Aalto University
School of Science
and Technology

Conformal field theory on the lattice: from discrete complex analysis to Virasoro algebra

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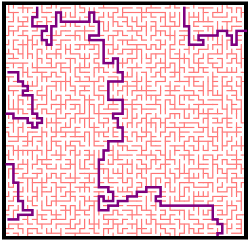
Outline

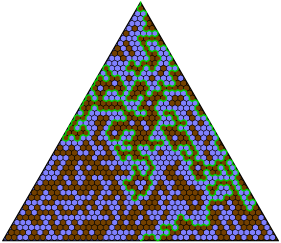
1. Introduction: Conformal Field Theory and Virasoro algebra
2. Main results: local fields of probabilistic lattice models form Virasoro representations
   - discrete Gaussian free field
   - Ising model
3. An algebraic theme and variations (Sugawara construction)
4. Proof steps (discrete complex analysis)

1. INTRODUCTION

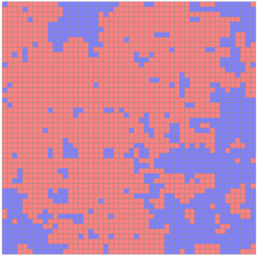
Intro: Two-dimensional statistical physics



(uniform spanning tree)



(percolation)



(Ising model)

etc. etc.

Intro: Conformally invariant scaling limits

Conventional wisdom: Any interesting scaling limit of any two-dimensional random lattice model is conformally invariant:

- *interfaces* $\longrightarrow$ SLE-type random curves
- *correlations* $\longrightarrow$ CFT correlation functions

Remarks:

- SLE: Schramm-Loewner Evolution
  - [cf. the other talks]
- CFT: Conformal Field Theory
  - powerful algebraic structures
    (Virasoro algebra, modular invariance, quantum groups, ...)
  - exact solvability (critical exponents, PDEs for correlation fns, ...)
  - mysteries — what is CFT, really?
- This talk: concrete probabilistic role for Virasoro algebra

Intro: The role of Virasoro algebra

Virasoro algebra: ∞-dim. Lie algebra, basis L_n ($n \in \mathbb{Z}$) and C

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{n^3-n}{12}\delta_{n+m,0}C$$
$$[C, L_n] = 0$$

(C a central element)

Role of Virasoro algebra in CFT?

- stress tensor T: first order response to variation of metric

  (in particular "infinitesimal conformal transformations")

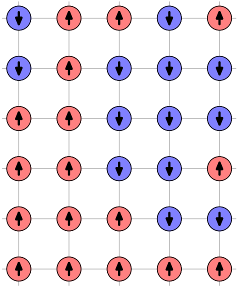
- Laurent modes of stress tensor $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-2-n}$
- C acts as $c \times \mathrm{id}$, with $c \in \mathbb{R}$ the "central charge" of the CFT
- action on local fields (effect of variation of metric on correlations)
  - local fields form a Virasoro representation
  - highest weights of the representation $\rightsquigarrow$ critical exponents
  - degenerate representations $\rightsquigarrow$ PDEs for correlations

(exact solvability & classification)

II. Local fields in lattice models

The critical Ising model on $\mathbb{Z}^2$

- domain $\Omega \subsetneq \mathbb{C}$ open, 1-connected
- $\delta > 0$ small mesh size
- lattice approximation $\Omega_\delta \subset \mathbb{C}_\delta := \delta\mathbb{Z}^2$



Ising model: random spin configuration

$$\sigma = \big(\sigma_z\big)_{z \in \mathbb{C}_\delta} \in \{+1, -1\}^{\mathbb{C}_\delta}$$

$$\sigma\big|_{\mathbb{C}_\delta \setminus \Omega_\delta} \equiv +1 \qquad \text{(plus-boundary conditions)}$$

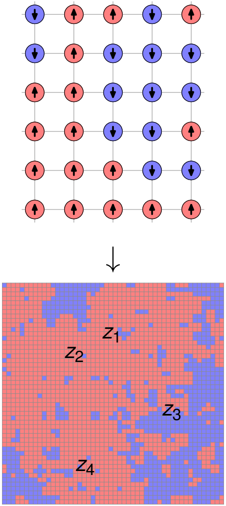
$$\mathsf{P}\big[\{\sigma\}\big] \propto \exp\Big(-\beta\, E(\sigma)\Big) \qquad \text{(Boltzmann-Gibbs)}$$

$$E(\sigma) = -\sum_{\|z-w\|=\delta} \sigma_z\, \sigma_w \qquad \text{(energy)}$$

$$\beta = \beta_c = \frac{1}{2}\log\big(\sqrt{2}+1\big) \qquad \text{(critical point)}$$

Celebrated scaling limits of Ising correlations

$\phi \colon \Omega \to \mathbb{H} = \{ z \in \mathbb{C} \mid \Im\mathfrak{m}(z) > 0 \}$ conformal map



Thm [Chelkak & Hongler & Izyurov, Ann. Math. 2015]

$$\lim_{\delta \to 0} \frac{1}{\delta^{k/8}} \mathsf{E}\Big[\prod_{j=1}^{k} \sigma_{z_j} \Big] = \prod_{j=1}^{k} |\phi'(z_j)|^{1/8} \times \mathcal{C}_k\big(\phi(z_1), \ldots, \phi(z_k)\big)$$

Thm [Hongler & Smirnov, Acta Math. 2013] [Hongler, 2011]

$$\lim_{\delta \to 0} \frac{1}{\delta^{m}} \mathsf{E}\Bigg[\prod_{j=1}^{m} \Big(-\sigma_{z_j}\sigma_{z_j+\delta} + \frac{1}{\sqrt{2}} \Big) \Bigg] = \prod_{j=1}^{m} |\phi'(z_j)| \times \mathcal{E}_m\big(\phi(z_1), \ldots, \phi(z_m)\big)$$

\+ [Gheissari & Hongler & Park, 2013 — Sung Chul's talk]
\+ [Chelkak & Hongler & Izyurov, 2018+ — Kostya's talk]

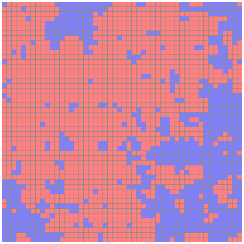
Local fields of the Ising model

Local fields $\mathfrak{F}(z)$ of Ising

- $V \subset \mathbb{Z}^2$ finite subset
- $P \colon \{+1,-1\}^V \to \mathbb{C}$ a function
- $\mathfrak{F}(z) = P\big((\sigma_{z+\delta x})_{x \in V}\big)$

$\rightsquigarrow$ $\mathcal{F}$ space of local fields

$\sigma = \big(\sigma_z\big)_{z \in \Omega_\delta}$ Ising



Null fields: "zero inside correlations"

- $\mathfrak{F}(z)$ null field:
  $\exists R > 0$ s.t. $\mathsf{E}\Big[\mathfrak{F}(z)\ \prod_{j=1}^{n} \sigma_{w_j}\Big] = 0$
  whenever $\|z - w_j\|_1 > R\delta \ \ \forall j$

$\rightsquigarrow$ $\mathcal{N} \subset \mathcal{F}$ space of null fields

Examples of local fields:

* $\mathfrak{F}(z) = \sigma_z$ (spin)
* $\mathfrak{F}(z) = -\sigma_z\,\sigma_{z+\delta}$ (energy)

$\mathcal{F}/\mathcal{N}$ — equivalence classes of local fields, "same correlations"

Main result 1: Virasoro action on Ising local fields

Theorem (Hongler & K. & Viklund, 2017)

The space $\mathcal{F}/\mathcal{N}$ of correlation equivalence classes of local fields of the critical Ising model on $\mathbb{Z}^2$ forms a representation of the Virasoro algebra with central charge $c = \frac{1}{2}$.

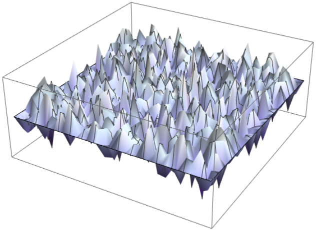
Discrete Gaussian Free Field on $\mathbb{Z}^2$

Discrete Gaussian Free Field (dGFF):

$$\Phi = \big(\Phi(z)\big)_{z \in \Omega_\delta}$$

Domain and discretization:

- $\Omega \subsetneq \mathbb{C}$ open, simply connected
- lattice approximation: $\Omega_\delta \subset \mathbb{C}_\delta := \delta\mathbb{Z}^2$



- centered Gaussian field on vertices of discrete domain Ω_δ

$$p(\phi) \propto \exp\Big(-\frac{1}{16\pi} E(\phi) \Big) \qquad \text{probability density}$$

$$E(\phi) = \sum_{\|z-w\|=\delta} \big(\phi(z) - \phi(w)\big)^2 \qquad \text{“Dirichlet energy”}$$

Local fields of the dGFF

Local fields $\mathfrak{F}(z)$ of dGFF

- $V \subset \mathbb{Z}^2$ finite subset
- $P \colon \mathbb{R}^V \to \mathbb{C}$ polynomial function
- $\mathfrak{F}(z) = P\big((\Phi(z+\delta x))_{x \in V}\big)$

$\rightsquigarrow$ $\mathcal{F}$ space of local fields

$$\Phi = \big(\Phi(z)\big)_{z \in \Omega_\delta} \qquad \text{dGFF}$$

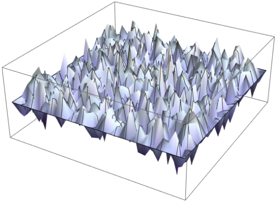
Examples of local fields:

* $\mathfrak{F}(z) = \Phi(z)$
* $\mathfrak{F}(z) = \frac{1}{2}\Phi(z+\delta) - \frac{1}{2}\Phi(z-\delta)$
* $\mathfrak{F}(z) = 361\,\Phi(z)^3$

Null fields: “zero inside correlations”

- $\mathfrak{F}(z)$ null field:
  $\exists R > 0$ s.t. $\mathsf{E}\Big[\mathfrak{F}(z)\ \prod_{j=1}^{n} \Phi(w_j)\Big] = 0$
  whenever $\|z - w_j\|_1 > R\delta \ \ \forall j$

$\rightsquigarrow$ $\mathcal{N} \subset \mathcal{F}$ space of null fields



$\mathcal{F}/\mathcal{N}$ — equivalence classes of local fields, “same correlations”

Main result 2: Virasoro action on dGFF local fields

Theorem (Hongler & K. & Viklund, 2017)

The space $\mathcal{F}/\mathcal{N}$ of correlation equivalence classes of local fields of the discrete Gaussian free field on $\mathbb{Z}^2$ forms a representation of the Virasoro algebra with central charge $c = 1$.

III. An algebraic theme and variations (Sugawara construction)

Bosonic Sugawara construction

commutator $[A, B] \; := \; A \circ B - B \circ A$

Proposition (bosonic Sugawara construction)

Suppose:

- V vector space and $\mathfrak{a}_j \colon V \to V$ linear for each $j \in \mathbb{Z}$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \quad : \quad j \geq N \implies \mathfrak{a}_j \, v = 0$
- $[\mathfrak{a}_i, \mathfrak{a}_j] = i \, \delta_{i+j,0} \, \mathrm{id}_V$

Define:

$$L_n \; := \; \frac{1}{2} \sum_{j<0} \mathfrak{a}_j \circ \mathfrak{a}_{n-j} + \frac{1}{2} \sum_{j \geq 0} \mathfrak{a}_{n-j} \circ \mathfrak{a}_j \qquad \text{for } n \in \mathbb{Z}$$

Then:

- $L_n \colon V \to V$ is well defined
- $[L_n, L_m] = (n-m) \, L_{n+m} + \frac{n^3 - n}{12} \, \delta_{n+m,0} \, \mathrm{id}_V$

$\therefore \quad V$ Virasoro representation, central charge $c = 1$

Fermionic Sugawara construction 1

commutator $[A, B] := A \circ B - B \circ A$

anticommutator $[A, B]_+ := A \circ B + B \circ A$

Proposition (fermionic Sugawara, Neveu-Schwarz sector)

Suppose:

- V vector space, $\mathfrak{b}_k \colon V \to V$ linear for each $k \in \mathbb{Z} + \frac{1}{2}$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \quad : \quad k \geq N \implies \mathfrak{b}_k \, v = 0$
- $[\mathfrak{b}_k, \mathfrak{b}_\ell]_+ = \delta_{k+\ell,0} \, \mathrm{id}_V$

Def.:
$$L_n := \frac{1}{2} \sum_{k>0} \Big(\frac{1}{2} + k\Big) \mathfrak{b}_{n-k} \mathfrak{b}_k - \frac{1}{2} \sum_{k<0} \Big(\frac{1}{2} + k\Big) \mathfrak{b}_k \mathfrak{b}_{n-k} \quad (n \in \mathbb{Z})$$

Then:

- $L_n \colon V \to V$ is well defined
- $[L_n, L_m] = (n - m) \, L_{n+m} + \frac{n^3 - n}{24} \, \delta_{n+m,0} \, \mathrm{id}_V$

$\therefore \quad V$ Virasoro representation, central charge $c = \dfrac{1}{2}$

Fermionic Sugawara construction 2

commutator $[A, B] := A \circ B - B \circ A$

anticommutator $[A, B]_+ := A \circ B + B \circ A$

Proposition (fermionic Sugawara, Ramond sector)

Suppose:

- V vector space, $\mathfrak{b}_j \colon V \to V$ linear for each $j \in \mathbb{Z}$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \; : \; j \geq N \implies \mathfrak{b}_j v = 0$
- $[\mathfrak{b}_i, \mathfrak{b}_j]_+ = \delta_{i+j,0} \, \mathrm{id}_V$

Def.:

$$L_n := \frac{1}{2} \sum_{j \geq 0} \Big(\frac{1}{2} + j\Big) \mathfrak{b}_{n-j} \mathfrak{b}_j - \frac{1}{2} \sum_{j<0} \Big(\frac{1}{2} + j\Big) \mathfrak{b}_j \mathfrak{b}_{n-j} \quad (n \in \mathbb{Z} \setminus \{0\})$$

$$L_0 := \frac{1}{2} \sum_{j>0} j \, \mathfrak{b}_{-j} \mathfrak{b}_j + \frac{1}{16} \, \mathrm{id}_V$$

Then:

- $L_n \colon V \to V$ is well defined
- $[L_n, L_m] = (n - m) \, L_{n+m} + \frac{n^3 - n}{24} \, \delta_{n+m,0} \, \mathrm{id}_V$

$\therefore \quad V$ Virasoro representation, central charge $c = \dfrac{1}{2}$

IV. Proof steps (discrete complex analysis)

Outline / steps

For the Ising model and discrete GFF:

✓ 0.) Define model and local fields

1.) Suitable discrete contour integrals and residue calculus

2.) Introduce discrete holomorphic observable

3.) Define Laurent modes of the observable

4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Outline / steps

For the Ising model and discrete GFF:

✓ 0.) Define model and local fields

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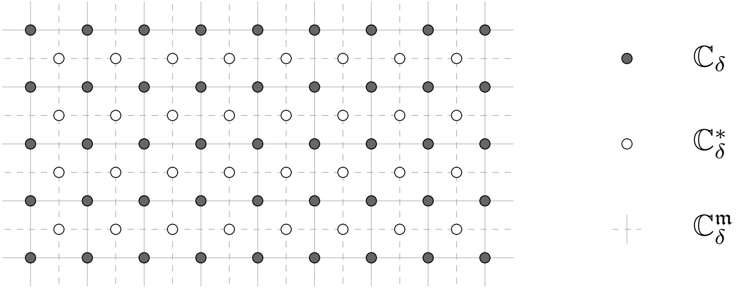
Lattices (square lattice and related lattices)

- fix small mesh size $\delta > 0$

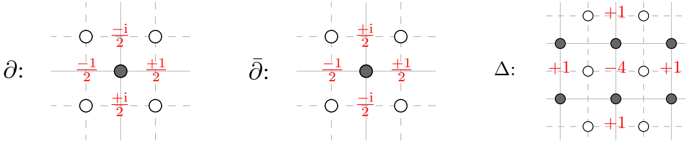


- square lattice $\mathbb{C}_\delta$
- dual lattice $\mathbb{C}_\delta^*$

$$\mathbb{C}_\delta = \delta\mathbb{Z}^2$$

- medial lattice $\mathbb{C}_\delta^{\mathfrak{m}}$
- diamond lattice $\mathbb{C}_\delta^{\diamond}$
- corner lattice $\mathbb{C}_\delta^{\mathfrak{c}}$

Lattices (discretization of differential operators)



- Discrete ∂ and $\bar{\partial}$:

$$\partial_\delta f(z) = \frac{1}{2}\Big(f\big(z+\frac{\delta}{2}\big) - f\big(z-\frac{\delta}{2}\big)\Big) - \frac{\mathbb{i}}{2}\Big(f\big(z+\frac{\mathbb{i}\delta}{2}\big) - f\big(z-\frac{\mathbb{i}\delta}{2}\big)\Big)$$

$$\bar{\partial}_\delta f(z) = \frac{1}{2}\Big(f\big(z+\frac{\delta}{2}\big) - f\big(z-\frac{\delta}{2}\big)\Big) + \frac{\mathbb{i}}{2}\Big(f\big(z+\frac{\mathbb{i}\delta}{2}\big) - f\big(z-\frac{\mathbb{i}\delta}{2}\big)\Big)$$

$$f \colon \mathbb{C}_\delta^{\mathfrak{m}} \to \mathbb{C} \quad \Rightarrow \quad \partial_\delta f, \bar{\partial}_\delta f \colon \mathbb{C}_\delta^{\diamond} \to \mathbb{C}$$

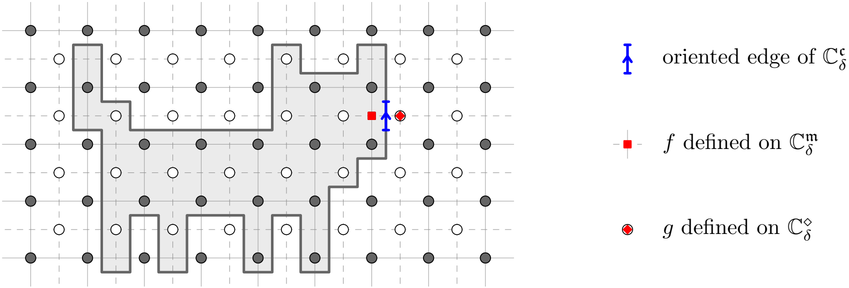
$$f \colon \mathbb{C}_\delta^{\diamond} \to \mathbb{C} \quad \Rightarrow \quad \partial_\delta f, \bar{\partial}_\delta f \colon \mathbb{C}_\delta^{\mathfrak{m}} \to \mathbb{C}$$

- Discrete Laplacian: $(\triangle_\delta = 4\,\bar{\partial}_\delta \partial_\delta = 4\,\partial_\delta \bar{\partial}_\delta)$

$$\triangle_\delta f(z) = f\big(z+\delta\big) + f\big(z-\delta\big) + f\big(z+\mathbb{i}\delta\big) + f\big(z-\mathbb{i}\delta\big) - 4\, f(z)$$

Discrete residue calculus (contour integral)

- two functions $f\colon \mathbb{C}_\delta^{\mathfrak{m}} \to \mathbb{C}$ and $g\colon \mathbb{C}_\delta^{\diamond} \to \mathbb{C}$
- γ path on the corner lattice $\mathbb{C}_\delta^{\mathfrak{c}}$



$$\oint_{[\gamma]} f(z_{\mathfrak{m}})\, g(z_{\diamond})\, [\mathrm{d}z]_\delta := \sum_{\vec{e}\in\gamma} f(e_{\mathfrak{m}})\, g(e_{\diamond}) \cdot \vec{e}$$

Discrete residue calculus (properties)

$$\oint_{[\gamma]} f(z_{\mathfrak{m}})\, g(z_{\diamond})\, [\mathrm{d}z]_\delta := \sum_{\vec{e}\in\gamma} f(e_{\mathfrak{m}})\, g(e_{\diamond}) \cdot \vec{e}$$

Proposition (properties of discrete contour integral)

- Green's formula (sum over $w_{\mathfrak{m}} \in \mathbb{C}_\delta^{\mathfrak{m}} \cap \mathrm{int}(\gamma)$ and $w_\diamond \in \mathbb{C}_\delta^{\diamond} \cap \mathrm{int}(\gamma)$)

$$\oint_{[\gamma]} f(z_m)\, g(z_\diamond)\, [\mathrm{d}z]_\delta = \mathbb{i} \sum_{w_{\mathfrak{m}}} f(w_{\mathfrak{m}})\, (\bar{\partial}_\delta g)(w_{\mathfrak{m}}) + \mathbb{i} \sum_{w_\diamond} (\bar{\partial}_\delta f)(w_\diamond)\, g(w_\diamond)$$

- contour deformation

γ_1, γ_2 two counterclockwise closed contours on $\mathbb{C}_\delta^{\mathfrak{c}}$
$\bar{\partial}_\delta f \equiv 0$ and $\bar{\partial}_\delta g \equiv 0$ on symm. diff. $\mathrm{int}(\gamma_1) \oplus \mathrm{int}(\gamma_2)$

$$\oint_{[\gamma_1]} f(z_m)\, g(z_\diamond)\, [\mathrm{d}z]_\delta = \oint_{[\gamma_2]} f(z_m)\, g(z_\diamond)\, [\mathrm{d}z]_\delta$$

- integration by parts

γ counterclockwise closed contour on $\mathbb{C}_\delta^{\mathfrak{c}}$
$\bar{\partial}_\delta f \equiv 0$ and $\bar{\partial}_\delta g \equiv 0$ on neighbors of γ

$$\oint_{[\gamma]} (\partial_\delta f)(z_m)\, g(z_\diamond)\, [\mathrm{d}z]_\delta = -\oint_{[\gamma]} f(z_m)\, (\partial_\delta g)(z_\diamond)\, [\mathrm{d}z]_\delta$$

Discrete monomial functions (defining properties)

Proposition (discrete monomial functions)

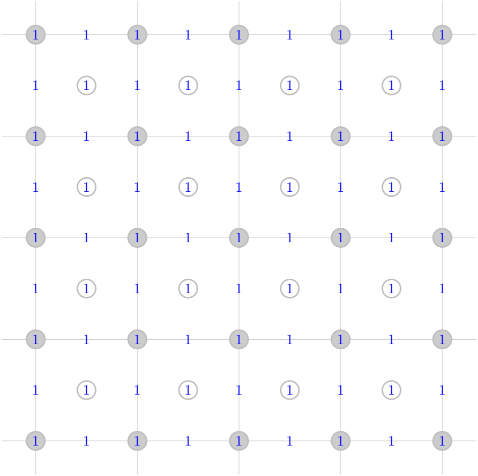
$\exists$ functions $z \mapsto z^{[p]}$, $p \in \mathbb{Z}$, defined on $\mathbb{C}^{\diamond}_{\delta} \cup \mathbb{C}^{\mathfrak{m}}_{\delta}$, such that

- $\bar{\partial}_{\delta} z^{[p]} = 0$ whenever ... "discrete holomorphicity"
  - $p \geq 0$ and $z \in \mathbb{C}^{\diamond}_{\delta} \cup \mathbb{C}^{\mathfrak{m}}_{\delta}$
  - $p < 0$ and $z \in \mathbb{C}^{\diamond}_{\delta} \cup \mathbb{C}^{\mathfrak{m}}_{\delta}$, $\|z\|_1 > R_p \, \delta$
- $z^{[0]} \equiv 1$ for all $z \in \mathbb{C}^{\diamond}_{\delta} \cup \mathbb{C}^{\mathfrak{m}}_{\delta}$ "constant function"
- $\bar{\partial}_{\delta} z^{[-1]} = 2\pi \, \delta_{z,0} + \frac{\pi}{2} \sum_{x \in \{\pm \frac{\delta}{2}, \pm \mathbb{i} \frac{\delta}{2}\}} \delta_{z,x}$ "$\bar{\partial}$ Green's function"
- $\partial_{\delta} z^{[p]} = p \, z^{[p-1]}$ "derivatives"
- $z^{[p]}$ has the same 90° rotation symmetry as z^p "symmetry"
- for $p < 0$ we have $z^{[p]} \to 0$ as $\|z\| \to \infty$ "decay"
- for any z there exists D_z such that $z^{[p]} = 0$ for $p \geq D_z$ "truncation"

For γ large enough counterclockwise closed contour surrounding the origin...

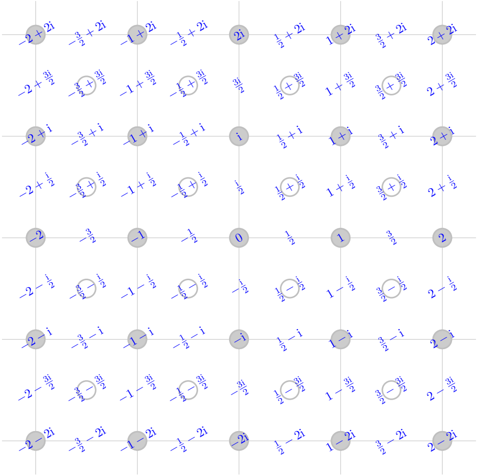
- $\oint_{[\gamma]} z_{\mathfrak{m}}^{[p]} \, z_{\diamond}^{[q]} \, [\mathrm{d}z]_{\delta} = 2\pi \mathbb{i} \; \delta_{p+q,-1}$ "residue calculus"
- $\oint_{[\gamma]} z_{\mathfrak{m}}^{\{p\}} \, z_{\diamond}^{[q]} \, [\mathrm{d}z]_{\delta} = 2\pi \mathbb{i} \; \delta_{p+q,-1}$ where $z_{\mathfrak{m}}^{\{p\}} = \frac{1}{4} \sum_{x \in \{\pm \frac{\delta}{2}, \pm \mathbb{i} \frac{\delta}{2}\}} (z_{\mathfrak{m}} - x)^{[p]}$

Discrete monomial functions (example 0)



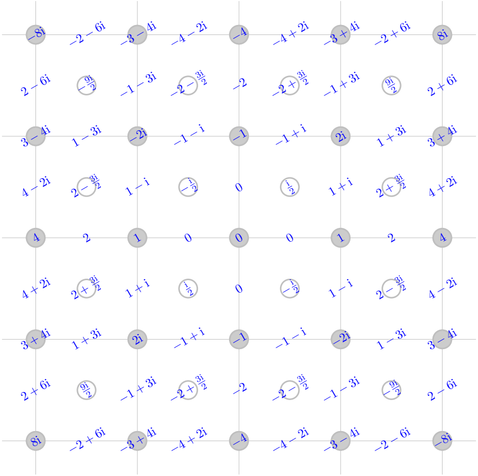
values of $z^{[0]}$

Discrete monomial functions (example 1)



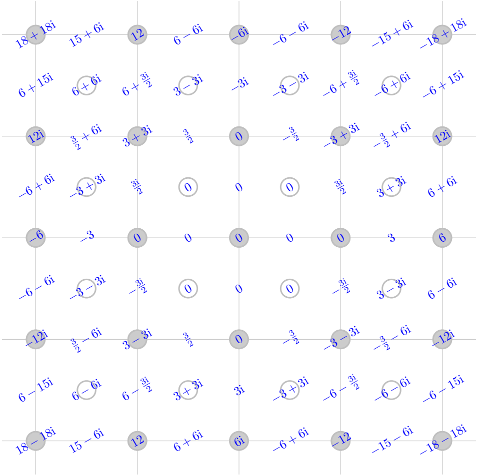
values of $z^{[1]}$

Discrete monomial functions (example 2)



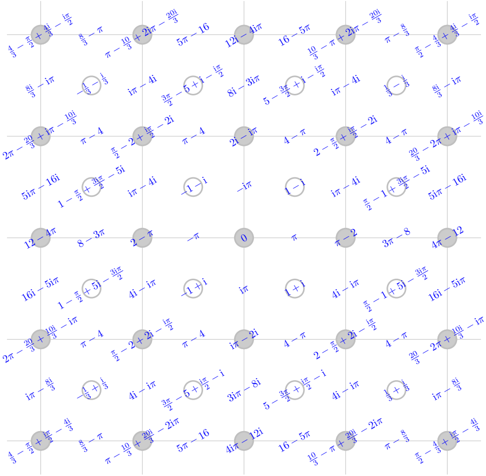
values of $z^{[2]}$

Discrete monomial functions (example 3)



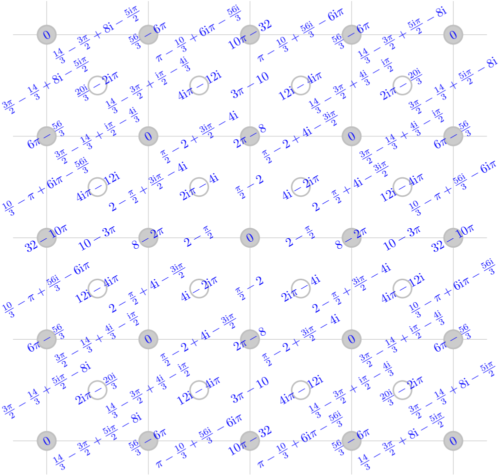
values of $z^{[3]}$

Discrete monomial functions (example -1)



values of $z^{[-1]}$

Discrete monomial functions (example -2)



values of $z^{[-2]}$

Outline / steps

For the ~~Ising model and~~ discrete GFF:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

2.) Introduce discrete holomorphic observable

3.) Define Laurent modes of the observable

4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

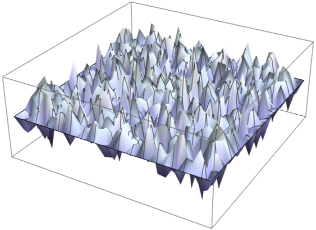
Discrete Gaussian Free Field (definition again)

Discrete Gaussian Free Field (dGFF):

$$\Phi = \big(\Phi(z)\big)_{z \in \Omega_\delta}$$

Domain and discretization:

- $\Omega \subsetneq \mathbb{C}$ open, simply connected
- lattice approximation
  $\Omega_\delta \subset \mathbb{C}_\delta, \; \Omega_\delta^\diamond \subset \mathbb{C}_\delta^\diamond, \; \Omega_\delta^\mathfrak{m} \subset \mathbb{C}_\delta^\mathfrak{m}, \; \Omega_\delta^\mathfrak{c} \subset \mathbb{C}_\delta^\mathfrak{c}$



- centered Gaussian field on vertices of discrete domain Ω_δ
- probability density $p(\phi) \propto \exp\big(-\frac{1}{16\pi} E(\phi)\big)$
  - $E(\phi) = \sum_{z \sim w} \big(\phi(z) - \phi(w)\big)^2$ "Dirichlet energy"
- covariance $\mathsf{E}\big[\Phi(z)\,\Phi(w)\big] = 8\pi\, G_{\Omega_\delta}(z, w)$
  - $G_{\Omega_\delta}(z, w) =$ expected time at w for random walk from z before exiting Ω_δ
  - $\triangle_\delta G(\cdot, w) = -\delta_w(\cdot)$ "$\triangle$ Green's function"

Local fields of the dGFF

$$\Phi = \big(\Phi(z)\big)_{z\in\Omega_\delta} \qquad \text{dGFF}$$

Local fields $\mathfrak{F}(z)$ of dGFF

- $V \subset \mathbb{Z}^2$ finite subset
- $P : \mathbb{R}^V \to \mathbb{C}$ polynomial function
- $\mathfrak{F}(z) = P\big((\Phi(z+\delta x))_{x\in V}\big)$
  (makes sense when Ω_δ is large enough)

$\rightsquigarrow$ $\mathcal{F}$ space of local fields

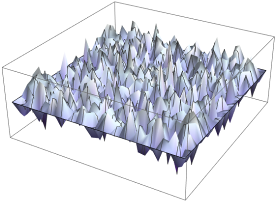
Examples of local fields:

* $\mathfrak{F}(z) = \Phi(z)$
* $\mathfrak{F}(z) = \frac{1}{2}\Phi(z+\delta) - \frac{1}{2}\Phi(z-\delta)$
* $\mathfrak{F}(z) = 361\,\Phi(z)^3$

Null fields: "zero inside correlations"

- $\mathfrak{F}(z)$ is null if for some R
  $\mathsf{E}\Big[\mathfrak{F}(z)\ \prod_{j=1}^{n}\Phi(w_j)\Big] = 0$
  whenever $\|z - w_j\|_1 > R\delta$ for all j

$\rightsquigarrow$ $\mathcal{N} \subset \mathcal{F}$ space of null fields

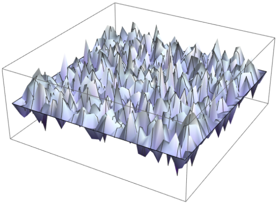


$\mathcal{F}/\mathcal{N}$ equivalence classes of local fields (same correlations)

Discrete holomorphic current (definition)

Discrete Gaussian Free Field (dGFF):

$$\Phi = \big(\Phi(z)\big)_{z \in \mathbb{C}_\delta}$$



- originally defined on $\Omega_\delta \subset \mathbb{C}_\delta$
- extend as zero to $\mathbb{C}_\delta \setminus \Omega_\delta$ and $\mathbb{C}_\delta^*$
- $\rightsquigarrow$ centered Gaussian field on $\mathbb{C}_\delta^\diamond$
- covariance $\mathsf{E}\big[\Phi(z)\,\Phi(w)\big] = 8\pi\, G_{\Omega_\delta}(z,w)$

Discrete holomorphic current $\mathfrak{J} = \big(\mathfrak{J}(z)\big)_{z \in \mathbb{C}_\delta^{\mathfrak{m}}}$

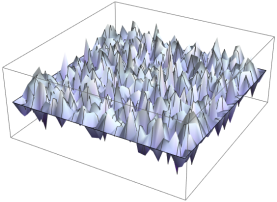
$$\mathfrak{J}(z) := \mathbb{i}\,\partial_\delta \Phi(z)$$

$$= \frac{\mathbb{i}}{2}\Big(\underbrace{\Phi\big(z+\tfrac{\delta}{2}\big) - \Phi\big(z-\tfrac{\delta}{2}\big)}_{\text{vanishes if } z \text{ on vertical edge}}\Big) + \frac{1}{2}\Big(\underbrace{\Phi\big(z+\tfrac{\mathbb{i}\delta}{2}\big) - \Phi\big(z-\tfrac{\mathbb{i}\delta}{2}\big)}_{\text{vanishes if } z \text{ on horizontal edge}}\Big)$$

- centered complex Gaussian field (...and a local field of dGFF!)
- purely real on vertical edges, imaginary on horizontal edges
- covariance $\mathsf{E}\big[\mathfrak{J}(z)\,\mathfrak{J}(w)\big] = -8\pi\,\partial_\delta^{(z)}\partial_\delta^{(w)}\,G_{\Omega_\delta}(z,w)$

Discrete holomorphic current (correlations)

$$\Phi = \big(\Phi(z)\big)_{z \in \mathbb{C}_\delta}$$



Wick's formula for centered Gaussians:

$$\mathsf{E}\Big[\prod_{j=1}^{n} \Phi(z_j)\Big] = \sum_{\substack{P \text{ pairing} \\ \text{of } \{1,\ldots,n\}}} \prod_{\{k,l\} \in P} \underbrace{\mathsf{E}\big[\Phi(z_k)\Phi(z_l)\big]}_{\propto G_{\Omega_\delta}(z_k, z_l)}$$

Discrete holomorphic current $\mathfrak{J} = \big(\mathfrak{J}(z)\big)_{z \in \mathbb{C}_\delta^{\mathfrak{m}}}$, $\mathfrak{J}(z) = \partial_\delta \Phi(z)$

Proposition (harmonicity of Φ, holomorphicity of $\mathfrak{J}$)

- $\mathsf{E}\Big[\big(\triangle_\delta \Phi\big)(z) \ \prod_{j=1}^{n} \Phi(w_j)\Big] = 0$ when $\|z - w_j\|_1 > \delta$ for all j
- $\mathsf{E}\Big[\big(\bar{\partial}_\delta \mathfrak{J}\big)(z) \ \prod_{j=1}^{n} \Phi(w_j)\Big] = 0$ when $\|z - w_j\|_1 > \delta$ for all j

$$\therefore \quad \bar{\partial}_\delta \mathfrak{J} = \frac{\mathbb{i}}{4} \triangle_\delta \Phi \text{ is a null field}$$

Outline / steps

For the ~~Ising model and~~ discrete GFF:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

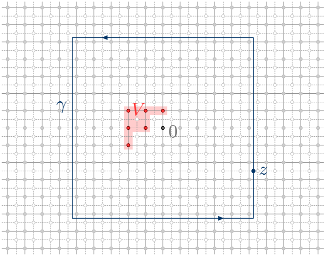
3.) Define Laurent modes of the observable

4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Discrete Laurent modes of the current

- $\mathfrak{F}(w) = F\big[(\Phi(w+x\delta))_{x\in V}\big]$ local field of dGFF
- γ sufficiently large counterclockwise closed path on $\mathbb{C}^{\mathfrak{c}}_{\delta}$ surrounding origin and $V\delta$



For $j \in \mathbb{Z}$ define a new local field of dGFF $(\mathfrak{J}_j\,\mathfrak{F})(w)$ by

$$(\mathfrak{J}_j\,\mathfrak{F})(0) \;:=\; \frac{1}{2\pi\mathbb{i}} \oint_{[\gamma]} \mathfrak{J}(z_{\mathfrak{m}})\, z_{\diamond}^{[j]}\, \mathfrak{F}(0)\; [\mathrm{d}z]_{\delta}$$

Lemma (discrete current modes)

$\mathfrak{J}_j\colon \mathcal{F}/\mathcal{N} \to \mathcal{F}/\mathcal{N}$ is well-defined — independent of choice of ...

- add null field to $\mathfrak{F}(0)$ $\rightsquigarrow$ add null field to $(\mathfrak{J}_j\,\mathfrak{F})(0)$ — ...representative
- change γ $\rightsquigarrow$ add null fields $\bar{\partial}_{\delta}\mathfrak{J}(z) \times (\cdots)$ to $(\mathfrak{J}_j\,\mathfrak{F})(0)$ — ...contour

Outline / steps

For the ~~Ising model and~~ discrete GFF:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

✓ 3.) Define Laurent modes of the observable

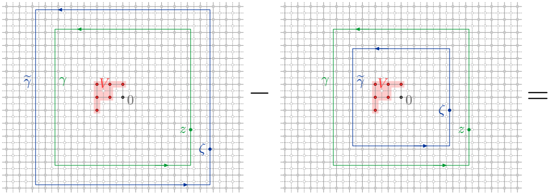
4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Commutation of modes of the discrete current

Proposition (commutation of discrete current modes)

$$[\mathfrak{J}_i, \mathfrak{J}_j] = i\,\delta_{i+j,0}\,\mathrm{id}_{\mathcal{F}/\mathcal{N}}$$



$$\mathsf{E}\Big[\Big(\mathfrak{J}_i\mathfrak{J}_j\,\mathfrak{F}(0) - \mathfrak{J}_j\mathfrak{J}_i\,\mathfrak{F}(0)\Big)\cdots\Big]$$

$$= \frac{1}{(2\pi\mathbb{i})^2}\mathsf{E}\Bigg[\oint_{[\widetilde{\gamma}_+]}\mathfrak{J}(\zeta_{\mathfrak{m}})\,\zeta_\diamond^{[i]}\Bigg(\oint_{[\gamma]}\mathfrak{J}(z_{\mathfrak{m}})\,z_\diamond^{[j]}\,\mathfrak{F}(0)\cdots\;[\mathrm{d}z]_\delta\Bigg)[\mathrm{d}\zeta]_\delta\Bigg]$$

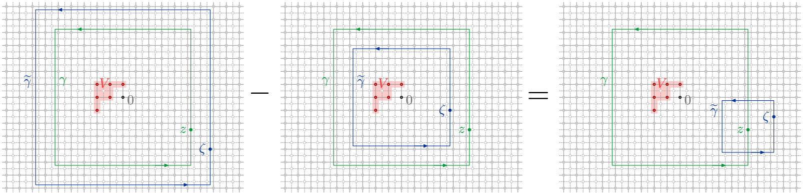
$$-\frac{1}{(2\pi\mathbb{i})^2}\mathsf{E}\Bigg[\oint_{[\gamma]}\mathfrak{J}(z_{\mathfrak{m}})\,z_\diamond^{[j]}\Bigg(\oint_{[\widetilde{\gamma}_-]}\mathfrak{J}(\zeta_{\mathfrak{m}})\,\zeta_\diamond^{[i]}\,\mathfrak{F}(0)\cdots\;[\mathrm{d}\zeta]_\delta\Bigg)[\mathrm{d}z]_\delta\Bigg]$$

Next: deform ζ integration contours for fixed z

Commutation of modes of the discrete current

Proposition (commutation of discrete current modes)

$$[\mathfrak{J}_i, \mathfrak{J}_j] = i\,\delta_{i+j,0}\ \mathrm{id}_{\mathcal{F}/\mathcal{N}}$$



$$\mathsf{E}\Big[\Big(\mathfrak{J}_i\mathfrak{J}_j\,\mathfrak{F}(0) - \mathfrak{J}_j\mathfrak{J}_i\,\mathfrak{F}(0)\Big)\cdots\Big]$$
$$= i\ \delta_{i+j,0}\ \mathsf{E}\Big[\mathfrak{F}(0)\cdots\Big]$$

(residue calculus)

Outline / steps

For the ~~Ising model and~~ discrete GFF:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

✓ 3.) Define Laurent modes of the observable

✓ 4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Sugawara construction with the dGFF current

Verify assumptions:

- V vector space

  space $\mathcal{F}/\mathcal{N}$ of local fields modulo null fields

- $\mathfrak{a}_j \colon V \to V$ linear for each $j \in \mathbb{Z}$

  discrete current Laurent mode $\mathfrak{J}_j \colon \mathcal{F}/\mathcal{N} \to \mathcal{F}/\mathcal{N}$

  $\big(\mathfrak{J}_j \mathfrak{F}\big)(0) := \frac{1}{2\pi} \oint_{[\gamma]} \partial_\delta \Phi(z_\mathfrak{m})\, z_\diamond^{[j]}\, \mathfrak{F}(0)\, [\mathrm{d}z]$

- $\forall v \in V \quad \exists N \in \mathbb{Z} \quad : \quad j \geq N \implies \mathfrak{a}_j\, v = 0$

  monomial truncation: $\forall z_\diamond \in \mathbb{C}_\delta^\diamond \ \ \exists D \quad : \quad j \geq D \Longrightarrow z_\diamond^{[j]} = 0$

- $[\mathfrak{a}_i, \mathfrak{a}_j] = i\, \delta_{i+j,0}\, \mathrm{id}_V$

  Laurent mode commutation $[\mathfrak{J}_i, \mathfrak{J}_j] = i\, \delta_{i+j,0}\, \mathrm{id}_{\mathcal{F}/\mathcal{N}}$

Theorem (Virasoro action for dGFF)

$$\mathfrak{L}_n := \frac{1}{2} \sum_{j<0} \mathfrak{J}_j \circ \mathfrak{J}_{n-j} + \frac{1}{2} \sum_{j \geq 0} \mathfrak{J}_{n-j} \circ \mathfrak{J}_j$$

defines Virasoro representation with $c = 1$ on the space $\mathcal{F}/\mathcal{N}$ of correlation equivalence classes of local fields of the dGFF.

Outline / steps

For the Ising model ~~and discrete GFF~~:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

2.) Introduce discrete holomorphic observable

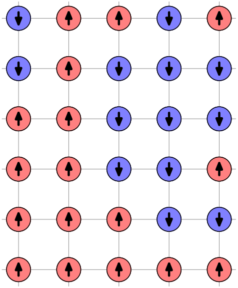
3.) Define Laurent modes of the observable

4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

The critical Ising model on $\mathbb{Z}^2$

- $\Omega \subsetneq \mathbb{C}$ open, 1-connected
- lattice approximation $\Omega_\delta \subset \mathbb{C}_\delta$, $\Omega_\delta^\diamond \subset \mathbb{C}_\delta^\diamond$, $\Omega_\delta^\mathfrak{m} \subset \mathbb{C}_\delta^\mathfrak{m}$, $\Omega_\delta^\mathfrak{c} \subset \mathbb{C}_\delta^\mathfrak{c}$



Ising model: random spin configuration

$$\sigma = \left(\sigma_z\right)_{z \in \mathbb{C}_\delta} \in \{+1, -1\}^{\mathbb{C}_\delta}$$

- $\sigma\big|_{\mathbb{C}_\delta \setminus \Omega_\delta} \equiv +1$ (plus-boundary conditions)

$$\mathsf{P}\big[\{\sigma\}\big] \propto \exp\Big(-\beta\, E(\sigma)\Big) \quad \text{(Boltzmann-Gibbs)}$$

$$E(\sigma) = -\sum_{z \sim w} \sigma_z \, \sigma_w \quad \text{(energy)}$$

$$\beta = \beta_c = \frac{1}{2} \log\left(\sqrt{2} + 1\right) \quad \text{(critical point)}$$

Local fields of the Ising model

Local fields $\mathfrak{F}(z)$ of Ising

- $V \subset \mathbb{Z}^2$ finite subset
- $P\colon \{+1,-1\}^V \to \mathbb{C}$ a function

$$\text{parity} \quad \begin{cases} P(-\sigma) = P(\sigma) & \text{even} \\ P(-\sigma) = -P(\sigma) & \text{odd} \end{cases}$$

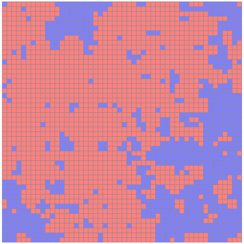
- $\mathfrak{F}(z) = P\big((\sigma_{z+\delta x})_{x \in V}\big)$

$\rightsquigarrow$ $\mathcal{F}$ space of local fields

$$\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^- \text{ by parity} \quad \begin{cases} \mathcal{F}^+ & \text{even} \\ \mathcal{F}^- & \text{odd} \end{cases}$$

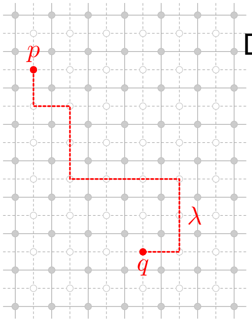
$\rightsquigarrow$ $\mathcal{N} \subset \mathcal{F}$ space of null fields
("zero in correlations")

$\sigma = (\sigma_z)_{z \in \Omega_\delta}$ Ising



$\mathcal{F}/\mathcal{N}$ equivalence classes of local fields (same correlations)

Disorder operators in Ising model



Disorder operator pair:

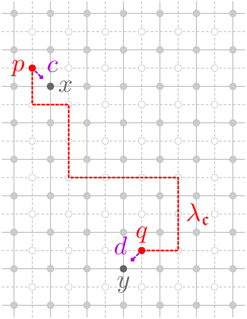
$$(\mu_p \mu_q)_\lambda := \exp\Big(-2\beta \sum_{\langle z,w\rangle^* \in \lambda} \sigma_z \sigma_w \Big)$$

- $p, q \in \mathbb{C}^*_\delta$ dual vertices
- λ path between p and q on $\mathbb{C}^*_\delta$ "disorder line"

Remark:

- a single disorder operator is NOT a local field
- a disorder operator pair is a local field (with fixed disorder line λ)

Corner fermions in Ising model



- $c, d \in \mathbb{C}_\delta^{\mathfrak{c}}$ corners
- $x, y \in \mathbb{C}_\delta$ adjacent to c, d, respectively
- $p, q \in \mathbb{C}_\delta^*$ adjacent to c, d, respectively
- $\nu(c) := \frac{x-p}{|x-p|}$ phase factor
- $\lambda_{\mathfrak{c}}$ path between c and d "on $\mathbb{C}_\delta^*$"
- $\mathcal{W}(\lambda_{\mathfrak{c}} : c \rightsquigarrow d)$ cumulative angle of turning of $\lambda_{\mathfrak{c}}$

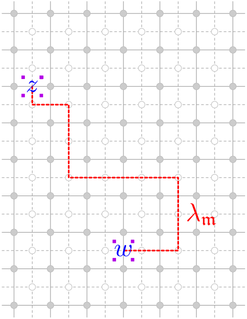
Corner fermion pair:

$$(\Psi_c^{\mathfrak{c}} \Psi_d^{\mathfrak{c}})_{\lambda_{\mathfrak{c}}} := -\overline{\nu(c)} \exp\left(-\frac{\mathbb{i}}{2} \mathcal{W}(\lambda_{\mathfrak{c}} : c \rightsquigarrow d) \right) (\mu_p \mu_q)_\lambda \, \sigma_x \sigma_y$$

Remark:

- one corner fermion is NOT a local field
- a corner fermion <u>pair</u> is a local field (with fixed disorder line)

Discrete holomorphic fermions in Ising model



- $z, w \in \mathbb{C}_\delta^{\mathfrak{m}}$ midpoints of edges
- $\lambda_{\mathfrak{m}}$ path between z and w "on $\mathbb{C}_\delta^*$"
- $c, d \in \mathbb{C}_\delta^{\mathfrak{c}}$ adjacent to z, w, respectively
- $\lambda_{\mathfrak{c}}^{c,d}$ path between c and d on $\mathbb{C}_\delta^*$ obtained by local modification of $\lambda_{\mathfrak{m}}$

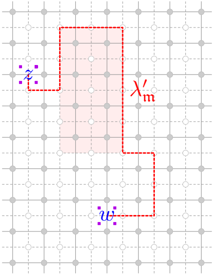
Holomorphic fermion pair:

$$\big(\Psi(z)\Psi(w)\big)_{\lambda_{\mathfrak{m}}} := \frac{1}{8\sqrt{2}} \sum_{c,d} \big(\Psi_c^{\mathfrak{c}} \Psi_d^{\mathfrak{c}}\big)_{\lambda_{\mathfrak{c}}^{c,d}}$$

Remark: (as before)

- one holomorphic fermion is NOT a local field
- a holomorphic fermion <u>pair</u> is a local field (with fixed disorder line)

Properties of the fermion pairs



Lemma (disorder line independence mod $\pm$)

If $\lambda_{\mathfrak{m}}, \lambda'_{\mathfrak{m}}$ are disorder lines between $z, \zeta \in \mathbb{C}^{\mathfrak{m}}_{\delta}$ then

$$\mathsf{E}\Big[(\Psi(\zeta)\Psi(z))_{\lambda_{\mathfrak{m}}} \prod_{j=1}^{n} \sigma_{w_j}\Big] = (-1)^{\mathcal{N}} \times \mathsf{E}\Big[(\Psi(\zeta)\Psi(z))_{\lambda'_{\mathfrak{m}}} \prod_{j=1}^{n} \sigma_{w_j}\Big]$$

where $\mathcal{N}$ is the number of points w_j in the area enclosed by $\lambda_{\mathfrak{m}}$ and $\lambda'_{\mathfrak{m}}$.

Lemma (antisymmetry of fermions)

$$(\Psi(\zeta)\Psi(z))_{\lambda_{\mathfrak{m}}} = -(\Psi(z)\Psi(\zeta))_{\lambda_{\mathfrak{m}}}$$

Lemma (holomorphicity and singularity of fermion)

$$\mathsf{E}\Big[(\bar{\partial}_{\delta}\Psi(\zeta_{\diamond})\Psi(z_{\mathfrak{m}})) \prod_{j=1}^{n} \sigma_{w_j}\Big] = \frac{-1}{4} \sum_{x \sim z_{\mathfrak{m}}} \delta_{\zeta_{\diamond}, x} \times \mathsf{E}\Big[\prod_{j=1}^{n} \sigma_{w_j}\Big]$$

Outline / steps

For the Ising model ~~and discrete GFF~~:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

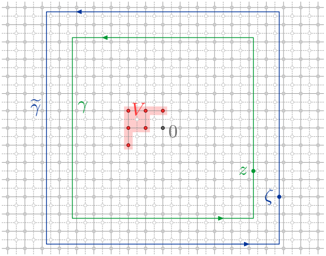
3.) Define Laurent modes of the observable

4.) Commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Laurent modes of fermions in even sector

- $\mathfrak{F}(w) = P\big[(\sigma_{w+x\delta})_{x\in V}\big]$ even local field of Ising
- $\gamma, \widetilde{\gamma}$ large nested counterclockwise closed paths on $\mathbb{C}_\delta^{\mathfrak{c}}$



For $k, \ell \in \mathbb{Z} + \frac{1}{2}$ define a new local field $\big((\Psi_k \Psi_\ell)\,\mathfrak{F}\big)(w)$ by

$$\big((\Psi_k \Psi_\ell)\,\mathfrak{F}\big)(0) \;:=\; \frac{1}{2\pi} \oint_{[\widetilde{\gamma}]} \oint_{[\gamma]} \zeta_\diamond^{[k-\frac{1}{2}]}\, z_\diamond^{[\ell-\frac{1}{2}]} \Big(\Psi\big(\zeta_{\mathfrak{m}}\big)\Psi\big(z_{\mathfrak{m}}\big)\Big)\,\mathfrak{F}(0)\;[\mathrm{d}z]_\delta\;[\mathrm{d}\zeta]_\delta$$

Lemma (discrete fermion mode pairs)

$(\Psi_k \Psi_\ell)\colon \mathcal{F}^+/\mathcal{N}^+ \to \mathcal{F}^+/\mathcal{N}^+$ is well-defined

Remark: (as before)

- one fermion Laurent mode is NOT defined
- a fermion Laurent mode pair is defined, and acts on (even) local fields

Outline / steps

For the Ising model ~~and discrete GFF~~:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

✓ 3.) Define Laurent modes of the observable on *even* sector

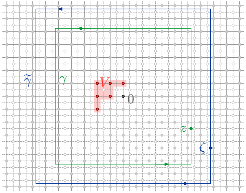
4.) *Anti* commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

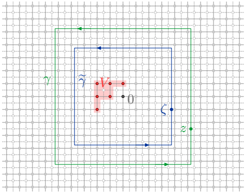
Anticommutation of fermion modes in even sector

Proposition (anticommutation of fermion modes)

$$(\Psi_k \Psi_\ell) + (\Psi_\ell \Psi_k) = \delta_{k+\ell,0} \, \mathrm{id}_{\mathcal{F}^+/\mathcal{N}^+}$$



$+$



$=$

$$\mathsf{E}\Big[\Big((\Psi_k \Psi_\ell)\, \mathfrak{F}(0) + (\Psi_\ell \Psi_k)\, \mathfrak{F}(0) \Big) \cdots \Big]$$

$$= \mathsf{E}\left[\frac{1}{2\pi} \oint_{[\widetilde{\gamma}_+]} \oint_{[\gamma]} \zeta_\diamond^{[k-\frac{1}{2}]} z_\diamond^{[\ell-\frac{1}{2}]} \, \Psi(\zeta_{\mathfrak{m}})\, \Psi(z_{\mathfrak{m}})\, \mathfrak{F}(0) \cdots \; [\mathrm{d}z]_\delta [\mathrm{d}\zeta]_\delta \right]$$

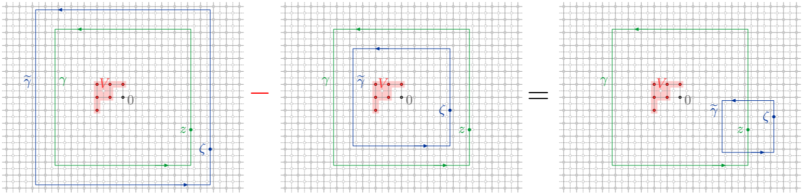
$$+ \, \mathsf{E}\left[\frac{1}{2\pi} \oint_{[\gamma]} \oint_{[\widetilde{\gamma}_-]} z_\diamond^{[\ell-\frac{1}{2}]} \zeta_\diamond^{[k-\frac{1}{2}]} \, \underset{\longleftrightarrow}{\Psi(z_{\mathfrak{m}})\, \Psi(\zeta_{\mathfrak{m}})}\, \mathfrak{F}(0) \cdots \; [\mathrm{d}\zeta]_\delta [\mathrm{d}z]_\delta \right]$$

Next: interchange fermion order by antisymmetry

Anticommutation of fermion modes in even sector

Proposition (anticommutation of fermion modes)

$$(\Psi_k \Psi_\ell) + (\Psi_\ell \Psi_k) = \delta_{k+\ell,0}\ \mathrm{id}_{\mathcal{F}^+/\mathcal{N}^+}$$



$$\mathsf{E}\Big[\Big((\Psi_k\Psi_\ell)\,\mathfrak{F}(0) + (\Psi_\ell\Psi_k)\,\mathfrak{F}(0)\Big)\cdots\Big]$$
$$= \delta_{k+\ell,0}\ \mathsf{E}\Big[\mathfrak{F}(0)\cdots\Big]$$

(residue calculus)

Outline / steps

For the Ising model ~~and discrete GFF~~:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

✓ 3.) Define Laurent modes of the observable

✓ 4.) *Anti*commutation relations of Laurent modes

5.) Virasoro action through Sugawara construction

Sugawara construction for Ising even local fields

- V vector space

  space $\mathcal{F}^+/\mathcal{N}^+$ of even local fields modulo null fields
- $\mathfrak{b}_k \colon V \to V$ linear for each $k \in \mathbb{Z} + \frac{1}{2}$

  fermion Laurent mode pairs $(\Psi_k \Psi_\ell) \colon \mathcal{F}^+/\mathcal{N}^+ \to \mathcal{F}^+/\mathcal{N}^+$

  $\frac{1}{2\pi} \oint_{[\tilde{\gamma}]} \oint_{[\gamma]} \zeta_\diamond^{[k-\frac{1}{2}]} z_\diamond^{[\ell-\frac{1}{2}]} \Big(\Psi(\zeta_\mathfrak{m}) \Psi(z_\mathfrak{m}) \Big) (\cdots) \, [\mathrm{d}z]_\delta \, [\mathrm{d}\zeta]_\delta$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \quad : \quad \ell \geq N \implies \mathfrak{b}_\ell \, v = 0$

  monomial truncation: $\forall z_\diamond \in \mathbb{C}_\delta^\diamond \;\; \exists D \quad : \quad \ell \geq D \Rightarrow z_\diamond^{[\ell-\frac{1}{2}]} = 0$
- $[\mathfrak{b}_k, \mathfrak{b}_\ell]_+ = \delta_{k+\ell,0} \, \mathrm{id}_V$

  anticommutation $(\Psi_k \Psi_\ell) + (\Psi_\ell \Psi_k) = \delta_{k+\ell,0} \, \mathrm{id}_{\mathcal{F}^+/\mathcal{N}^+}$
  and $(\Psi_p \Psi_k)(\Psi_\ell \Psi_q) + (\Psi_p \Psi_\ell)(\Psi_k \Psi_q) = \delta_{k+\ell,0} \, (\Psi_p \Psi_q)$

Theorem (Virasoro action for Ising even sector)

$$\mathfrak{L}_n := \frac{1}{2} \sum_{k>0} \Big(\frac{1}{2} + k\Big) (\Psi_{n-k} \Psi_k) - \frac{1}{2} \sum_{k<0} \Big(\frac{1}{2} + k\Big) (\Psi_k \Psi_{n-k})$$

defines Virasoro repr. with $c = \frac{1}{2}$ on the space $\mathcal{F}^+/\mathcal{N}^+$ of correlation equivalence classes of Ising even local fields.

Outline / steps

For the Ising model ~~and discrete GFF~~:

✓ 0.) Define model and local fields

✓ 1.) Suitable discrete contour integrals and residue calculus

✓ 2.) Introduce discrete holomorphic observable

3.) Define Laurent modes of the observable

4.) *Anti*commutation relations of Laurent modes

5.) Apply Sugawara construction to define Virasoro action on *odd* local fields

Odd sector: Discrete half-integer monomials

Proposition (discrete half-integer monomial functions)

$\exists$ functions $z \mapsto z^{[p]}$, $p \in \mathbb{Z} + \frac{1}{2}$, defined on the double cover $[\mathbb{C}^{\diamond}_{\delta}; 0] \cup [\mathbb{C}^{\mathfrak{m}}_{\delta}; 0]$ ramified at the origin, such that

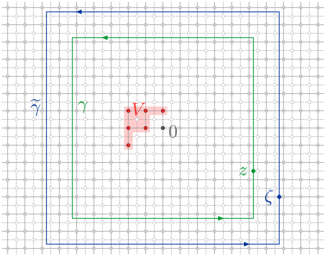
- $\bar{\partial}_{\delta} z^{[p]} = 0$ whenever ... "discrete holomorphicity"
  - $p > 0$ and $z \in [\mathbb{C}^{\diamond}_{\delta}; 0] \cup [\mathbb{C}^{\mathfrak{m}}_{\delta}; 0]$
  - $p < 0$ and $z \in [\mathbb{C}^{\diamond}_{\delta}; 0] \cup [\mathbb{C}^{\mathfrak{m}}_{\delta}; 0]$, $\|z\|_1 > R_p\,\delta$
- $\partial_{\delta} z^{[p]} = p\, z^{[p-1]}$ "derivatives"
- $z^{[p]}$ has the same 90° rotation symmetry as z^p "symmetry"
- for $p < 0$ we have $z^{[p]} \to 0$ as $\|z\| \to \infty$ "decay"
- for any z there exists D_z such that $z^{[p]} = 0$ for $p \geq D_z$ "truncation"

For γ large enough counterclockwise closed contour surrounding the origin...

- $\oint_{[\gamma]} z_{\mathfrak{m}}^{[p]}\, z_{\diamond}^{[q]}\, [\mathrm{d}z]_{\delta} = 2\pi\mathbb{i}\; \delta_{p+q,-1}$ "residue calculus"
- $\oint_{[\gamma]} z_{\mathfrak{m}}^{\{p\}}\, z_{\diamond}^{[q]}\, [\mathrm{d}z]_{\delta} = 2\pi\mathbb{i}\; \delta_{p+q,-1}$ where $z_{\mathfrak{m}}^{\{p\}} = \frac{1}{4}\sum_{x \in \{\pm\frac{\delta}{2}, \pm\mathbb{i}\frac{\delta}{2}\}} (z_{\mathfrak{m}} - x)^{[p]}$

Odd sector: Laurent modes of fermions

- $\mathfrak{F}(w) = P\big[(\sigma_{w+x\delta})_{x\in V}\big]$
  odd local field of Ising
- $\gamma, \widetilde{\gamma}$ large nested counterclockwise closed paths on $\mathbb{C}^{\mathfrak{c}}_{\delta}$



For $i, j \in \mathbb{Z}$ define a new local field $\big((\Psi_i\Psi_j)\,\mathfrak{F}\big)(w)$ by

$$\big((\Psi_i\Psi_j)\,\mathfrak{F}\big)(0) \;:=\; \frac{1}{2\pi}\oint_{[\widetilde{\gamma}]}\oint_{[\gamma]} \zeta_{\diamond}^{[\,i-\frac{1}{2}\,]}\, z_{\diamond}^{[\,j-\frac{1}{2}\,]} \Big(\Psi\big(\zeta_{\mathfrak{m}}\big)\Psi\big(z_{\mathfrak{m}}\big)\Big)\,\mathfrak{F}(0)\;[\mathrm{d}z]_{\delta}\;[\mathrm{d}\zeta]_{\delta}$$

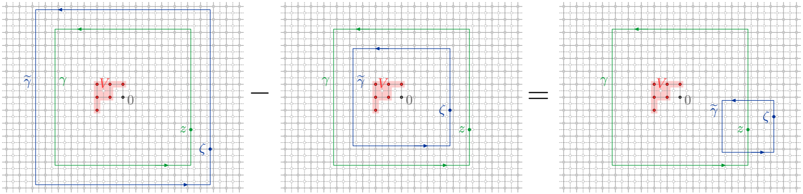
Lemma (discrete fermion mode pairs)

$(\Psi_i\Psi_j)\colon \mathcal{F}^-/\mathcal{N}^- \to \mathcal{F}^-/\mathcal{N}^-$ is well-defined

Odd sector: Anticommutation of fermion modes

Proposition (anticommutation of fermion modes)

$$(\Psi_i \Psi_j) + (\Psi_j \Psi_i) = \delta_{i+j,0} \, \mathrm{id}_{\mathcal{F}^-/\mathcal{N}^-}$$



Odd sector: Fermionic Sugawara construction

Proposition (fermionic Sugawara, Ramond sector)

Suppose:

- V vector space, $\mathfrak{b}_j \colon V \to V$ linear for each $j \in \mathbb{Z}$
- $\forall v \in V \quad \exists N \in \mathbb{Z} \quad : \quad j \geq N \implies \mathfrak{b}_j \, v = 0$
- $[\mathfrak{b}_i, \mathfrak{b}_j]_+ = \delta_{i+j,0} \, \mathrm{id}_V$

Def.:

$$L_n := \frac{1}{2} \sum_{j \geq 0} \Big(\frac{1}{2} + j\Big) \mathfrak{b}_{n-j} \mathfrak{b}_j - \frac{1}{2} \sum_{j<0} \Big(\frac{1}{2} + j\Big) \mathfrak{b}_j \mathfrak{b}_{n-j} \quad (n \in \mathbb{Z} \setminus \{0\})$$

$$L_0 := \frac{1}{2} \sum_{j>0} j \, \mathfrak{b}_{-j} \mathfrak{b}_j + \frac{1}{16} \, \mathrm{id}_V$$

Then:

- $L_n \colon V \to V$ is well defined
- $[L_n, L_m] = (n-m) \, L_{n+m} + \frac{n^3-n}{24} \, \delta_{n+m,0} \, \mathrm{id}_V$

Theorem (Virasoro action on Ising odd local fields)

The space of odd Ising local fields modulo null fields becomes Virasoro representation with central charge $c = \frac{1}{2}$.

Conclusions and outlook

✓ Lattice model fields of finite patterns form Virasoro repr.

- discrete Gaussian free field: L_n on $\mathcal{F}/\mathcal{N}$ by bosonic Sugawara
- Ising model: $\mathfrak{L}_n$ on $\underbrace{\mathcal{F}^+/\mathcal{N}^+ \oplus \mathcal{F}^-/\mathcal{N}^-}_{\text{“Neveu-Schwarz} \oplus \text{Ramond”}}$ by fermionic Sugawara

TODO Many CFT ideas rely on variants of Sugawara construction

- Wess-Zumino-Witten models
- symplectic fermions
- coset conformal field theories $\rightsquigarrow$ CFT minimal models
- Coulomb gas formalism

TODO CFT fields $\longleftrightarrow$ lattice model fields of finite patterns

- 1-1 correspondence via the Virasoro action on lattice model fields?
- correlations of lattice model fields with appropriate renormalization converge in scaling limit to CFT correlations?
- conceptual derivation of PDEs for limit correlations via singular vectors?

THANK YOU!