First passage sets of the 2D GFF and Minkowski content

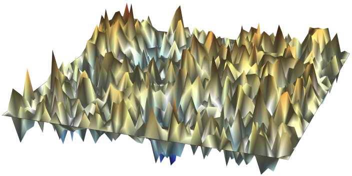
(with Juhan Aru (ETHZ) and Avelio Sepùlveda (Lyon 1))

Titus Lupu

CNRS/Sorbonne Université

Random Conformal Geometry and Related Fields, KIAS, Seoul, June 18th, 2018

Continuum Gaussian free field in 2D



$D \subset \mathbb{C}$ open (simply connected) domain with non-polar boundary.

Φ continuum Gaussian free field in D with zero boundary condition.

Informally, field with distribution:

$$Z^{-1} \exp\Big(-\frac{1}{4\pi} \int_D \|\nabla \varphi\|^2 \Big) \mathcal{D}(\varphi),$$

with $\mathcal{D}(\varphi)$ "uniform measure on fields" (which does not exist).

Rigorously:
Φ random Gaussian generalized function (in Sobolev space $H^{-\varepsilon}$, $\varepsilon > 0$).
$\mathbb{E}[\Phi] = 0$, $\qquad$ $\mathrm{Cov}((\Phi, f_1), (\Phi, f_2)) = \iint_{D\times D} f_1(x) G_D(x,y) f_2(y) dxdy$.
$G_D = (-\Delta/2\pi)^{-1}$ Green's function.

Conformal invariance: $\Phi \circ \psi \overset{(d)}{=} \Phi$, $\psi : D \to D$ conformal.
$u : D \to \mathbb{R}$ harmonic. $\Phi + u$ GFF with non-zero b.c.

Markov property of the GFF

K closed subset of $\overline{D}$.
$H_0^1(D) = \mathrm{Harm}(D\backslash K) \overset{\perp}{\oplus} H_0^1(D\backslash K)$, $\qquad \|f\|^2_{H_0^1} = \int \|\nabla f\|^2$,
$\mathrm{Harm}(D\backslash K)$ subspace of functions harmonic on $D\backslash K$.

Weak Markov property: $\Phi = \Phi_K + \Phi^K$, $\Phi_K \amalg \Phi^K$,
Φ_K harmonic on $D\backslash K$, Φ^K 0 b.c. GFF on $D\backslash K$.

Strong Markov property:
A stopping set: A closed, $\forall U$ open, $\{A \subset U\}$ measurable w.r.t. $\Phi 1_U$.
$\Phi = \Phi_A + \Phi^A$, $\Phi_A \amalg \Phi^A$ cond. on A,
Φ_A harmonic on $D\backslash A$ (harmonic part h_A),
cond. on A, Φ^A 0 b.c. GFF on $D\backslash A$.

Schramm-Sheffield:
Local set of the GFF = a generalization of stopping sets.
Sets along which Φ has a Markovian decomposition.
Does not assume measurability w.r.t. Φ.

Thin and non-thin local sets

A local set: $\Phi = \Phi_A + \Phi^A$.
$\Phi_A \amalg \Phi^A$ cond. on A,
Φ_A harmonic on $D \backslash A$ (harmonic part h_A),
cond. on A, Φ^A 0 b.c. GFF on $D \backslash A$.

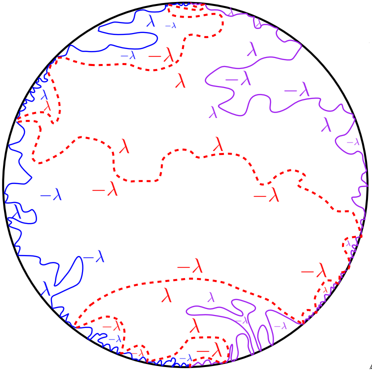
Thin local set: $\Phi_A = h_A$.

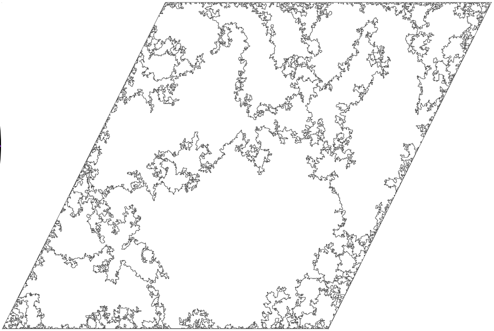
Non-thin local set: $\Phi_A \neq h_A$.

Examples of thin local sets:

- level lines - $\text{SLE}_4(\underline{\rho})$ processes (Schramm-Sheffield 2006), dim $= 3/2$;
- flow lines - $\text{SLE}_\kappa(\underline{\rho})$, $0 < \kappa < 8$ (Miller-Sheffield 2012), dim $= 1 + \kappa/8$;
- CLE_4 gasket (Miller-Sheffield coupling), dim $= 15/8$;
- two-valued sets $\mathbb{A}_{-a,b}$, $a + b \geq 2\lambda$ (Aru-Sepúlveda-Werner 2016), dim $= 2 - 2\lambda^2/(a+b)^2$. Can be seen as points of "$\Phi \in [-a, b]$" accessible from the boundary. Constructed by iterating level lines.

Example of thin local sets: level lines

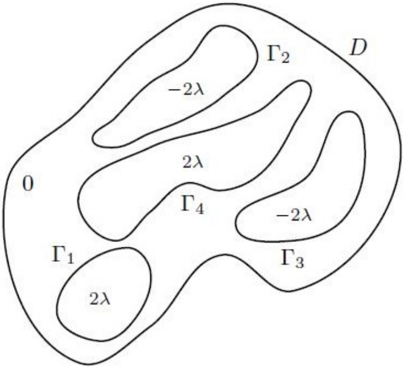


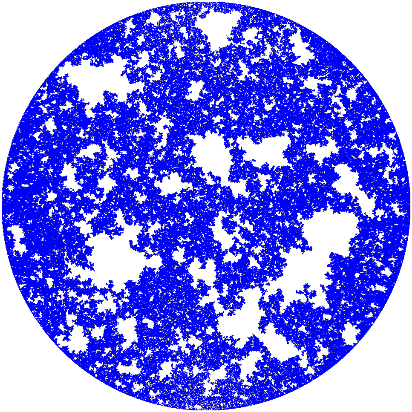


Level lines of Φ: $\mathrm{SLE}_4(\rho_1, \rho_2)$ curves.

$2\lambda = \pi$: height gap (Schramm-Sheffield, 2010).

Example of thin local sets: CLE_4 gasket





Conformal Loop Ensemble CLE_4.
Miller-Sheffield coupling of CLE_4 and GFF:
in each hole Γ_i an independent 0 b.c. GFF + $\sigma_i 2\lambda$,
$\sigma_i \in \{-1, 1\}$ i.i.d., $\mathbb{P}(\sigma_i = -1) = \mathbb{P}(\sigma_i = 1) = 1/2$.

Example of non-thin local set: First passage sets.

Trivial examples of non-thin local sets: A has non-empty interior.

More interesting example of non-thin local sets:
First passage sets (Aru, L., Sepúlveda, 2017).
$-a < 0$. $\mathbb{A}_{-a}$ local set s.t.

- $h_{\mathbb{A}_{-a}} = -a$ on $D \backslash \mathbb{A}_{-a}$.
- $\Phi_{\mathbb{A}_{-a}} + a$ is a non-negative measure (supported on $\mathbb{A}_{-a}$).

$\mathbb{A}_{-a}$ essentially the only local set to satisfies these properties.
$\mathrm{Leb}(\mathbb{A}_{-a}) = 0$ a.s. In particular $\mathbb{A}_{-a}$ has empty interior.

Informally:

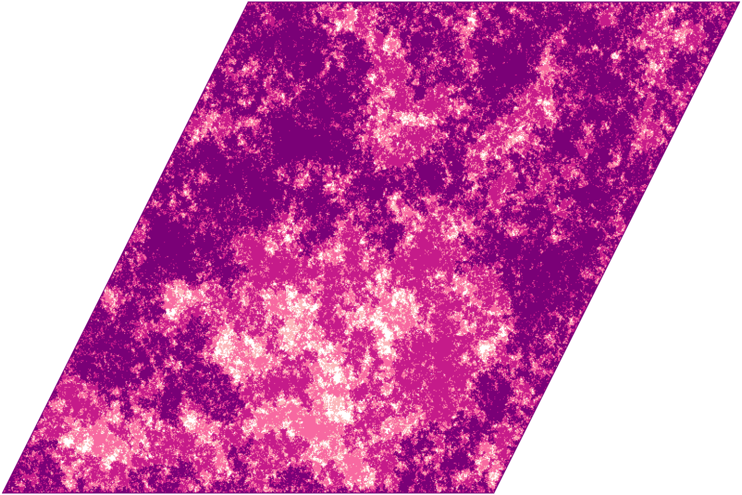
$$\mathbb{A}_{-a} = \text{"}\{z \in \overline{D} | \exists \gamma \text{ path } z \overset{\gamma}{\leftrightarrow} \partial D, \Phi \geq -a \text{ on } \gamma\}\text{"}.$$

$\mathbb{A}_{-a}$ cannot be thin:

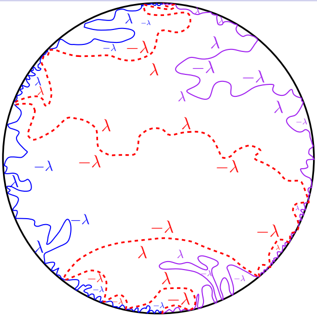
$$\mathbb{E}[(\Phi_{\mathbb{A}_{-a}}, f)] = \mathbb{E}[(\Phi, f)] - \mathbb{E}[(\Phi^{\mathbb{A}_{-a}}, f)] = \mathbb{E}[(\Phi, f)] = 0$$
$$\neq \mathbb{E}[(h_{\mathbb{A}_{-a}}, f)] = -a \int_D f.$$

A simulation of first passage sets

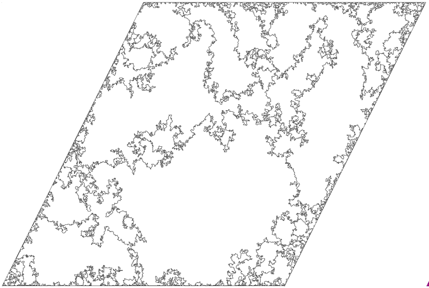
Computer simulation of $\mathbb{A}_{-\lambda}, \mathbb{A}_{-2\lambda}, \mathbb{A}_{-3\lambda}$ and $\mathbb{A}_{-4\lambda}$.

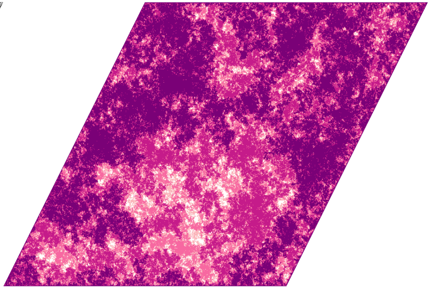


Continuum FPS and level lines of the GFF



Construction of first passage set $\mathbb{A}_{-\lambda}$.
First step: tree of level lines $-\lambda, \lambda$ ($\mathrm{SLE}_4(-1,-1)$ curves).





Expected size of first passage sets

D simply connected. $z \in D$ fixed. CR: conformal radius.
$(\eta_t)_{t \geq 0}$ continuous growing family of local sets.
$u_t(z)$ harmonic extension to z of values of Φ induced by η_t. If

$$t = \log(\mathrm{CR}(z, D) / \mathrm{CR}(z, D \backslash \eta_t)),$$

then $(u_t(z))_{t \geq 0}$ is a standard Brownian motion.
Consequence:

$$\log(\mathrm{CR}(z, D) / \mathrm{CR}(z, D \backslash \mathbb{A}_{-a})) \overset{(d)}{=} T_{-a},$$

T_{-a} first time B.M. starting from 0 hits $-a$.

$\mathbb{P}(T_{-a} > t) \asymp t^{-\frac{1}{2}}$.

$+$ Koebe quarter theorem:

$$\mathbb{E}[\mathrm{Area}(\mathbb{A}_{-a} + B(0, r))] \asymp |\log(r)|^{-\frac{1}{2}}.$$

$\dim(\mathbb{A}_{-a}) = 2$ a.s.

Minkowksi content

$\Phi = \nu_{\mathbb{A}_{-a}} - a + \Phi^{\mathbb{A}_{-a}}$, $\nu_{\mathbb{A}_{-a}}$ positive measure supported on $\mathbb{A}_{-a}$.

Theorem (Aru, L., Sepúlveda, 2018)

$\nu_{\mathbb{A}_{-a}}$ is a Minkowski content measure in the gauge $|\log(r)|^{\frac{1}{2}} r^2$. More precisely, for all $f \in \mathcal{C}_c(D)$,

$$(\nu_{\mathbb{A}_{-a}}, f) = \lim_{r \to 0} \sqrt{\frac{\pi}{2}} |\log(r)|^{\frac{1}{2}} \int_{d(z, \mathbb{A}_{-a}) < r} f(z) d|z|^2.$$

The result extends to all local sets contained in some first passage set, that is to say the restriction of the GFF to such local set is a Minkowski content.

The result extends to multiply finitely connected domains.

Main tool: Liouville Quantum Gravity measure.

$\gamma \in (-2,2)$. $e^{\gamma\Phi}$ exponential of the Gaussian free field, Liouville Quantum Gravity measure (Kahane's Gaussian Multiplicative Chaos 1985, Polyakov 80s, Duplantier-Sheffield 2011).

$$e^{\gamma\Phi} = \lim_{\varepsilon\to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma\Phi_\varepsilon},$$

Φ_ε regularization by circle average.

Aru, Powell, Sepúlveda, 2017: For all $\gamma \in [0,2)$,

$$e^{-\gamma\Phi} = e^{\gamma a} e^{-\gamma\Phi^{\mathbb{A}_{-a}}},$$

Uses that Φ is a positive measure on $\mathbb{A}_{-a}$ and that $\mathrm{Leb}(\mathbb{A}_{-a}) = 0$.

In particular,

$$\mathbb{E}[e^{-\gamma\Phi}|\mathbb{A}_{-a}] = e^{\gamma a}\mathbb{E}[e^{-\gamma\Phi^{\mathbb{A}_{-a}}}|\mathbb{A}_{-a}] = e^{\gamma a}\,\mathrm{CR}(z, D\backslash\mathbb{A}_{-a})^{\frac{\gamma^2}{2}} d|z|^2.$$

Intermediate description of the measure

Let $f \in \mathcal{C}_c(D)$. $\gamma \mapsto (e^{\gamma\Phi}, f)$ is a.s. analytic on $(-\sqrt{2}, \sqrt{2})$ (and has a holomorphic extension to $B(0,\sqrt{2})$). For instance,

$$e^{\gamma\Phi} = CR(z,D)^{\frac{\gamma^2}{2}}(1+\gamma\Phi+\sum_{n\geq 2}\frac{\gamma^n}{n!}:\Phi^n:).$$

$$\text{Moreover,}\quad \frac{d}{d\gamma}(e^{\gamma\Phi}, f)_{|\gamma=0} = (\Phi, f).$$

Differentiation in γ and conditional expectation commute:

$$\mathbb{E}[\frac{d}{d\gamma}(e^{\gamma\Phi}, f)|\mathbb{A}_{-a}] = \frac{d}{d\gamma}\mathbb{E}[(e^{\gamma\Phi}, f)|\mathbb{A}_{-a}].$$

$$\mathbb{E}[\frac{d}{d\gamma}(e^{\gamma\Phi}, f)_{|\gamma=0}|\mathbb{A}_{-a}] = (\nu_{\mathbb{A}_{-a}}, f) - a\int_D f.$$

$$\begin{aligned}(\nu_{\mathbb{A}_{-a}}, f) &= \lim_{\gamma\to 0^+}\gamma\int_D(-\log(\mathrm{CR}(z,D\backslash\mathbb{A}_{-a})))\,\mathrm{CR}(z,D\backslash\mathbb{A}_{-a})^{\frac{\gamma^2}{2}}f(z)d|z|^2\\ &= \lim_{\gamma\to 0^+}\gamma\int_D|\log(d(z,\mathbb{A}_{-a}))\wedge 0|d(z,\mathbb{A}_{-a})^{\frac{\gamma^2}{2}}f(z)d|z|^2\end{aligned}$$

The density tends to 0, but the total mass does not. The mass concentrates around $\mathbb{A}_{-a}$.

Scaling

$F_\gamma(x) = \gamma|\log(x) \wedge 0|x^{\frac{\gamma^2}{2}}$.
$I_r(x) = |\log(r)|^{\frac{1}{2}} 1_{0<x<r}, \ r \in (0,1)$.

Want:

$$\lim_{\gamma \to 0} \int_D F_\gamma(d(z, \mathbb{A}_{-a}))f(z)d|z|^2 = cste \lim_{r \to 0} \int_D I_r(d(z, \mathbb{A}_{-a}))f(z)d|z|^2.$$

$F_{\gamma'}(x) = \beta^{-\frac{1}{2}} F_\gamma(x^\beta)$, with $\beta = \gamma'^2/\gamma^2$.
$I_{r'}(x) = \beta^{-\frac{1}{2}} I_r(x^\beta)$, with $\beta = |\log(r)|/|\log(r')|$.
Same scaling in both cases.

To show that the Minkowski content converges, approximate $I_{1/2}$ by linear combinations $\sum_{i=1}^N a_i F_{\gamma_i}$ and use the scaling.

Boundedness of the Minkowski content

Denote

$$\mathfrak{M}(F) = 1_{D\setminus\mathbb{A}_{-a}} F(d(z,\mathbb{A}_{-a}))dz.$$

Let $q_r(x) = 1_{x\in(0,1)} \frac{|\log(r\vee x)|}{|\log r|^{1/2}}$. Then, for all f continuous compactly supported in D, $(\mathfrak{M}(q_r), f)$ stays bounded as $r \to 0$.

Let $\gamma_0 > 0$. There is $C > 0$ such that

$$1_{x\in(1/4,1/2)} \leq CF_{\gamma_0}(x) \quad \text{and} \quad 1_{x\in(1/2,1)}|\log x| \leq CF_{\gamma_0}(x).$$

Thus, by the scaling of F_γ,

$$1_{(2^{-2^{k+1}},2^{-2^k})}(x) = 1_{(1/4,1/2)}(x^{2^{-k}}) \leq CF_{\gamma_0}(x^{2^{-k}}) = C2^{-\frac{k}{2}} F_{2^{-\frac{k}{2}}\gamma_0}(x).$$

$$q_{1/2} \leq C \sum_{k=1}^{+\infty} 2^{-\frac{k}{2}} F_{2^{-\frac{k}{2}}\gamma_0}.$$

As q_r satisfies the scaling $q_r(x) = \beta^{-\frac{1}{2}} q_{1/2}(x^\beta)$ with $\beta = \frac{\log 2}{|\log r|}$,

$$q_r \leq C \sum_{k=1}^{+\infty} 2^{-\frac{k}{2}} F_{2^{-\frac{k}{2}}\beta\gamma_0}.$$

Existence of the Minkowski content

$I_{1/2,\rho}$ smooth function with values in $[0, (\log 2)^{1/2}]$, which coincides with $I_{1/2}$ on $[\rho, 1/2 - \rho]$ and is 0 on $[0, \rho/2] \cup [1/2 - \rho/2, 1]$.

$$I_{r,\rho}(x) = \beta^{-\frac{1}{2}} I_{1/2,\rho}(x^\beta) \text{ with } \beta = \frac{\log 2}{|\log r|}.$$

The convergence of $\mathfrak{M}(I_{r,\rho})$ for all $\rho > 0$ implies the convergence of $\mathfrak{M}(I_r)$ as $r \to 0$.

Uniform approximation $|I_{1/2,\rho} - \sum_{i=1}^N a_i F_{\gamma_i}| \leq \varepsilon q_{1/2}$.

Comes from approximation of $|\log(x)|^{-1} I_{1/2,\rho}(x)$ by polynomials.

Computing the constant

Given that the Minkoski content exists, how to show that

$$\lim_{\gamma \to 0} \int_D F_\gamma(d(z, \mathbb{A}_{-a})) f(z) d|z|^2 = cste \lim_{r \to 0} \int_D I_r(d(z, \mathbb{A}_{-a})) f(z) d|z|^2$$

and identify the constant?

$$\begin{aligned} F_\gamma(x) &= -\int_0^1 F'_\gamma(r) 1_{0<x<r} dr \\ &= -\int_0^1 F'_\gamma(r) |\log(r)|^{-\frac{1}{2}} I_r(x) dr \\ &= \int_0^1 \gamma r^{\frac{\gamma^2}{2}-1} (1 + \gamma^2 \log(r)/2) |\log(r)|^{-\frac{1}{2}} I_r(x) dr \\ &= \int_0^{+\infty} \sqrt{2} e^{-t} (1-t) t^{-\frac{1}{2}} I_{e^{-2t/\gamma^2}}(x) dt. \end{aligned}$$

$$cste = \int_0^{+\infty} \sqrt{2} e^{-t} (1-t) t^{-\frac{1}{2}} dt = \sqrt{\frac{\pi}{2}}.$$

Le Jan's isomorphism

Framework: $\mathcal{G} = (V, E)$ undirected graph. $C(e) > 0$ conductances on edges. Boundary $\partial\mathcal{G} \subseteq V$. Continuous time Markov jump process on $\mathcal{G}$, jump rates = conductances.

Measure on loops:

$$\mu^{\mathcal{G}}_{\mathrm{loop}}(\cdot) := \sum_{x \in V \setminus \partial\mathcal{G}} \int_0^{+\infty} \mathbb{P}^t_{x,x}(\cdot, t < T_{\partial\mathcal{G}}) p_t(x,x) \frac{dt}{t}.$$

Essentially same measure that appears in Symanzik and in Brydges, Frölich, Spencer. Loop-soup: $\mathcal{L}_\alpha$ P.P.P. with intensity $\alpha\mu^{\mathcal{G}}_{\mathrm{loop}}$.

Le Jan's isomorphism, 2007:

$$L(\mathcal{L}_{1/2}) := \sum_{\gamma \in \mathcal{L}_{1/2}} L_\gamma \overset{(d)}{=} \frac{1}{2}\phi_0^2, \ \phi_0 \text{ DGFF, b.c. 0 on } \partial\mathcal{G}.$$

An extension of Dynkin's isomorphism (1984).

A version with positive b.c.

$u \geq 0$ on $\partial\mathcal{G}$. E_u P.P.P of boundary-to-boundary excursions with intensity:

$$\frac{1}{2} \sum_{(x,y)\in\partial\mathcal{G}} u(x)u(y)H_{\mathcal{G}}(x,y)\mathbb{P}^{\mathrm{exc}}_{x,y}.$$

$H_{\mathcal{G}}(x,y)$ boundary Poisson kernel = an effective conductance.
$E_u \amalg \mathcal{L}_{1/2}$. Then:

$$L(\mathcal{L}_{1/2} \cup E_u) \overset{(d)}{=} \frac{1}{2}\phi_u^2, \ \phi_u \text{ DGFF, b.c. u on } \partial\mathcal{G}.$$

In continuum 2D: Wick square and measures on Brownian loops and excursions

Φ continuum GFF on $D \subset \mathbb{C}$, 0 b.c. u harmonic on D. Wick square:

$$: \Phi^2 := \lim_{\varepsilon \to 0} \Phi_\varepsilon^2 - \mathbb{E}[\Phi_\varepsilon^2], \qquad : (\Phi + u)^2 :=: \Phi^2 : + 2u\Phi.$$

Brownian loop measure and excursion measure (for $u \geq 0$):

$$\mu_{\mathrm{loop}}^D(\cdot) := \int_D \int_0^{+\infty} \mathbb{P}_{z,z}^t(\cdot, t < T_{\partial D}) p_t(z,z) \frac{dt}{t} d|z|^2.$$

$$\mu_{\mathrm{exc}}^{D,u}(\cdot) := \frac{1}{2} \iint_{\partial D \times \partial D} u(x)u(y)H_D(x,y)\mathbb{P}_{x,y}^{\mathrm{exc}}(\cdot)dxdy,$$

$H_D(x,y) = \partial_{n_x}\partial_{n_y} G_D(x,y)$ boundary Poisson kernel.

Isomorphism for continuum GFF

2 indep. P.P.P.: critical Brownian loop-soup $\mathcal{L}_{1/2}$ (intensity $1/2\mu_{\mathrm{loop}}^{D}$) and E_u (intensity $\mu_{\mathrm{exc}}^{D,u}$).

Occupation measure $L(\mathcal{L}_{1/2})$ not locally finite.
Centred occupation measure:

$$L^{\mathrm{ctr}}(\mathcal{L}_{1/2}) = \lim_{\varepsilon\to 0} L(\{\gamma \in \mathcal{L}_{1/2} | \operatorname{diam}(\gamma) \geq \varepsilon\}) - \mathbb{E}[L(\{\gamma \in \mathcal{L}_{1/2} | \operatorname{diam}(\gamma) \geq \varepsilon\})].$$

Isomorphism:

$$L^{\mathrm{ctr}}(\mathcal{L}_{1/2}) + L(E_u) \overset{(d)}{=} \frac{1}{2} : \Phi^2 : + u\Phi + \frac{1}{2}u^2,$$

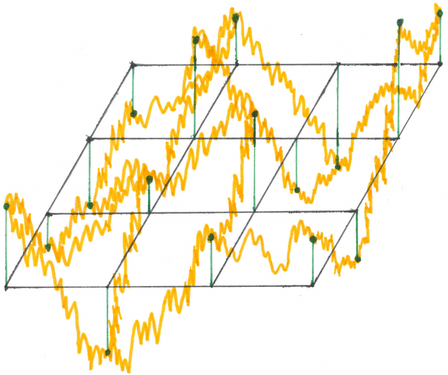
$$L^{\mathrm{ctr}}(\mathcal{L}_{1/2}) + L^{\mathrm{ctr}}(E_u) \overset{(d)}{=} \frac{1}{2} : (\Phi + u)^2 : .$$

The GFF on a metric graph

$\mathcal{G} = (V, E)$ undirected graph. $C(e) > 0$ conductances on edges.

$\widetilde{\mathcal{G}}$ metric graph associated to $\mathcal{G}$: each edge $e \in E$ is replaced by a continuous line of length $C(e)^{-1}$.
Discrete resistor $\longrightarrow$ continuous conducting wire.

Metric graph GFF $\tilde{\phi}$: interpolate the discrete GFF ϕ with independent Brownian bridges inside edges.



Why do we like metric graph GFF?

- Gaussian.
- Markov.
- **Satisfies the intermediate value property.**

Signed isomorphism on metric graph

$\widetilde{\phi}_u$ metric graph GFF with b.c. $u \geq 0$ on $\partial \mathcal{G}$.

Metric graph Brownian motion. Metric graph loop measure.
Metric graph measure on boundary-to-boundary excursions.
Restriction to vertices: measures on discrete-graph paths.

2 indep. P.P.P: metric graph loop-soup $\widetilde{\mathcal{L}}_{1/2}$ and excursions $\widetilde{E}_u$.
Occupation field $L(\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_u)$: sum of local times.

Signed isomorphism (L. 2014):

- Sample $\widetilde{\mathcal{L}}_{1/2}$ and $\widetilde{E}_u$.
- Sample $\sigma_{\mathcal{C}} \in \{-1, 1\}$ for $\mathcal{C}$ cluster of $\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_u$.
  $\sigma_{\mathcal{C}} = 1$ if $\mathcal{C}$ contains an excursion ($\widetilde{E}_u$).
  If not, $\sigma_{\mathcal{C}}$ independent, $\mathbb{P}(\sigma_{\mathcal{C}} = -1) = \mathbb{P}(\sigma_{\mathcal{C}} = 1) = 1/2$.
- $(\sigma_{\mathcal{C}(x)} \sqrt{2L(\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_u)})_{x \in \widetilde{\mathcal{G}}} \overset{(d)}{=} \widetilde{\phi}_u$.

Clusters of $\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_u$ exactly connected components of $\{\tilde{\phi}_u \neq 0\}$.

First passage sets (FPS) of a metric graph GFF

$\tilde{\phi}_0$ metric graph GFF on $\widetilde{\mathcal{G}}$ with b.c. 0 on $\partial\mathcal{G}$.

First passage set of level $-a \leq \inf u$:

$$\widetilde{\mathbb{A}}_{-a} := \{x \in \widetilde{\mathcal{G}} | \exists\gamma \text{ path } x \overset{\gamma}{\leftrightarrow} \partial\mathcal{G}, \tilde{\phi}_0 \geq -a \text{ on } \gamma\}.$$

Stopping set. Analogue of a first passage bridge of a Brownian motion.

In the signed isomorphism of $\tilde{\phi}_0$ with $\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_a$,
$\widetilde{\mathbb{A}}_0$ is the union of topological closures $\overline{\mathcal{C}}$ of clusters $\mathcal{C}$ of $\widetilde{\mathcal{L}}_{1/2} \cup \widetilde{E}_a$ that contain at least an excursion $(\widetilde{E}_a)$.

Metric graph approximation and isomorphism

Proposition

Metric graph FPS converge in the fine mesh limit to continuum FPS.

Corollary

$a > 0$. *2 indep. P.P.P of 2D Brownian loops and excursions:* $\mathcal{L}_{1/2}$ *and* E_a. *One can couple* Φ *and* $\mathcal{L}_{1/2} \cup E_a$ *on the same probability space such that*

- $L^{\text{ctr}}(\mathcal{L}_{1/2}) + L(E_a) = \frac{1}{2} :\Phi^2: + a\Phi + \frac{1}{2}a^2$,
- *The FPS* $\mathbb{A}_{-a}$ *is the union of topological closures of all clusters of* $\mathcal{L}_{1/2} \cup E_a$ *that contain at least an excursion (*E_a*).*

Corollary

The clusters in a critical 2D Brownian loop-soup $\mathcal{L}_{1/2}$ *have a non-trivial Minkowski content in gauge* $|\log(r)|^{\frac{1}{2}} r^2$

The gauge for clusters of a subcritical Brownian loop-soup is unknown.

Thank you for your attention!