \propto \sum_{c\sim e} F_{[\Omega_\delta,0]}(\vec{e}, c)$

Core idea: discrete-continuous correspondence

- Discrete fermions are discrete meromorphic
- Discrete fermions should converge to their continuous counterparts, which are typically meromorphic and have characteristic b.c.
- Define full-plane discrete meromorphic fermions and subtract from the domain fermions to remove the poles, then take them separately to limit
- Once convergence in bulk is obtained, model the continuous picture with known discrete full-plane functions

Analogies

(Hongler and Smirnov, 2013) : $F_{\Omega_\delta}(\vec{e}\delta, \cdot)$

- $\delta^{-1} F_{\mathbb{C}_\delta}(\vec{e}\delta, z) \to 1/z$
- $\delta^{-1} F_{\Omega_\delta}(\vec{e}\delta, \cdot) \to f_\Omega$:

  $f_\Omega(z) - 1/z$ holomorphic in Ω, $f_\Omega \in \nu_{out}^{-\frac{1}{2}}\mathbb{R}$ on $\partial\Omega$.

$$f_\Omega(z) = \frac{1}{z} + r_\Omega^{-1}(0) + O(z)$$

$$F_{\Omega_\delta}(\vec{e}\delta, z) = F_{\mathbb{C}_\delta}(\vec{e}\delta, z) + \left[F_{\Omega_\delta} - F_{\mathbb{C}_\delta}\right](\vec{e}\delta, z)$$

$$\mathbb{E}_{\Omega_\delta}^{\beta_c,+}\left[\sigma_{(1+i)\delta}\sigma_{2\delta}\right] = \frac{\sqrt{2}}{2} + \frac{r_\Omega^{-1}(0)}{\pi}\delta + o(\delta)$$

Analogies

(Chelkak, Hongler, Izyurov, 2015) : $f_{[\Omega_\delta,0]}(\cdot)$

- $\delta^{-1}F_{[\mathbb{C}_\delta,0]}(z) \to 1/\sqrt{z}$
- $\delta^{-1/2}G_\delta(z) \to \sqrt{z}$
- $\delta^{-1}F_{[\Omega_\delta,0]} \to f_{[\Omega,0]}$:

  $f_{[\Omega,0]}(z) - 1/\sqrt{z}$ holomorphic in $[\Omega, 0]$, $f_\Omega \in \nu_{out}^{-\frac{1}{2}}\mathbb{R}$ on $\partial[\Omega, 0]$.

$$f_{[\Omega,0]}(z) = \frac{1}{\sqrt{z}} + 2\mathcal{A}\sqrt{z} + O(\sqrt{z}^3)\ (\mathcal{A} = -\frac{1}{4}\partial_z \log r_z(0))$$

$$F_{[\Omega_\delta,0]}(z) = F_{[\mathbb{C}_\delta,0]}(z) + 2\mathrm{Re}\mathcal{A}G_\delta(z) + \left(2\mathrm{Im}\mathcal{A}\tilde{G}_\delta^-(z)\right) + o(\delta)$$

$$\frac{\mathbb{E}_{\Omega_\delta}^{\beta_c,+}[\sigma_{2\delta}]}{\mathbb{E}_{\Omega_\delta}^{\beta_c,+}[\sigma_0]} = 1 + 2\mathrm{Re}\mathcal{A}\cdot\delta + o(\delta)$$

Analogies

(Gheissari, Hongler, P., 2017) : $f_{[\Omega,0]}(\vec{e}\delta,\cdot)$
Continuous limits are simply a real constant times the [CHI15] case, but...

$$f_{[\Omega,0]}(\vec{e},z) = \frac{C_e}{\sqrt{z}} + 2C_e\mathcal{A}\sqrt{z} + O(\sqrt{z}^3)$$

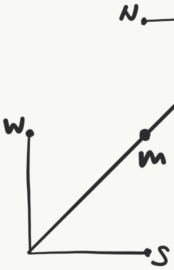
$$\begin{aligned} F_{[\Omega_\delta,0]}(\vec{e}\delta,z) &= F_{[\mathbb{C}_\delta,0]}(\vec{e}\delta,z) + C_e\left[2\mathrm{Re}\mathcal{A}G_\delta + 2\mathrm{Im}\mathcal{A}\tilde{G}_\delta^-\right](z) \\ &+ 2i\nu_{\vec{e}}^{1/2}\left[\mathrm{Re}\mathcal{A}G_1 + \mathrm{Im}\mathcal{A}\tilde{G}_1^-\right](e)\left[\tilde{G}_\delta^+ - \tilde{G}_\delta^-\right](z) \\ &+ o(\delta) \end{aligned}$$

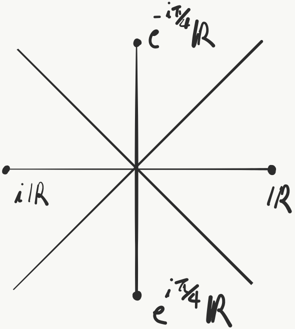
$$\frac{\mathbb{E}_{\Omega_\delta}^{\beta_c,+}[\sigma_0\epsilon_{e\delta}]}{\mathbb{E}_{\Omega_\delta}^{\beta_c,+}[\sigma_0]} = \frac{\mathbb{E}_{\mathbb{C}_1}^{\beta_c,+}[\sigma_0\epsilon_e]}{\mathbb{E}_{\mathbb{C}_1}^{\beta_c,+}[\sigma_0]} + \mathrm{Re}\left[P'(\{e\})\partial_z\log r_\Omega(0)\right]\delta + o(\delta)$$

S-holomorphicity

- Fermions satisfy *s-holomorphicity* away from $0, a$:

$$f(m) = f(S) + f(N) = f(E) + f(W)$$





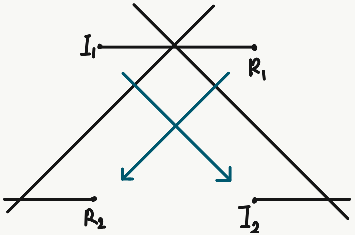
$(\Leftrightarrow f(S) = \mathrm{Re}\, f(m), \cdots)$

Corner phases $\Leftarrow e^{-\frac{i}{2}W(\gamma)}$; Edge-corner relation $\Leftarrow$ XOR bijection

- Similarly, on a boundary edge $\vec{e_{out}}$: $F_{\Omega_\delta}(e_{out}) \in \nu_{out}{}^{-\frac{1}{2}}\mathbb{R}$

S-holomorphicity

- S-holomorphic functions are *discrete holomorphic* on $\{\mathbb{R}, i\mathbb{R} - \text{corners}\}$:



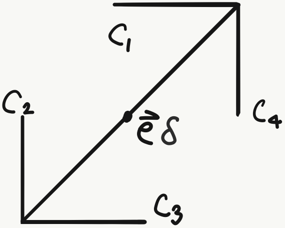
Discrete Cauchy-Riemann:

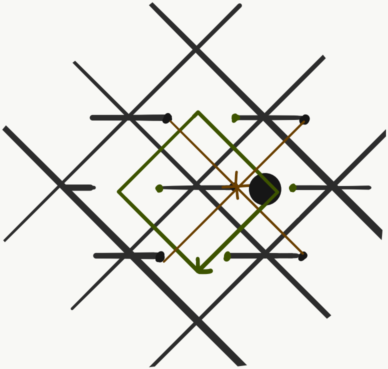
$$f(R_1) - f(R_2) + i(f(I_1) - f(I_2)) = 0$$

- D. holomorphic functions are discrete harmonic on $\{\mathbb{R} - \text{corners}\}$.

Singularities

- S-holomorphicity fails at $e\delta$, harmonicity fails near 0:





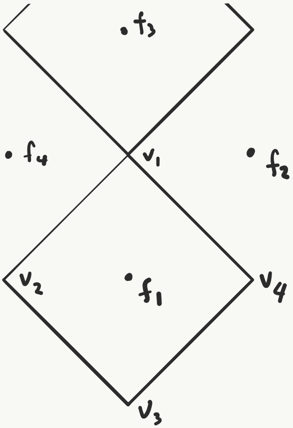
$$F_{\Omega_\delta}(\vec{e}\delta, c_1) + F_{\Omega_\delta}(\vec{e}\delta, c_3)$$

$$-F_{\Omega_\delta}(\vec{e}\delta, c_2) - F_{\Omega_\delta}(\vec{e}\delta, c_4) = \sqrt{\frac{\nu_{\vec{e}}}{2}} e^{-3\pi i/4}$$

- Save: zero on any corner on monodromy face

Integration of the square

- S-holomorphic functions can be square-integrated on $\{\text{faces}\} \cup \{\text{vertex}\}$:



$$\bar{\partial}^\delta F^2(v_1) = \frac{1}{i}\sum F^2(f_i f_{i+1})\,\overrightarrow{(f_i f_{i+1})},\quad \bar{\partial}^\delta F^2(f_1) \in \mathbb{R}$$

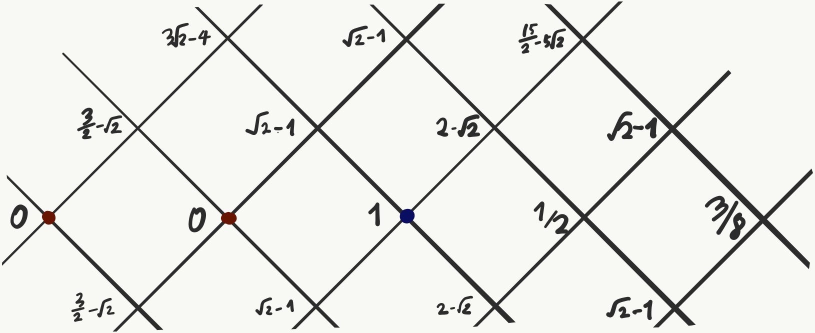
$$\Rightarrow H := \mathrm{Re}\int F^2\,dz$$

- $H(f_1) - H(v_1) = 2\delta\,|H(v_1 f_1)|^2$
- $\Delta^\delta H(f) \geq 0, \quad \Delta^\delta H(v) \leq 0$
- $0 \geq \sum_v \Delta^\delta H(v) = \sum_{\vec{e}_{out}} \partial^\delta_{\nu_{out}} H(e_{out})$

$$= \sum_{\vec{e}_{out}} \left[\left|\mathrm{Proj}_{\nu_{out}^{-1/2}\mathbb{R}} F(e_{out})\right|^2 - \left|\mathrm{Proj}_{i\nu_{out}^{-1/2}\mathbb{R}} F(e_{out})\right|^2 \right]$$

- The square integral yields boundary-to-bulk estimates

Harmonic measure of the slit plane



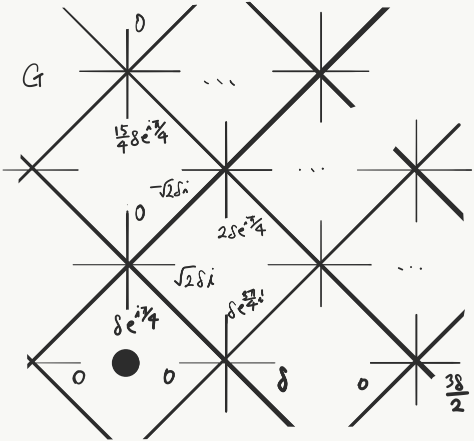
$$hm_0^{\mathbb{C}_1\setminus\mathbb{R}_-}(s+ik) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\cos(\theta)}{1+|\sin\theta|}\right)^{|k|}\frac{e^{is\theta}}{\sqrt{1-e^{-2i\theta}}}d\theta$$

$$\sqrt{\delta}^{-1} hm(\delta^{-1}z) \xrightarrow{\delta\to 0} \mathrm{Re}\sqrt{\frac{2}{\pi z}}$$

Discrete functions

- On real corners, $G_\delta(z) := \sum_{n\geq 0} hm(\delta^{-1}(z-2n\delta))\delta$
- Harmonic conjugate to imaginary corners, propagate to rest by s-holomorphicity
- $G_\delta(z) \xrightarrow{\delta\to 0} \sqrt{\frac{2z}{\pi}}$
- $\tilde{G}^{\pm}_\delta(z) := iG_\delta(z\pm\delta)$
- *A posteriori*

$$e^{\pi i/4}G_\delta(e^{\pi i/2}z) = \frac{1}{2}\left[\tilde{G}^{+}_\delta(z)+\tilde{G}^{-}_\delta(z)\right]$$

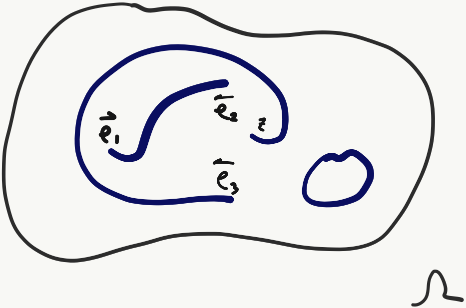


Identification

- In addition to precompactness, square integral IDs the limit in bulk

$$F_{[\Omega_\delta,0]}(e_{out}) \in \nu_{out}^{-1/2}\mathbb{R} \Leftrightarrow \partial^\delta_{\nu_{out}} H(e_{out}) \geq 0, \partial^\delta_{\nu_{tan}} H(e_{out}) = 0$$

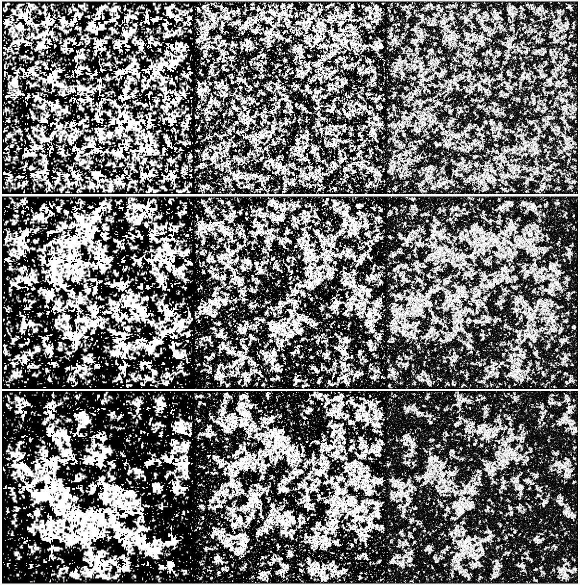
- Bulk-to-singularity uses Beurling estimate on symmetrised versions
- n energy densities are given by $2n$-point fermions, further broken down into 2-point fermions by identifying poles and b.c.



$$F_{\Omega_\delta}(\vec{e_1},\vec{e_2},\vec{e_3},z) =$$
$$-F_{\Omega_\delta}(\vec{e_1},\vec{e_2})F_{\Omega_\delta}(\vec{e_3},z)$$
$$+F_{\Omega_\delta}(\vec{e_1},\vec{e_3})F_{\Omega_\delta}(\vec{e_2},z)$$
$$-F_{\Omega_\delta}(\vec{e_2},\vec{e_3})F_{\Omega_\delta}(\vec{e_1},z)$$

$$F_{\Omega_\delta}(\vec{e_1},\vec{e_2},\dots,\vec{e_{2n}})$$
$$= \mathrm{Pf}[F_{\Omega_\delta}(\vec{e_i},\vec{e_j})]_{ij}$$

Near-critical regime



Massive s-holomorphicity

- Take the scaling limit with $\beta_c - \beta = m\delta$ for fixed m.
- Massive s-holomorphicity:
  $f(m) = \phi_\delta \left[f(S) + f(N)\right] = \phi_\delta^{-1} \left[f(E) + f(W)\right], |\phi_\delta| = 1, \phi_\delta \xrightarrow{\delta\to 0} 1$
- Massive s-holomorphicity leads to massive versions of aforementioned notions, giving analogues of:
  $\partial_{\bar{z}} F = m\bar{F}$ (Vekua equation), $(\Delta - m^2)F = 0$ (massive harmonicity)
- Vekua-Bers theory motivates various generalisations of critical constructions; $(\Delta - m^2)$ generates extinguished Brownian motion
- Square integral exists, since $\partial_{\bar{z}} F^2 = 2m|F|^2 \in \mathbb{R}$:

$$\Delta^\delta H(f) \geq 2m|F|^2(f); \Delta^\delta H(v) \leq 2m|F|^2(v)$$

- Note that if $m \leq 0$, $\Delta^\delta H(v) \leq 0$, and we have a priori bounds.

Current work

Theorem (Hongler and P., 2018)

On Ω with smooth boundary and for $m < 0$, massive versions of the fermions in [HoSm13] and [CHI15] converge to their continuous counterparts in bulk and near monodromy.

In progress:

- Cardy and Mussardo in 1990 conjectured a perturbed Virasoro structure on the space of fields in massive Ising QFT
- Lattice-level Virasoro structure (Hongler, Kytölä, Viklund, 2017) together with multipoint local correlation convergence could provide discrete building blocks
- Variable mass case; massive SLE?

Thank you for listening!