Using techniques from SDEs in the study of Schramm-Loewner Evolution

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Outline

1. A radius independent SDE in the context of backward Loewner differential equation
   - How to characterize the law of the tip of the *SLE* trace at a fixed capacity time?

2. The Bessel SDE in the context of SLE
   - (Backward) Bessel SDE: Are the conformal welding homeomorphisms continuous in κ?
   - (Forward) Bessel SDE: Are the nullsets outside of which the trace exists changing when varying κ?

     – towards the continuity in κ of the SLE_κ traces – in progress

A time change of the Loewner differential equation

- Let us do the identification $h_t(z) - \sqrt{\kappa} B_t = z_t$, with $z_t = x_t + iy_t$. From the Loewner differential equation, we obtain the coupled equations

$$dx_t = \frac{-2x_t}{x_t^2 + y_t^2} dt - \sqrt{\kappa} dB_t \,, \qquad dy_t = \frac{2y_t}{x_t^2 + y_t^2} dt \,.$$

Itô's formula for the function $f(x, y) = \frac{x}{y}$ gives

$$d\frac{x_t}{y_t} = -\frac{\frac{4x_t}{y_t}}{x_t^2 + y_t^2} dt - \frac{\sqrt{\kappa} dB_t}{y_t} \,.$$

- Using a time change and Lévy's characterization of Brownian motion, we have

$$u(s) = \int_0^s \frac{dt}{y_t^2} \,, \qquad \tilde{B}_{u(s)} = \int_0^s \frac{dB_t}{y_t}$$

- We obtain the following SDE in the random time u

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u \,.$$

Sequence of scaled one dimensional SDEs

- For $n \in \mathbb{N}$, when considering the scaling of the maps $h_t(z)$, we obtain the following sequence of diffusion processes

$$d\tilde{T}^{(n)}_{\tilde{u}_n} = -4\frac{\tilde{T}^{(n)}_{\tilde{u}_n}}{1+\kappa(\tilde{T}^{(n)}_{\tilde{u}_n})^2} d\tilde{u}_n - dB_{\tilde{u}_n}\,,$$
$$\tilde{T}^{(n)}_{\tilde{u}_n} = 0\,.$$

  where $d\tilde{u}_n = d(\tilde{t}n^2)/y^2_{\tilde{t}n^2}$ and $dB_{\tilde{u}_n}$ is a Brownian motion with respect to the $\tilde{u}_n$ time.
- We have that in distribution, for all fixed $\tilde{t} > 0$

$$\lim_{n\to\infty} ctg(arg(\hat{h}_{\tilde{t}n^2}(i))) = \lim_{n\to\infty} ctg(arg(\hat{h}_{\tilde{t}}(i/n)))$$

- Thus, in distribution

$$\lim_{n\to\infty} ctg(arg(\hat{h}_{\tilde{t}}(i/n))) = ctg(arg(\lim_{n\to\infty} \hat{h}_{\tilde{t}}(i/n))) = ctg(arg(\hat{h}_{\tilde{t}}(0+))).$$

Control of the random time

- Let $F(s,t) = 2 + min\{s, t-s\}$. Following G. Lawler and F.J. Viklund [2011], we study the events $E_{t,l}$ on which
$$e^{as - c\log F(s,t)} \leq y_{e^{2as}} \leq ce^{as}\,.$$
- On $E_{t,l}$, for $\tilde{u}_n(s,\omega) = \int_0^{sn^2} \frac{d\tilde{t}n^2}{y^2_{\tilde{t}n^2}(\omega)}$ we obtain that
$$a_n = \log(K_1 + \hat{C}_1 n) \leq \log \tilde{u}_n(s,\omega) \leq \log\left(K_2 + \hat{C}_2 n^{2c+1}\right) = b_n\,.$$
- Moreover,
$$\lim_{l\to\infty} \inf_{t>0} \mathbb{P}_*(E_{t,l}) = 1\,,$$
where $\mathbb{P}_*$ is a Wiener measure weighted with a specific martingale.

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where $\mathbb{P}_*$ is a Wiener measure weighted with a specific martingale.

Proposition

The stationary measure of the recurrent $T(u)$ process has the density given by

$$\rho(T) = C\left(\kappa T^2 + 1\right)^{-4/\kappa}.$$

Law of the tip of the SLE trace at a fixed capacity time

There is previous work that answers different questions by D. Zhan: Ergodicity of the tip of an SLE curve [2013].

- In our study, we consider the splitting

$$\frac{1}{u_n(\omega)}\int_0^{u_n(\omega)} f(T_s(\omega_1))ds$$
$$\leq \frac{a_n}{u_n(\omega)}\frac{1}{a_n}\int_0^{a_n} f(T_s(\omega_1))ds + \frac{b_n-a_n}{u_n(\omega)}\frac{1}{b_n-a_n}\int_{a_n}^{b_n} f(T_s(\omega_1))ds$$

- We can estimate these quantities using only the marginals $\mathbb{P}_*$ and $\mathbb{P}$ in the joint probability space, i.e. for one of the terms, we estimate

$$A^c(\omega,\omega_1) = A^c(\omega) = \{a_n/u_n(\omega) > 1\}$$

and

$$B^c(\omega,\omega_1) = B^c(\omega_1) = \left\{\left|\frac{1}{a_n}\int_0^{a_n} f(T_s(\omega_1))ds - \mu(f)\right| \geq \epsilon\right\}.$$

Law of the tip of the SLE trace at a fixed capacity time

Proposition

Let $f \in L^1(\mu) \cap C^\infty_{b,+}(\mathbb{R})$, *where* $\mu(dx) = \frac{dx}{(1+x^2)^{4/\kappa}}$ *is the stationary measure corresponding to the diffusion* $T(u)$.
Let $g(n) = 2^{2n}$ *and let* $u_n(S,\omega) = \log \int_0^{Sg(n)} \frac{g(n)dt}{y^2_{tg(n)}}$, *with* $y_0 = 1$. *Then, for any fixed capacity time* $S > 0$ *and* $\kappa < 8$, *the law of the tip of the SLE trace* $ctg(arg(\lim_{n\to\infty} f_S(i/n))) = ctg(arg(\gamma(S)))$ *is the limiting law* $\lim_{n\to\infty} T(u_n(S))$ *of the diffusion process* $T(u)$, *in the following sense:*

$$\left| \frac{1}{u_n(S,\omega)} \int_0^{u_n(S,\omega)} f(T_s(\omega_1))ds - (1+2c)\mu(f) \right| \xrightarrow{n\to\infty} 0$$

in probability with respect to the joint law with marginals $\mathbb{P}$ *and* $\mathbb{P}_*$, *for an explicit constant* $c > 0$.

Second part: The Bessel SDE in the context of SLE

- (Backward) Bessel SDE: Are the conformal welding homeomorphisms continuous in κ?

- (Forward) Bessel SDE: Are the nullsets outside of which the trace exists changing when varying κ?
  – towards the continuity in κ of the SLE_κ traces –

The Bessel SDE in the context of SLE

- Consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_B)$, with the Wiener measure associated with the Brownian paths.
- The Bessel SDE has strong solutions, thus we build the Bessel processes as functions of the driver and we consider a coupling of these processes with the same Brownian motion.
- For fixed index i, we take

$$f_{\kappa_i}^{x_0} : (\Omega, \mathcal{F}_t, \mathbb{P}_B) \to (\Omega, \mathcal{F}'_t, f_{\kappa_i}^{x_0} * \mathbb{P}_B)$$

$$f_{\kappa_i}^{x_0}(B_t(\omega)) = X_t^{x_0,\kappa_i}(\omega),$$

  with $X_t^{x_0,\kappa_i}$ a Bessel process of dimension $d(\kappa_i)$ starting from $x_0 > 0$.
- A natural rigidity when extending the maps to the boundary: SLE_κ trace exists $\to$ real Bessel process of dimension $d(\kappa)$ exists.

- Previous work: S. Sheffield [2010]; K. Astala, P. Jones, A. Kupiainen, E. Saksman [2011]; S. Rohde and D. Zhan [2013]; W.Qian, J. Miller [2018].

We study the conformal welding homeomorphism: two points $x > 0$ and $y < 0$ are to be identified if they hit zero simultaneously under the backward Loewner differential equation.

Proposition

The welding homeomorphism induced by the backward Loewner differential equation on the real line when driven by $\sqrt{\kappa}B_t$ is a.s. (sequentially) continuous in the parameter κ, for $\kappa \in [0,4]$.

- We prove that these random homeomorphisms, as functions from $\mathbb{R}$ to $\mathbb{R}$ depending on the parameters κ_i, $i \in I$, are point-wise convergent a.s..

Elements of the proof

The Lamperti relation gives that $T^\kappa(x_0) = x_0^2 \int_0^\infty exp[2(\tilde{B}_s + \mu(\kappa)s)]ds$, with law

$$x_0^2 \int_0^\infty exp2(\tilde{B}_s + \mu(\kappa)s)ds \overset{(d)}{=} \frac{x_0^2}{2Z_{-\mu(\kappa)}},$$

where $Z_{-\mu(\kappa)}$ is a Gamma random variable with index $\mu(\kappa)$.

Lemma

For almost every Brownian path, no two points on the same side of the singularity can give rise to solutions to the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ that will hit the origin at the same time.

[Sketch of the proof] Let us consider $0 < x_0 < y_0$. For a.e. Brownian path, we have that

$$\frac{d}{dt}(y(t) - x(t)) = 2\frac{y(t) - x(t)}{(y(t) - \sqrt{\kappa}B_t)(x(t) - \sqrt{\kappa}B_t)}.$$

Elements of the proof

Lemma

Let us consider $X_t^{\kappa_i}(x_0)$ a collection of Bessel processes started from fixed $x_0 > 0$ coupled by driving them with the same Brownian paths.
Let $(\kappa_i)_{i \in I}$ be a strictly increasing sequence of values of $\kappa \in \mathbb{R}_+$. Then for all starting points $x_0 > 0$, $T^{\kappa_i}(x_0) \leq T^{\kappa_j}(x_0)$ for $i \leq j$, for almost every Brownian path.

- Considering the Laplace transforms of these times, we have that

$$\mathcal{L}(T^{\kappa_0 -}(x_0)) \leq \mathcal{L}(T^{\kappa_0}(x_0)) \leq \mathcal{L}(T^{\kappa_0 +}(x_0)) .$$

- Thus, using the convergence of the laws in $\mu(\kappa)$, we obtain that

$$\mathcal{L}(T^{\kappa_0 +}(x_0)) = \lim_{\kappa_i \to \kappa_0} \mathcal{L}(T^{\kappa_i}(x_0)) = \mathcal{L}(T^{\kappa_0}(x_0)) .$$

- We have that a.s. $[T^{\kappa_0 +}(x_0) - T^{\kappa_0}(x_0)](\omega) \geq 0$. Thus, we have a.s.

$$T^{\kappa_0 +}(x_0) = T^{\kappa_0}(x_0) = T^{\kappa_0 -}(x_0) .$$

Elements of the proof

- We use the Lemma to argue that for the fixed value κ_{i_1}, there is a.s. only one point on the negative part of the real line that will hit the origin in time $T^{\kappa_{i_1}}(x_0)$. We call this point $y_0^{\kappa_{i_1}}$.
- If we keep the starting point fixed and we change the parameters, we obtain that the hitting time $T^{\kappa_i}(x_0) \to T^{\kappa}(x_0)$. The same applies on the negative part of the real line.
- We consider the Lemma in order to get that for the fixed value κ the point y_0^{κ} that hits simultaneously with $x_0 > 0$ should be the same with the limiting $y_0^{\lim_i \kappa_i}$, where $\kappa_i \to \kappa$.

Forward Bessel processes and the control of the nullsets when varying $\kappa \in \mathbb{R}, \kappa \neq 8$.

- Previous non-constant κ analysis: S. Rohde, F. J. Viklund, C. Wong [2012]; P. Friz, A. Shekhar [2017].
- Our approach: The absolute continuity of the laws of the Bessel processes with indexes $\mu(\kappa_1)$ and $\mu(\kappa_2)$ is given by the following relation:

$$\frac{d\mathbb{P}_{x_0}^{\mu(\kappa_1)}}{d\mathbb{P}_{x_0}^{\mu(\kappa_2)}}|_{\mathcal{F}_t^{\mu}} = \left(\frac{X_t}{x_0}\right)^{\mu(\kappa_1)-\mu(\kappa_2)} exp\left(-\frac{\mu^2(\kappa_1)-\mu^2(\kappa_2)}{2}\int_0^t \frac{ds}{X_s^2}\right),$$

  for any starting point $x_0 > 0$, $\mathbb{P}_{x_0}^{\mu(\kappa_1)}$- a.s..
- We use the rigidity in the coupled picture to control the dependence on the starting point.

Strategy of the proof

Theorem (Tran)

Suppose the driving function of the Loewner chain $g_t(z)$ is weakly Hölder $\frac{1}{2}$ with sub-power function ϕ and suppose that $|f_t'(iy)| \leq c_0 y^{-\beta}$ is satisfied. Then, there exists a sub-power function $\tilde{\phi}(n)$ that depends on ϕ, c_0 and β, such that for all $n \geq 1/y_0^2$ and $t \in [0,1]$ we have that $|\gamma^n(t) - \gamma(t)| \leq \frac{\tilde{\phi}(n)}{n^{\frac{1}{2}\left(1-\sqrt{\frac{1+\beta}{2}}\right)}}$, where γ^n is the curve generated by the algorithm.

- One challenge is to control the nullset outside of which the estimate $|f_t'(iy)| \leq c_0 y^{-\beta}$ holds when varying $8 \neq \kappa \in \mathbb{R}_+$. We do this using the forward Bessel flow on the real line.

Strategy of the proof

- We assume that the nullset outside of which the estimate $|f_t'(iy)| \leq c_0 y^{-\beta(\kappa_i)}$ is satisfied is changing as we vary κ_i with $i \in \mathbb{R}$.
- For fixed i, when $|f_t'(iy)| \leq c_0 y^{-\beta(\kappa_i)}$ holds, using Rohde-Schramm Theorem we have a.s. the existence of the trace. Using the rigidity, we also have that a.s. the collection of real Bessel processes of dimension $d(\kappa_i)$ exists.
- To control the dependence on the starting point of the result, we use the rigidity of the dynamics in the coupled picture.
- We obtain a contradiction with the absolutely continuous laws of the Bessel processes of different parameters.

Towards the continuity in κ of the SLE_κ traces for $\kappa \neq 8$.

- The same contradiction holds on the event $\{T_x > t\}$ for $8 \neq \kappa > 4$.
- We estimate

$$|\gamma^{\kappa_1(n)}(t) - \gamma^{\kappa_2}(t)| \leq |\gamma^{\kappa_1(n)}(t) - \gamma^n_{\kappa_2}(t)| + |\gamma^n_{\kappa_2}(t) - \gamma^{\kappa_2}(t)|\,.$$

- The second term is estimated by the convergence of the algorithm by Huy Tran. For the first term, we have $|(f^{\kappa_1(n)}_{t_k})'(z)| \leq cy^{-\beta(\kappa(n))}$ and $|\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_1(n)}B_t| \leq |\lambda^n_{\kappa_2}(t) - \sqrt{\kappa_2}B_t| + |\sqrt{\kappa_2}B_t - \sqrt{\kappa_1(n)}B_t|\,.$

Summary

- Law of the tip of the SLE trace at a fixed capacity time $ctg(arg(\lim_{n\to\infty} f_S(i/n))) = ctg(arg(\gamma(S)))$ is the limiting law $\lim_{n\to\infty} T(u_n(S))$ of the diffusion process $T(u)$, in the following sense:

$$\left| \frac{1}{u_n(S,\omega)} \int_0^{u_n(S,\omega)} f(T_s(\omega_1))ds - (1+2c)\mu(f) \right| \xrightarrow{n\to\infty} 0$$

  in probability.
- The welding homeomorphism induced by the backward Loewner differential equation on the real line when driven by $\sqrt{\kappa}B_t$ is *a.s.* (sequentially) continuous in the parameter κ, for $\kappa \in [0,4]$.
- The nullsets outside of which the trace exists can be controlled when changing the parameter $\kappa \neq 8$.
  Using this and the approximation of the traces by Huy Tran, we performed the analysis presented in order to study the continuity in κ of the SLE_κ traces.

Thank you for your attention!