Random Trees via Conformal Welding

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- Joint work with Steffen Rohde

Conformal embeddings of triangulations

Let G be a triangulation of S^2.

Glue together equilateral triangles according to G to get a Riemann surface S which is topologically equivalent to S^2.

By uniformization, there is a unique$^{\text{(up to Möbius transformation)}}$ conformal isomorphism $\varphi : S \to \hat{\mathbb{C}}$.

Conformal embeddings of triangulations

The previous procedure gives a canonical way to embed G into $\hat{\mathbb{C}}$.

The measure structure on G pushes forward to a measure structure on $\hat{\mathbb{C}}$ via this embedding.

Problem: Let G be a uniform random triangulation with n faces. Prove that as $n \to \infty$ the measure converges (to LQG).

Obstacle: Prove that the diameter of embedded triangles goes to 0 as $n \to \infty$.

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Related: Gwynne, Miller, Sheffield (2017) have shown that Tutte embedding of mated-CRT map converges to LQG.

Conformal embedding of trees

For the rest of this talk: consider the case when G arises from a large random tree.

Let T_n a plane tree with n edges.

We can view T_n as a random triangulation of S^2.

Combining this with the previous procedure gives a way of canonically embedding the tree in $\hat{\mathbb{C}}$.

Examples of Conformal embedding of trees

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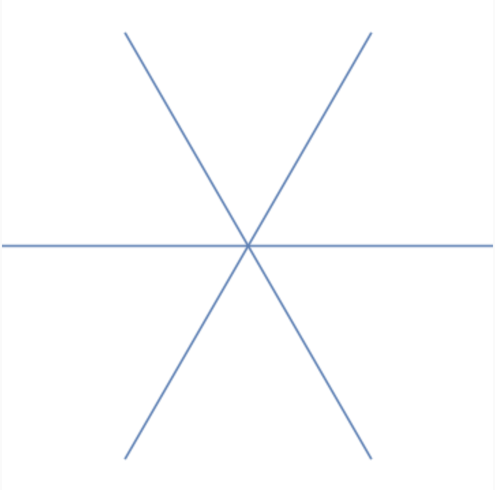
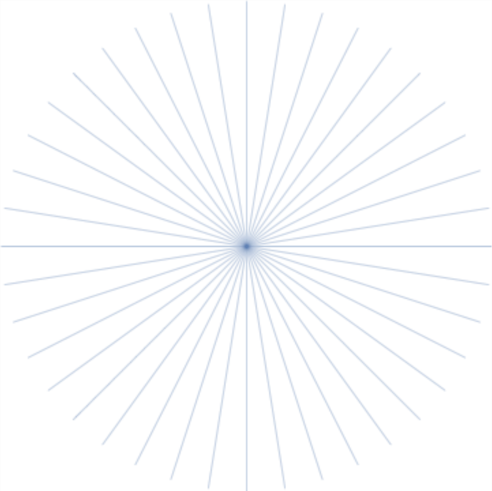
Degree 6 star:



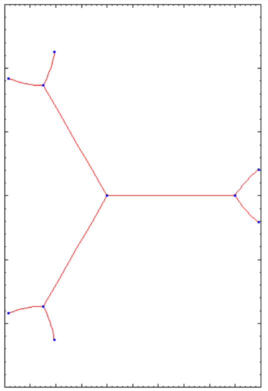
Examples of Conformal embedding of trees

Degree 80 star:



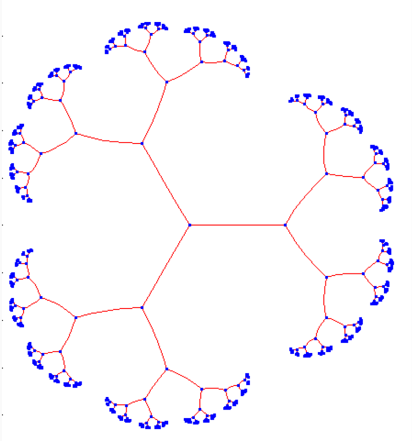
Examples of Conformal embedding of trees

Trinary tree of depth 2:



Examples of Conformal embedding of trees

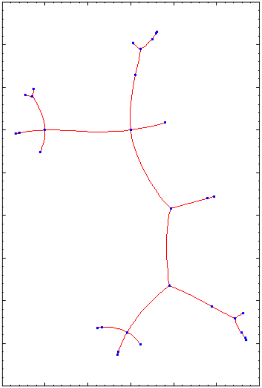
Trinary tree of depth 9:



Donald Marshall's zipper software

Examples of Conformal embedding of trees

Tree with 30 edges:



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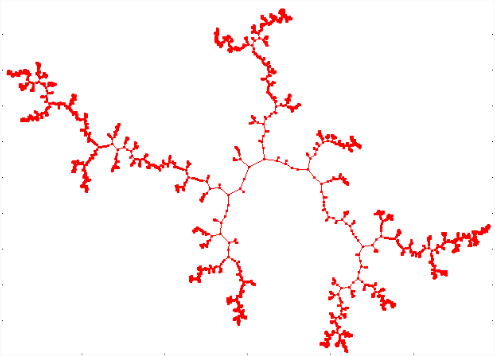
Examples of Conformal embedding of trees

Tree with 13000 edges:



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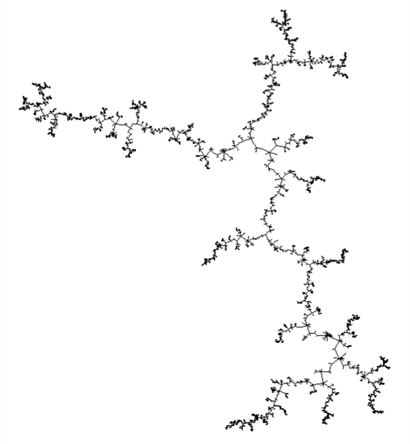
Examples of Conformal embedding of trees

Tree with 50000 edges:



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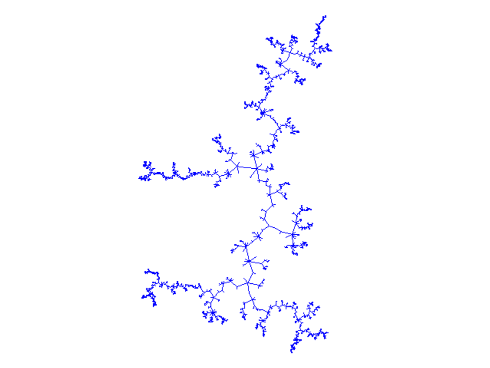
Examples of Conformal embedding of trees

Tree with 30000 edges:



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Convergence of conformally embedded trees

Let T_n be a uniformly random tree with n edges.

Does the embedding converge as $n \to \infty$?

Convergence of conformally embedded trees

Let $\varphi_n : \bigsqcup_{i=1}^{2n} \Delta / \sim \to \hat{\mathbb{C}}$ be the embedding of T_n into $\hat{\mathbb{C}}$, normalized so that $\varphi_n(\infty) = \infty$ and $\varphi_n'(\infty) = 1$.

Let f_n be the conformal map from $\hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ to $\hat{\mathbb{C}} \backslash \varphi_n(T_n)$, normalized in the same way.

Theorem (L, Rohde)
As $n \to \infty$, f_n converges in distribution with respect to the sup norm on $C(\hat{\mathbb{C}} \backslash \mathbb{D})$.

In particular the size of the edges goes to 0 as $n \to \infty$.

Convergence of conformally embedded trees

Proof sketch

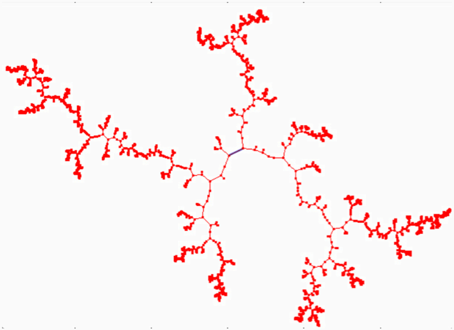
Proof Sketch

Want to show: Size of edges goes to 0.

We do this by showing that w.h.p, there are **lots** of **thick** annuli surrounding the edge.

How to construct the annuli

Fix $\lambda > 1$.



$\mathcal{T}$ = random tree equipped with graph distance.

Let X_k be the points on $\mathcal{T}$ which are distance λ^k from the root.

Join these points with geodesics in the complement of the tree.

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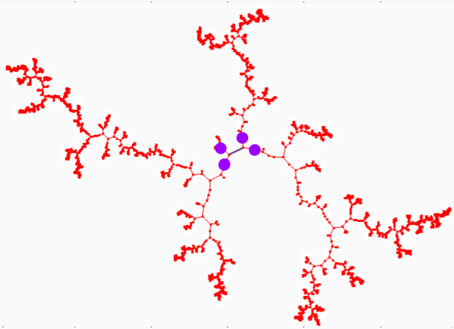

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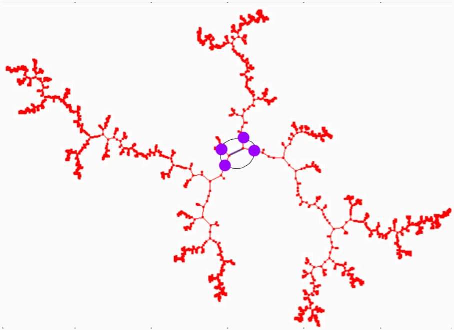

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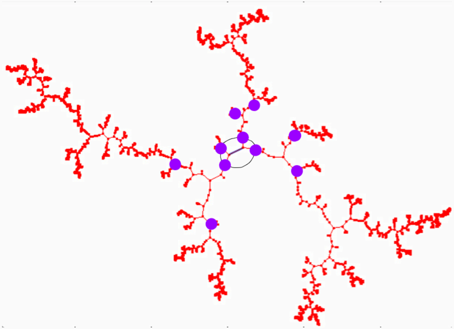

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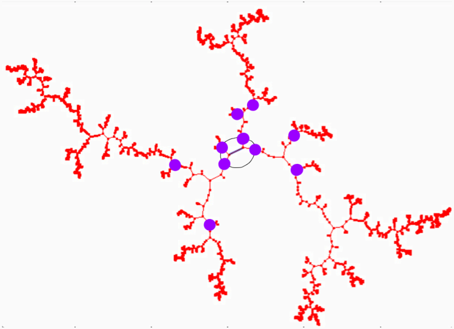

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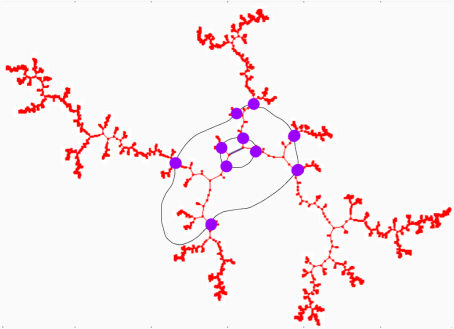

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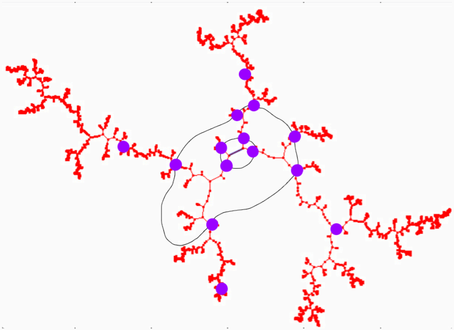

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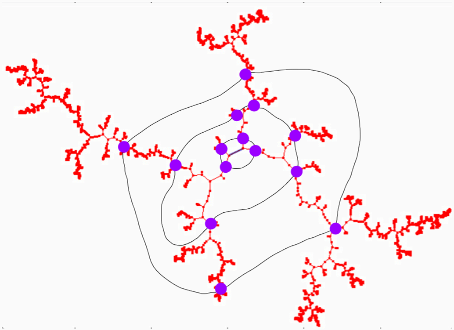

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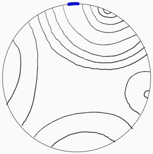
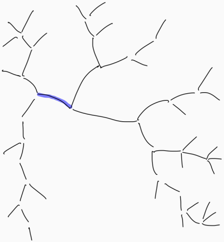
Need to find lots of thick annuli to bound size of edge (blue).





Create annuli by joining points by geodesics (red).

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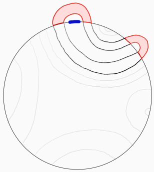

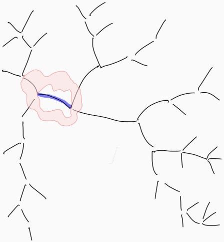

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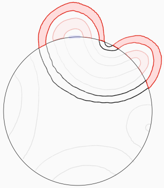

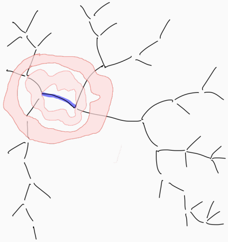

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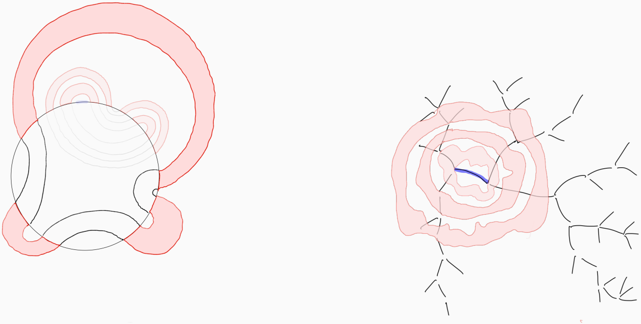

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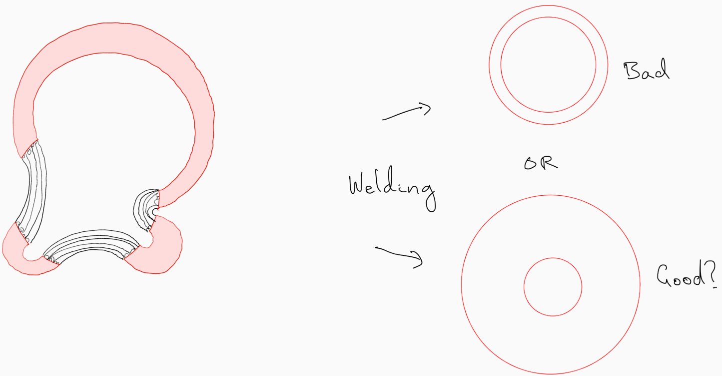
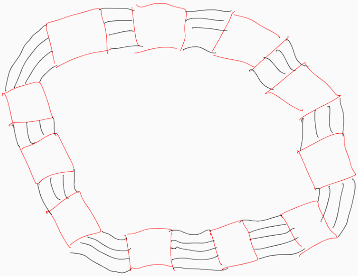
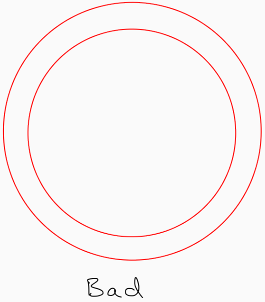
Wishlist to make thick annuli



Conditions to ensure annulus is thick after welding?

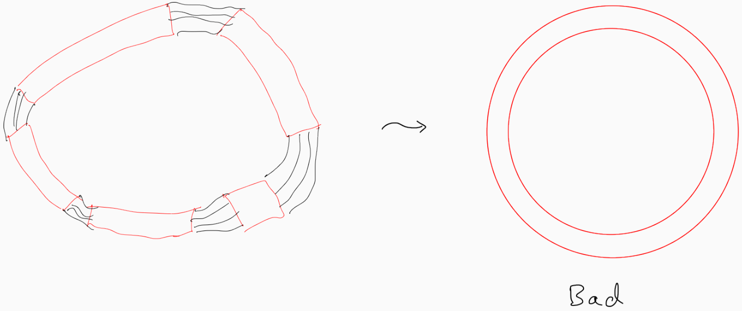
Wishlist to make thick annuli





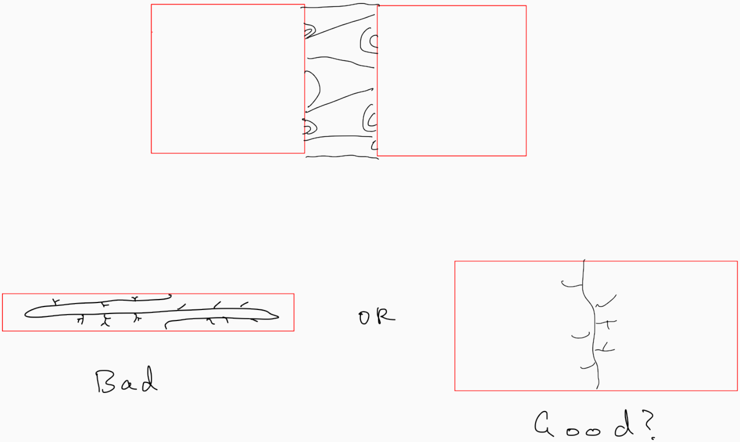
1. Need bounded number of rectangles.

Wishlist to make thick annuli



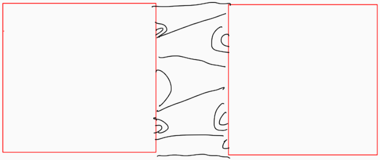
2. Need bounded geometry for rectangles.

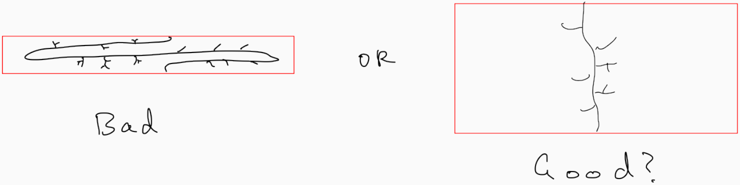
Wishlist to make thick annuli



3. Need to know *something* about the welding map.

How to ensure that rectangle is still thick after welding

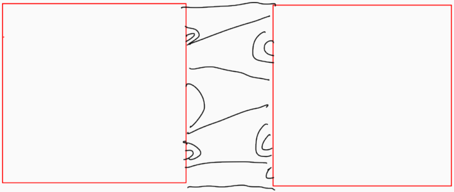




3. Need to know *something* about the equivalence relation.

If we can control the *quality* of the equivalence relation on a *large* subset of the edge then we get control on the modulus.

How to ensure that rectangle is still thick after welding



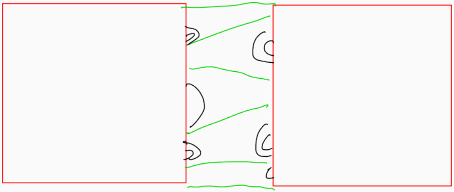
If we can control the *quality* of the equivalence relation on a *large* subset of the edge then we get control on the modulus.

It's not enough that the sets on each side are large.

The equivalence relation should also behave nicely.

We know that quasisymmetry is good enough, however, in this random setting, there is no hope of getting quasisymmetry. Need something more flexible.

How to ensure that rectangle is still thick after welding



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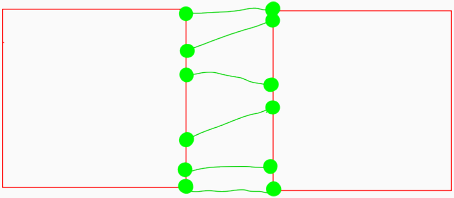

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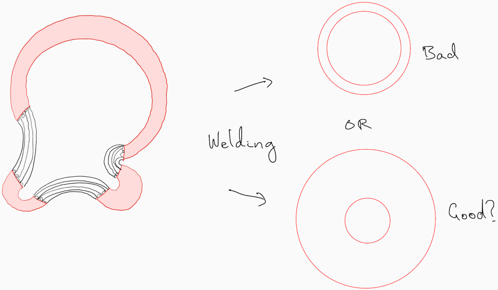


To get thick annuli, want

1. Bounded number of rectangles
2. Bounded geometry rectangles
3. Control over $\sim$ on interface

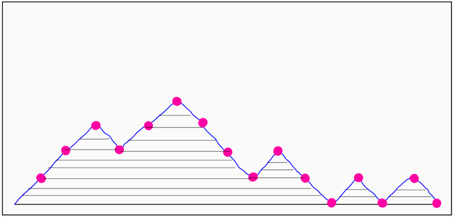
Recap: how we show that an edge is small

For each edge, we

- Construct annuli by drawing circles of radius λ^k.
- Hope that many of the annuli we encounter satisfies the following wishlist:
    1. Bounded number of rectangles
    2. Bounded geometry rectangles
    3. Control over regularity of $\sim$ on the interface

To analyze this process and get the desired probabilistic bounds, it is helpful to translate everything to the excursion picture.

Excursions $\leftrightarrow$ tree



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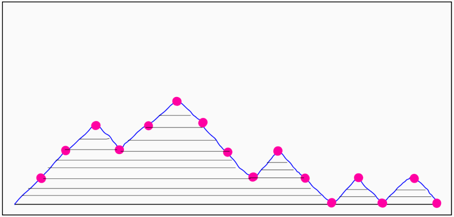

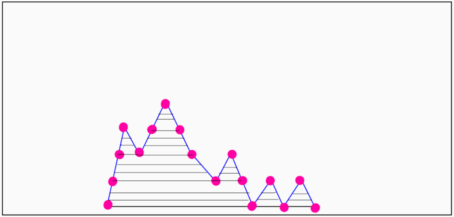

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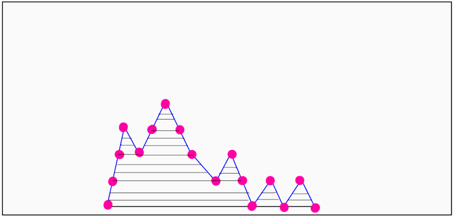

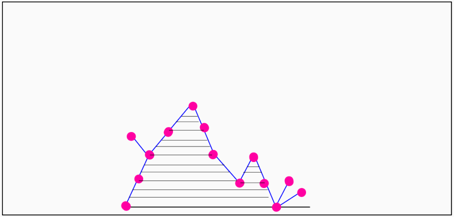

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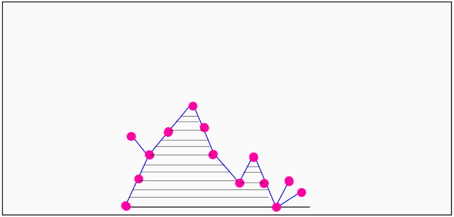

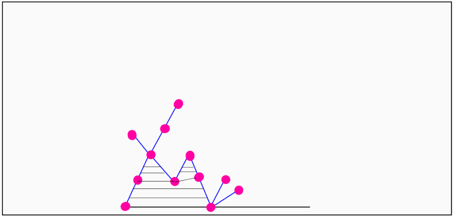

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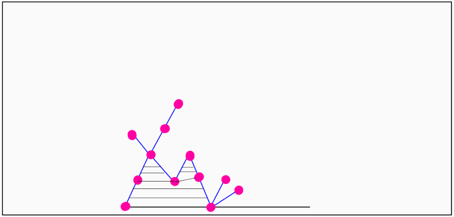

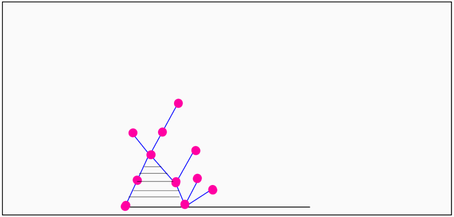

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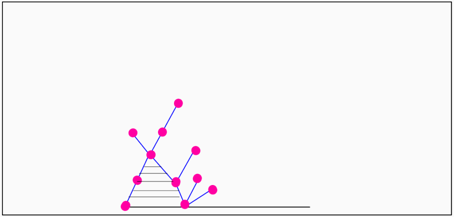

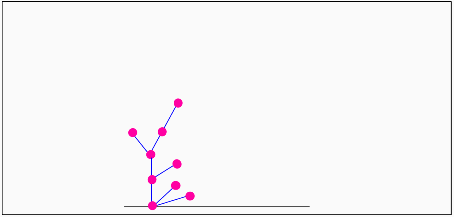

Translating to excursion picture:

Construct annuli by drawing circles of radius $\lambda^k \leftrightarrow$ Construct annuli by exploring upwards from $y = 0$: Look at excursions away from height λ^k.

1. Bounded number of rectangles $\leftrightarrow$ Bounded number of excursions away from λ^k.
2. Bounded geometry rectangles $\leftrightarrow$ Bounded ratios for excursion interval lengths.
3. Control over regularity of $\sim$ on the interface $\leftrightarrow$ control over Hölder regularity of excursion.

Translating to excursion picture:

Construct annuli by exploring upwards from $y = 0$: Look at excursions away from λ^k that reach λ^{k+1}.

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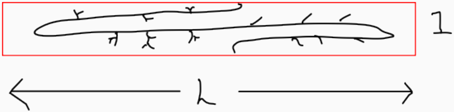
This exploration process is a Markov chain.

The 'wishlist' corresponds to a subset of the state space.

In the rest of the proof, we get large deviations estimates on the amount of time that the Markov chain spends in the good part of the state space.

Modulus and Gluing of rectangles



Lemma

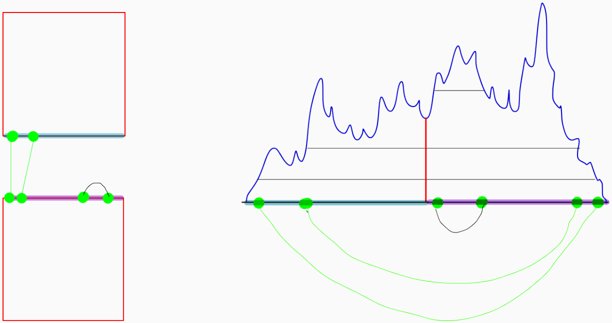
We have

$$L \lesssim \inf_{\mu^-,\mu^+} \mathcal{E}(\mu^-) + \mathcal{E}(\mu^+).$$

- μ^- ranges over probability measures on $I^- \cap \operatorname{supp}(\sim)$
- μ^+ ranges over probability measures on $I^+ \cap \operatorname{supp}(\sim)$
- μ^- and μ^+ must be be *equivalent* with respect to $\sim$.
- Energy:

$$\mathcal{E}(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y).$$

How to use this lemma for our problem:



Consider a toy model where we glue two squares together using the equivalence relation from a random excursion.

What should we take for μ^- and μ^+?

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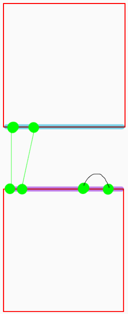
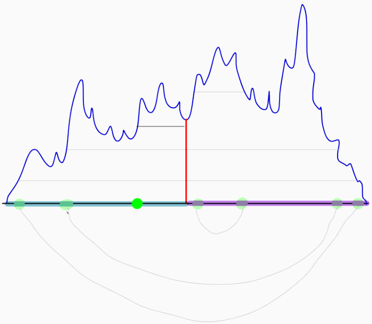


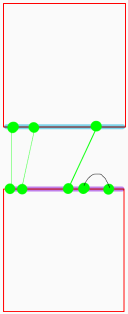


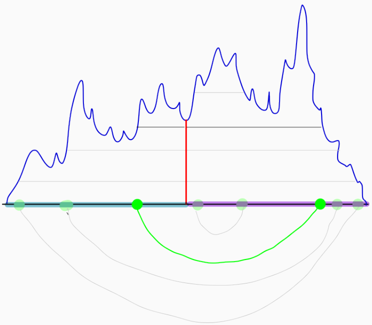
Consider a toy model where we glue two squares together using the equivalence relation from a random excursion.

What should we take for μ^- and μ^+?

Notice that the support of $\sim$ is exactly the images of the left sided and right sided inverse map of the excursion $\mathbb{e}$ on $[0, \mathbb{e}(1/2)]$.

How to use this lemma for our problem:



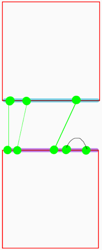


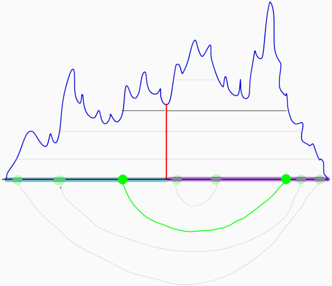
Consider a toy model where we glue two squares together using the equivalence relation from a random excursion.

What should we take for μ^- and μ^+?

Notice that the support of $\sim$ is exactly the images of the left-sided and right-sided inverse map of the excursion $\mathbb{e}$ on $[0, \mathbb{e}(1/2)]$.

How to use this lemma for our problem:





Thus we should take μ^-, μ^+ to be pullback of Lebesgue measure on $[0, \mathbb{e}(1/2)]$ via $\mathbb{e}$.

This demonstrates that the modulus L is controlled by the Hölder regularity of $\mathbb{e}$.

Thank you!