Scaling limits of critical Ising correlations: convergence, fusion rules, applications to SLE.

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The Ising model notation

- For a **finite graph** with a vertex set V, define the

$$\text{State space: } \{\sigma : V \to \pm 1\}\,;$$

- ... and the Hamiltonian

$$\mathcal{H}(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y .$$

  sum over edges of the graph = pairs of nearest neighbors;

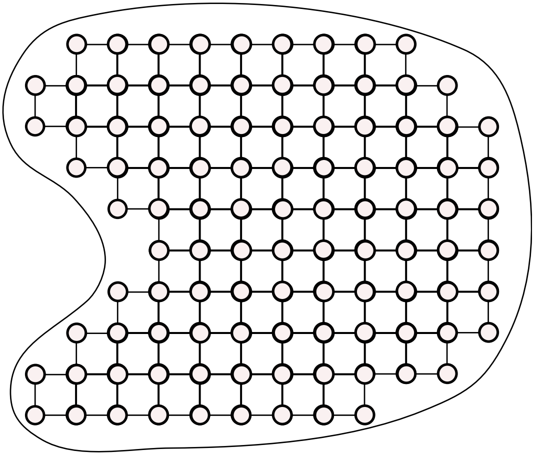
- ...and the Gibbs-Boltsmann distribution

$$\mathbb{P}(\sigma) = e^{-\beta \mathcal{H}(\sigma)}/Z, \quad \text{where } Z = \sum_{\sigma : V \to \pm 1} e^{-\beta \mathcal{H}(\sigma)}$$

  and $\beta = \frac{k}{T}$ is the inverse temperature.

The phase transition

The graph: finite subset $\Omega^\delta \subset \delta\mathbb{Z}^2$ for small δ approximating a fixed domain Ω:



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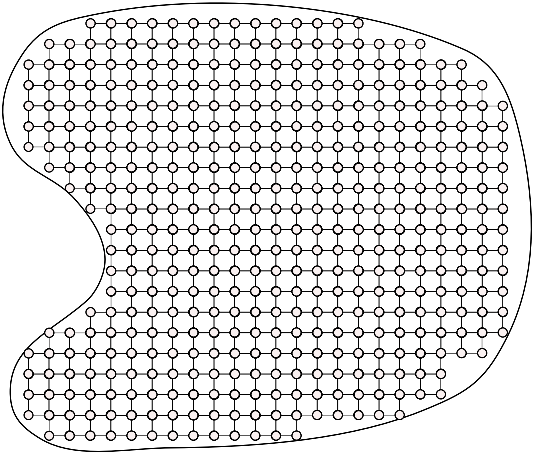
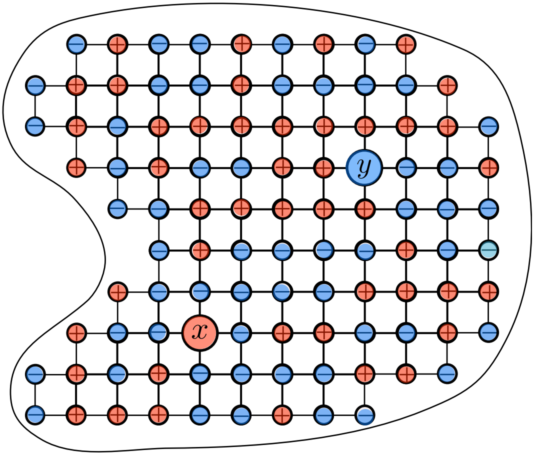


The scaling limit

The graph: finite subset $\Omega^\delta \subset \delta\mathbb{Z}^2$ for small δ approximating a fixed domain Ω:



We are interested in the correlations at criticality: $\beta = \beta_c = \log(\sqrt{2}+1)$.

Spin correlations (Chelkak–Hongler–K. I., 2015)

There is a function $\langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega : \Omega^{\times n} \setminus \mathrm{diag} \to \mathbb{R}$ such that

$$\mathbb{E}_{\Omega^\delta} \left(\sigma_{v_1} \dots \sigma_{v_n} \right) \sim C^n \cdot \delta^{\frac{n}{8}} \cdot \langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega \text{ as } \delta \to 0;$$

The correlation function $\langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega$ is conformally covariant:

$$\langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega = \prod_{i=1}^{n} |\varphi'(v_i)|^{\frac{1}{8}} \cdot \langle \sigma_{\varphi(v_1)} \dots \sigma_{\varphi(v_n)} \rangle_{\varphi(\Omega)}.$$

In the upper half-plane $\mathbb{H}$, there is an explicit formula:

$$\langle \sigma_{v_1} \dots \sigma_{v_n} \rangle_\Omega = \prod_{i=1}^{n} |\mathfrak{Im}\, v_i|^{-\frac{1}{8}} \left(\sum_{s \in \{\pm 1\}^n} s_1 \dots s_n \prod_{i<j} \left| \frac{v_i - v_j}{v_i - \bar{v}_j} \right|^{\frac{s_i s_j}{2}} \right)^{\frac{1}{2}}.$$

Similar results hold for plus/Dobrushin/etc. boundary conditions.

More general correlations

How about more general random variables ("lattice fields")

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\right) \sim \delta^{\Delta_1+\cdots+\Delta_n}\langle\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_\Omega$$

where $\mathcal{O}_j(x_j)$ only depends on spins at distance $O(\delta)$ from x_j (finite number of lattice steps)?
In general, we expect much *more complicated covariance rules*.

General fields: example

Example: it is natural to expect

$$\mathbb{E}_{\Omega^\delta}\left((\sigma_{v_1+\delta}-\sigma_{v_1})\cdot\sigma_{v_2}\ldots\sigma_{v_n}\right)\sim\delta^{1+\frac{n}{8}}\partial_{\mathfrak{Re}\,v_1}\langle\sigma_{v_1}\ldots\sigma_{v_n}\rangle.$$

Since

$$\langle\sigma_{v_1}\ldots\sigma_{v_n}\rangle_\Omega=\prod_{i=1}^n|\varphi'(v_i)|^{\frac{1}{8}}\cdot\langle\sigma_{\varphi(v_1)}\ldots\sigma_{\varphi(v_n)}\rangle_{\varphi(\Omega)}$$

we get

$$\partial_{\mathfrak{Re}\,v_1}\langle\sigma_{v_1}\ldots\sigma_{v_n}\rangle=\partial_{\mathfrak{Re}\,v_1}\left(\frac{1}{8}\log|\varphi'(v_1)|+\log\langle\sigma_{\varphi(v_1)}\ldots\sigma_{\varphi(v_n)}\rangle_{\varphi(\Omega)}\right)\cdot$$
$$\cdot\prod_{i=1}^n|\varphi'(v_i)|^{\frac{1}{8}}\cdot\langle\sigma_{\varphi(v_1)}\ldots\sigma_{\varphi(v_n)}\rangle_{\varphi(\Omega)}$$

Primary fields

Primary fields are those for which the simplest possible covariance rule holds:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_\Omega = \prod_{i=1}^{n} \varphi'(x_i)^{\Delta'_i} \prod_{i=1}^{n} \overline{\varphi'(x_i)}^{\Delta''_i} \cdot \langle \mathcal{O}_1(\varphi(x_1)) \dots \mathcal{O}_n(\varphi(x_n)) \rangle_{\varphi(\Omega)}.$$

We will focus on *four* primary fields in the Ising model:

- **spins** σ_x, indexed by *vertices* of Ω^δ ($\Delta' = \Delta'' = \frac{1}{16}$);
- **energies** $\epsilon_e = \sigma_{e_+}\sigma_{e_-} - \frac{\sqrt{2}}{2}$, indexed by *edges* of Ω^δ ($\Delta' = \Delta'' = \frac{1}{2}$);
- **disorders** μ_u, indexed by *faces* of Ω^δ ($\Delta' = \Delta'' = \frac{1}{16}$);
- **fermions** ψ_z, indexed by *corners* of Ω^δ ($\Delta' = \frac{1}{2}, \Delta'' = 0$) (and its "conjugate" $\psi_z^\star$ with Δ'=0, Δ=$\frac{1}{2}$).

Primary fields

We will focus on *four* primary fields in the Ising model:

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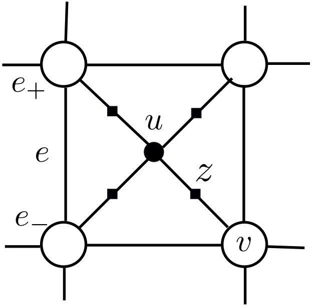


Convergence theorem

- As $\delta \to 0$, one has

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\right) \sim \prod_{i=1}^{n} C_i \cdot \delta^{\Delta_1+\cdots+\Delta_n} \cdot \langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_\Omega,$$

  where each $\mathcal{O}_i$ can be any of σ, ϵ, μ, ψ.

- The *correlation functions* $\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_\Omega$ are conformally covariant:

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle_\Omega = \prod_{i=1}^{n} \varphi'(x_i)^{\Delta_i'} \prod_{i=1}^{n} \overline{\varphi'(x_i)}^{\Delta_i''} \cdot \langle \mathcal{O}_1(\varphi(x_1))\ldots\mathcal{O}_n(\varphi(x_n))\rangle_{\varphi(\Omega)}.$$

- In the upper half-plane or an annulus, the formulas are (in principle) explicit.

- Boundary conditions: free, plus, minus, or combinations thereof.

General local fields

Gheissari–Hongler–Park, Hongler–Kytölä–Viklund: many (all?) local fields can be expressed in terms of discrete contour integrals involving

$$\mathbb{E}(\sigma_{v_1}\sigma_{v_2}\psi_{z_1}\ldots\psi_{z_{2k}}) \quad \text{and} \quad \mathbb{E}(\psi_{z_1}\ldots\psi_{z_{2k}}).$$

Therefore, when combined with their results, convergence of spin-fermion correlation yields a more general result.

Conformal field theory

A. Belavin, A. Polyakov, A. Zamolodchikov (1984), ... :

- **postulated** existence and conformal invariance of the scaling limit;
- deduced **PDE** for the correlation functions (**BPZ equations**);
- deduced **fusion rules** (asymptotics as some of the points collide together);
- derived many **explicit formulae** for the correlations.

What is μ_u

.

- Given an arbitrary collection γ of **dual** edges, define

$$\mu_\gamma := e^{-2\beta \sum_{(xy)\cap\gamma\neq\emptyset} \sigma_x\sigma_y}$$

- Define

$$\mathbb{E}(\mu_{u_1}\dots\mu_{u_m}\sigma_{v_1}\dots\sigma_{v_n}) := \mathbb{E}(\mu_\gamma\sigma_{v_1}\dots\sigma_{v_n}),$$

  where γ is **any** subset of edges with

$$\{u_1,\dots,u_m\} = \partial\gamma \bmod 2 = \{u : \#\{e\in\gamma : e\sim u\} \text{ is odd}\}$$

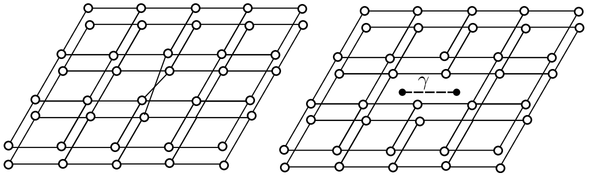
- Note that up to sign, this **does not** depend on the choice of γ.

$|\mathbb{E}(\mu_{u_1} \dots \sigma_{v_n})|$ does not depend on γ

$$\mathbb{E}_{\Omega^\delta}(\mu_{u_1}\mu_{u_2}\sigma_{v_1}\dots\sigma_{v_n}) = \mathbb{E}_{\Omega^\delta}(\mu_\gamma\sigma_{v_1}\dots\sigma_{v_n})$$

$$= \frac{1}{Z}\sum_{\sigma:\Omega^\delta\to\{\pm1\}} \sigma_{v_1}\dots\sigma_{v_n}\exp\left(-\beta\sum_{(xy)\cap\gamma\neq\emptyset}\sigma_x\sigma_y + \beta\sum_{(xy)\cap\gamma=\emptyset}\sigma_x\sigma_y\right)$$

$$= \frac{1}{Z}\sum_{\substack{\sigma:\Omega^\delta_{[u_1,u_2]}\to\{\pm1\}\\ \sigma(v)=-\sigma(v^\star)}} \sigma_{v_1}\dots\sigma_{v_n}\exp\left(\frac{\beta}{2}\sum_{x\approx y}\sigma_x\sigma_y\right)$$



Spinor structure of $\mathbb{E}(\mu_{u_1} \dots \mu_{u_m} \sigma_{v_1} \dots \sigma_{v_n})$

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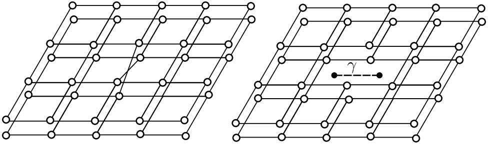


Moreover, there is a natural way to track the signs as $v_1, \dots, v_n$ and $u_1, \dots, u_m$ move around in the lattice:

- If v_i moves to $v_i' \sim v_i$, just lift the move to the double cover.
- If u_i moves to $u_i' \sim u_i$, replace γ with $\gamma + (u_i u_i') \bmod 2$.

With this choice,

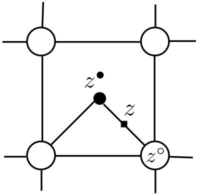
$$\mathbb{E}(\mu_{u_1} \dots \mu_{u_m} \sigma_{v_1} \dots \sigma_{v_n}) \prod_{i,j} (u_i - v_j)^{\frac{1}{2}}$$

*is a **well-defined function** of* $(v_1, \dots, u_m) \in (\Omega^\delta)^{\times n} \times (\Omega^{\delta,\star})^{\times m}$.

What is ψ_z

We define, formally,

$$\chi_z := \sigma_{z^\circ}\mu_{z^\bullet}$$
$$\psi_z := (z^\bullet - z^\circ)^{-\frac{1}{2}}\delta^{\frac{1}{2}}\chi_z$$



This is understood by inserting identities into correlations: any expression of the form

$$\mathbb{E}(\psi_{z_1}\dots\psi_{z_k}\mu_{u_1}\dots\mu_{u_m}\sigma_{v_1}\dots\sigma_{v_n})$$

is well defined

- up to sign at any particular point;
- as a (multi-valued) function of $z_1, \dots v_m$ living on the Riemann surface of

$$\prod(z_i-u_j)^{\frac{1}{2}}\prod(z_i-v_j)^{\frac{1}{2}}\prod(u_i-v_j)^{\frac{1}{2}}.$$

Convergence theorem

- As $\delta \to 0$, one has

$$\mathbb{E}_{\Omega^\delta}\left(\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right) \sim \prod_{i=1}^{n} C_i \cdot \delta^{\Delta_1+\dots+\Delta_n} \cdot \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\Omega,$$

  where each $\mathcal{O}_i$ can be any of σ, ϵ, μ, ψ.

- The *correlation functions* in the right-hand side are conformally covariant:

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\Omega = \prod_{i=1}^{n} \varphi'(x_i)^{\Delta_i'} \prod_{i=1}^{n} \overline{\varphi'(x_i)}^{\Delta_i''} \cdot \langle \mathcal{O}_1(\varphi(x_1))\dots\mathcal{O}_n(\varphi(x_n))\rangle_{\varphi(\Omega)}.$$

- In the upper half-plane or an annulus, the formulas are (in principle) explicit.
- Boundary conditions: free, plus, minus, or combinations thereof.

Fusion rules (or Operator Product Expansions)

Fusion rules are a collection of *asymptotic expansions* of correlation functions as marked point collide together.
Example:

$$\sigma_v \sigma_{\hat{v}} = |v - \hat{v}|^{-\frac{1}{4}} \left(1 + \frac{1}{2} \epsilon_w |v - \hat{v}| + o(v - \hat{v}) \right), \quad v \to \hat{v}.$$

This is understood as follows:

$$\begin{aligned} \langle \sigma_v \sigma_{\hat{v}} \mathcal{O} \rangle_\Omega = |v - \hat{v}|^{-\frac{1}{4}} \langle \sigma_v \sigma_{\hat{v}} \mathcal{O} \rangle_\Omega \\ + \frac{1}{2} |v - \hat{v}|^{\frac{3}{4}} \langle \epsilon_v \mathcal{O} \rangle + o(v - \hat{v})^{\frac{3}{4}}, \quad v \to \hat{v}, \end{aligned}$$

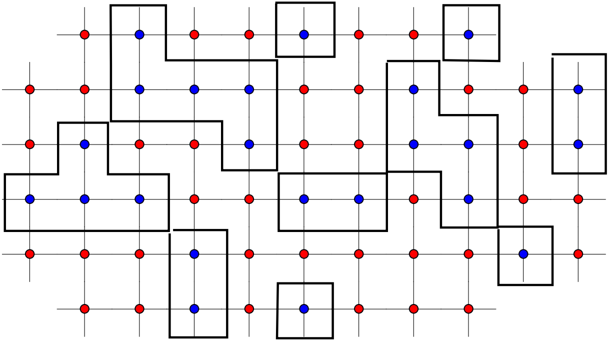
where $\mathcal{O}$ is anything (containing spins, energies, disorders and fermions) away from v.

The rules

$$
\begin{aligned}
\psi_{\hat{w}}\psi_w &= 2(\hat{w}-w)^{-1} + O(\hat{w}-w),\\
\psi_{\hat{w}}\psi^\star_w &= -2i\epsilon_w + O(\hat{w}-w),\\
\psi_{\hat{w}}\epsilon_w &= i(\hat{w}-w)^{-1}\psi^\star_w + O(1),\\
\psi_{\hat{w}}\mu_w &= e^{-\frac{i\pi}{4}}(\hat{w}-w)^{-\frac{1}{2}}\left(\sigma_w + O(\hat{w}-w)\right),\\
\psi_{\hat{w}}\sigma_w &= e^{\frac{i\pi}{4}}(\hat{w}-w)^{-\frac{1}{2}}\left(\mu_w + O(\hat{w}-w)\right),\\
\sigma_{\hat{w}}\sigma_w &= |\hat{w}-w|^{-\frac{1}{4}}\left(1 + \frac{1}{2}\epsilon_w|\hat{w}-w| + o(\hat{w}-w)\right),\\
\mu_{\hat{w}}\mu_w &= |\hat{w}-w|^{-\frac{1}{4}}\left(1 - \frac{1}{2}\epsilon_w|\hat{w}-w| + o(\hat{w}-w)\right),\\
\mu_{\hat{w}}\sigma_w &= |\hat{w}-w|^{\frac{1}{4}}(\psi_w^{\eta_{\hat{w}w}} + O(\hat{w}-w)),\\
\epsilon_{\hat{w}}\epsilon_w &= |\hat{w}-w|^{-2} + O(1),\\
\epsilon_{\hat{w}}\sigma_w &= \frac{1}{2}|\hat{w}-w|^{-1}\sigma_w + O(1),\\
\epsilon_{\hat{w}}\mu_w &= -\frac{1}{2}|\hat{w}-w|^{-1}\mu_w + O(1).
\end{aligned}
$$

Application to SLE_3 variants

Spins configurations can be put in correspondence to loop configurations:



Configurations := $\{S \subset \text{Edges}((\Omega^\delta)^\star) : \partial S = \emptyset \mod 2\}$.
$\mathbb{P}(S) = \frac{1}{Z} x^{|S|}$, where $x = e^{-2\beta} = \sqrt{2} - 1$.

Application to SLE_3 variants

- It is natural to generalize this to:

$$\text{Configurations} := \{S \subset \text{Edges}((\Omega^\delta)^\star) : \partial S = u_1, \ldots, u_m \mod 2\}.$$

$$\mathbb{P}(S) = \frac{1}{Z} x^{|S|}$$

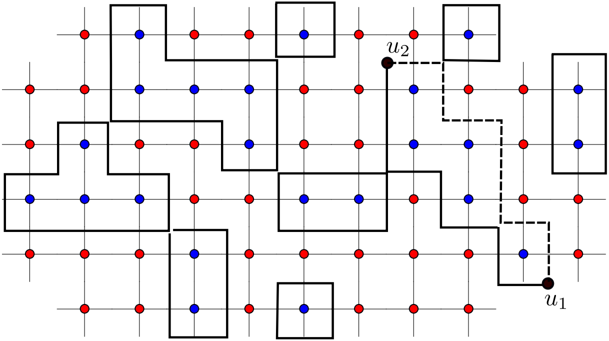
- Apart from loops, there are now *interfaces* connecting $u_1, \ldots, u_m$ in some order.

- If $u_1, \ldots, u_m$ are on the boundary, then this corresponds to imposing $+/-/\ldots/+/-$ boundary conditions.

Application to SLE_3 variants

- D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, S. Smirnov, 2007–2013: interface with $+/-$ boundary conditions converges to SLE_3;
- C. Hongler, K. Kytölä, 2013: interface with $+/-/$free boundary contitions converges to dipolar SLE_3;
- S. Benoist, C. Hongler, 2016 (based on Hongler-Kytölä): the whole collection of loops converges to CLE_3.
- K. I., 2013-2017: any number of boundary arcs with $+$, $-$ and free boundary conditions, multiply connected domains, radial SLE's, ...
- V. Beffara, E. Peltola, H. Wu, 2017: axiomatic approach.

Application to SLE$_3$ variants



Bold: a **random** condiguration S with $\partial S = \{u_1, u_2\}$.
Dashed: a **non-random** "disorder line" γ with $\partial\gamma = \{u_1, u_2\}$.
$S \triangle \gamma$ gives rise to a spin configuration with Ising probability measure tilted by

$$\mu_\gamma = e^{-2\beta \sum_{(xy)\cap\gamma\neq\emptyset} \sigma_x \sigma_y}.$$

Martingale observables

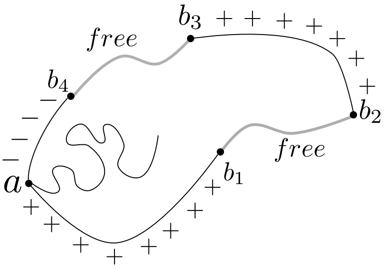
Let $\beta_{[n]}$ be the initial segment of the interface starting from u_1.
Then,

$$\frac{\mathbb{E}_{\Omega^\delta\setminus\beta_{[n]}}(\mathcal{O}\mu_{\gamma\triangle\beta_{[n]}})}{\mathbb{E}_{\Omega^\delta\setminus\beta_{[n]}}(\mu_{\gamma\triangle\beta_{[n]}})} = \mathbb{E}_{\Omega^\delta\setminus\beta_{[n]},\mathrm{tilted}}(\mathcal{O}) = \mathbb{E}_{\Omega^\delta,\mathrm{tilted}}(\mathcal{O}|\mathfrak{F}(\beta_{[n]}))$$

is a martingale with respect to $\mathfrak{F}(\beta_{[n]})$.
This is enough to characterize the scaling limit of γ
Usually, the most convenient choice is $\mathcal{O} = \psi_z\psi_w$ with $w \sim u_j$ for some $j \neq 1$ (as is the case for the original Smirnov's observable for chordal SLE)

Limits of the interfaces

This is enough to characterize the scaling limit of γ



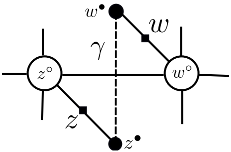
$$da(t) = \sqrt{3}dB_t - \frac{3/2}{a(t)-b_1}dt - \frac{3/2}{a(t)-b_2}dt - \frac{3/2}{a(t)-b_3}dt$$
$$+ 3\left(a(t) - \frac{b_1\sqrt{b_3-b_2}+b_2\sqrt{b_3-b_1}}{\sqrt{b_3-b_2}+\sqrt{b_3-b_1}}\right)^{-1} dt.$$

Overview of the proof

- It suffices to consider $\mathbb{E}(\psi_{z_1} \dots \psi_{z_k} \sigma_{v_1} \dots \sigma_{v_n})$:

$$\psi_z \stackrel{\mathsf{def}}{=} (z^\bullet - z^\circ)^{-\frac{1}{2}} \delta^{\frac{1}{2}} \sigma_{z^\circ} \mu_{z^\bullet} \Longrightarrow \mu_{z^\bullet} = (z^\bullet - z^\circ)^{\frac{1}{2}} \delta^{-\frac{1}{2}} \psi_z \sigma_{z^\circ};$$

$$\psi_z \psi_w = i(z^\bullet - z^\circ)^{-1} \sigma_{w^\circ} \sigma_{z^\circ} e^{-2\beta \sigma_{w^\circ} \sigma_{z^\circ}} = i(z^\bullet - z^\circ)^{-1} \left(1 - \sqrt{2} \sigma_{w^\circ} \sigma_{z^\circ}\right).$$



- "Ising model is a free fermion": *pfaffian structure* of the correlations

$$\frac{\mathbb{E}(\psi_{z_1} \dots \psi_{z_k} \sigma_{v_1} \dots \sigma_{v_n})}{\mathbb{E}(\sigma_{v_1} \dots \sigma_{v_n})} = \mathrm{Pf}\, \frac{\mathbb{E}(\psi_{z_i} \psi_{z_j} \sigma_{v_1} \dots \sigma_{v_n})}{\mathbb{E}(\sigma_{v_1} \dots \sigma_{v_n})}$$

(caveat: z_i is allowed to be one lattice step away from z_j or v_i.)

Overview of the proof: properties of ψ_z

- Discrete holomorphicity: $\mathbb{E}(\psi_z \mathcal{O})$ is discrete holomoprhic in z (away from other marked points)

$\Longrightarrow$ (any subsequential) limit $<\psi_z \mathcal{O}>$ is holomorphic.

- Boundary conditions: (any subsequential) limit satisfies

$$<\psi_z \mathcal{O}> \in \begin{cases} \tau_z^{-\frac{1}{2}} \cdot \mathbb{R}, & \text{fixed b. c.} \\ \tau_z^{-\frac{1}{2}} \cdot i\mathbb{R}, & \text{free b. c.} \end{cases}$$

($\tau_z \in \mathbb{C}$ is the unit tangent vector at z to the boundary.)

- Tractable behaviour at singularities: (any subsequential) limit satisfies the OPEs

$$\psi_{\hat{w}} \psi_w = 2(\hat{w} - w)^{-1} + O(\hat{w} - w);$$
$$\psi_{\hat{w}} \psi_w^\star = O(1);$$
$$\psi_{\hat{w}} \sigma_w = e^{\frac{i\pi}{4}} (\hat{w} - w)^{-\frac{1}{2}} \left(\mu_w + O(\hat{w} - w)\right);$$

Convergence of discrete holomoprhic functions

- When points are apart, we deduce

$$\delta^{-1}\frac{\mathbb{E}_{\Omega^\delta}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1}\dots\sigma_{v_n})}\to C\cdot\frac{\langle\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega}{\langle\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega},$$

  where the RHS is a holomoprhic spinor solving a well-posed, conformally covariant boundary-value problem, with singularities of the type

$$(z-w)^{-1}\quad\text{and}\quad e^{i\pi/4}\alpha_j(z-v_j)^{-\frac{1}{2}},\ \alpha_j\in\mathbb{R}.$$

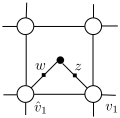
- When z is adjacent to v_1 (that is, at distance δ from v_1), we expect an additional factor of $\delta^{-\frac{1}{2}}$. This turns out to be is indeed true:

$$\delta^{-1}(z-v_1)^{\frac{1}{2}}\frac{\mathbb{E}_{\Omega^\delta}(\psi_z\psi_w\sigma_{v_1}\dots\sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1}\dots\sigma_{v_n})}\to C'\cdot\frac{\langle\psi_w\mu_{v_1}\dots\sigma_{v_n}\rangle_\Omega}{\langle\sigma_{v_1}\dots\sigma_{v_n}\rangle_\Omega},$$

  where the fraction in the RHS is equal to α_1. Similarly when $w\sim v_j$ and/or $w\sim z$.

Pure spin correlations

- Let $\hat{v}_1$ be adjacent to v_1, and take z, w as follows:



$$\frac{\mathbb{E}_{\Omega^\delta}(\psi_z \psi_w \sigma_{v_1} \ldots \sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \ldots \sigma_{v_n})} = \delta(z - v_1)^{-\frac{1}{2}}(w - v_1)^{-\frac{1}{2}} \frac{\mathbb{E}_{\Omega^\delta}(\sigma_{\hat{v}_1} \sigma_{v_2} \ldots \sigma_{v_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \sigma_{v_2} \ldots \sigma_{v_n})}$$

- This allows one to compute the limits of ratios

$$\frac{\mathbb{E}_{\Omega^\delta}(\sigma_{\hat{v}_1} \ldots \sigma_{\hat{v}_n})}{\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \ldots \sigma_{v_n})} \to \frac{\langle \sigma_{\hat{v}_1} \ldots \sigma_{\hat{v}_n} \rangle_\Omega}{\langle \sigma_{v_1} \ldots \sigma_{v_n} \rangle_\Omega}.$$

- Finally, we use that

$$\mathbb{E}_{\Omega^\delta}(\sigma_{v_1} \ldots \sigma_{v_{2n}}) \sim \mathbb{E}_{\mathbb{C}^\delta}(\sigma_{v_1} \sigma_{v_2}) \ldots \mathbb{E}_{\mathbb{C}^\delta}(\sigma_{v_{2n-1}} \sigma_{v_{2n}})$$

as $v_1 \to v_2, \ldots, v_{2n-1} \to v_{2n}$.

Thank you!