Tree Embedding via the Generalized Loewner Equation

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Preview: Main Idea

Just heard from Peter Lin:

Embedding the CRT as a limit of embeddings of finite trees.

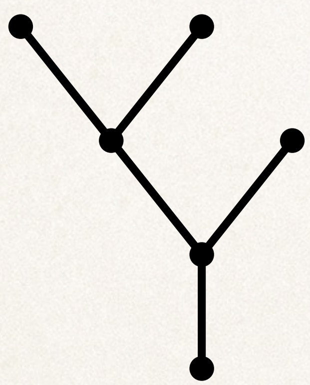
This talk:

- Originally motivated by same question (embedding CRT)
- Different approach: embed trees as growth processes
- Use Loewner equation
- Approach adds geometric difficulty
- Benefits:
  - Links embedding problem to SLE
  - Geometric and analytic properties of independent interest
  - Hope: useful for scaling limit of discrete processes?

Preview: Main Idea

<u>Galton-Watson trees</u>

Describe the genealogy of birth-death processes.



<u>Continuum Random Tree</u>

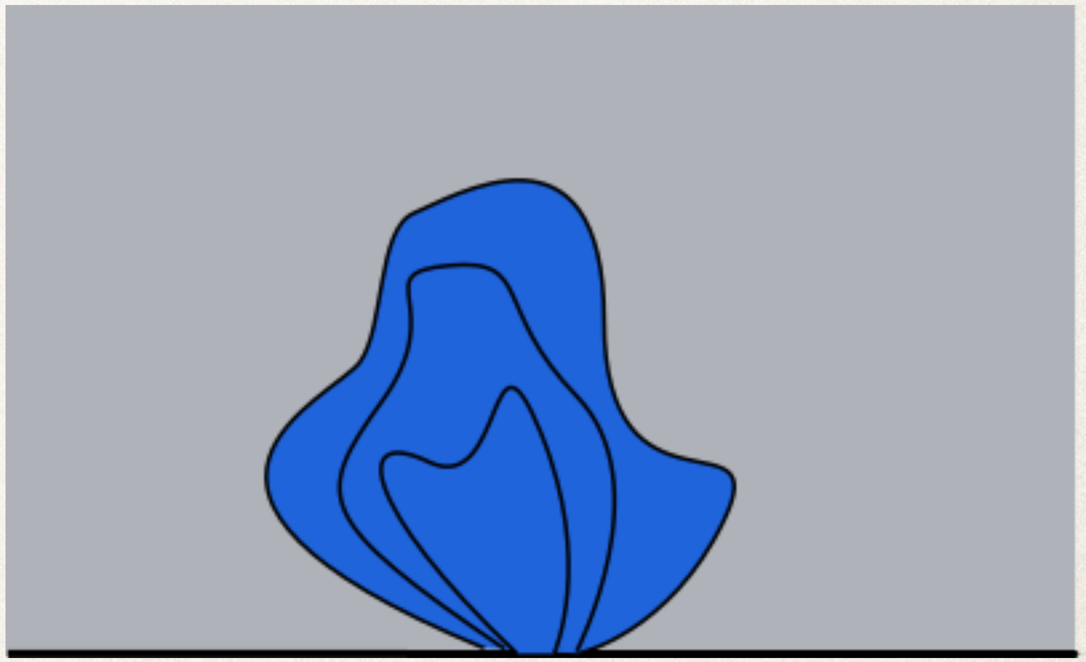
scaling limit of GW trees, conditioned to be large

<u>Chordal Loewner equation</u>

Growing hulls

$\Longleftrightarrow$

Evolving real measures



Preview: Main Idea

<u>Question 1:</u> Can we use the Loewner equation to construct embeddings of Galton-Watson trees in the upper half plane (as growth processes)?

<u>Question 2:</u> Can we construct an embedding of the CRT as a limit of these embeddings of finite Galton-Watson trees?

Preview: Main Idea

Answer to Q1:

Let μ_t be the evolving real measure:

- supp(μ_t) is a particle system on the real line
- branching determined by a tree T
  - "birth" in T: particle duplicates
  - "death" in T: particle disappears
- repulsion ~ $(x_i - x_j)^{-1}$

"Theorem": Let K_s = hull generated by the Loewner equation with driving measure μ above.

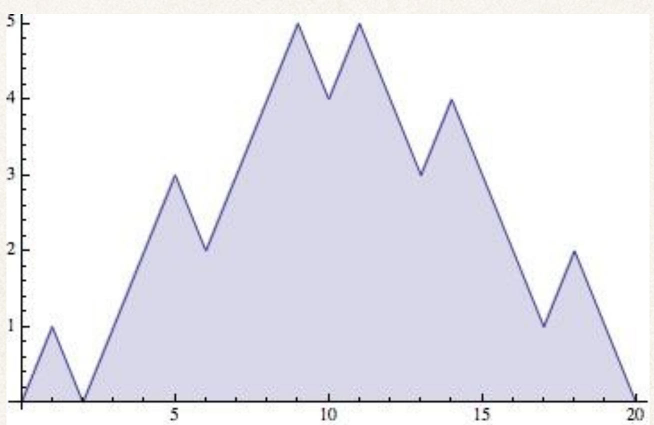
- K_s is a graph embedding of the subtree of T_s
- $K_s \subset K_{s'}$ if $s < s'$.

Contents

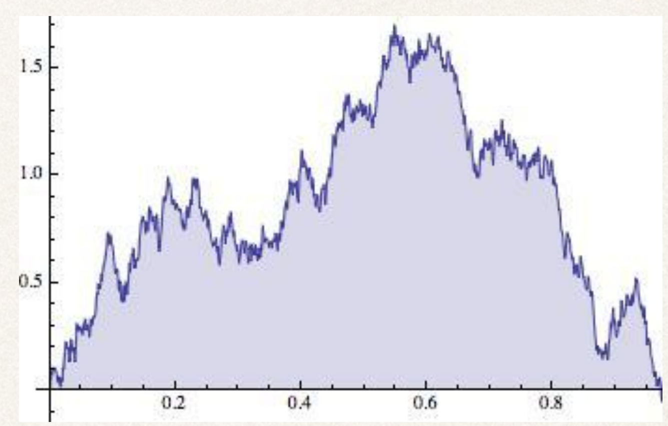
The Continuum Random Tree

<u>Definition</u>: The continuum random tree (CRT) is the random metric tree coded by the normalized Brownian excursion $\mathbb{e}$.

Uniform distribution on Dyck paths (length $2n$) $\longrightarrow$ $\mathbb{e}$.
Uniform distribution on plane trees (n edges) $\longrightarrow$ CRT.



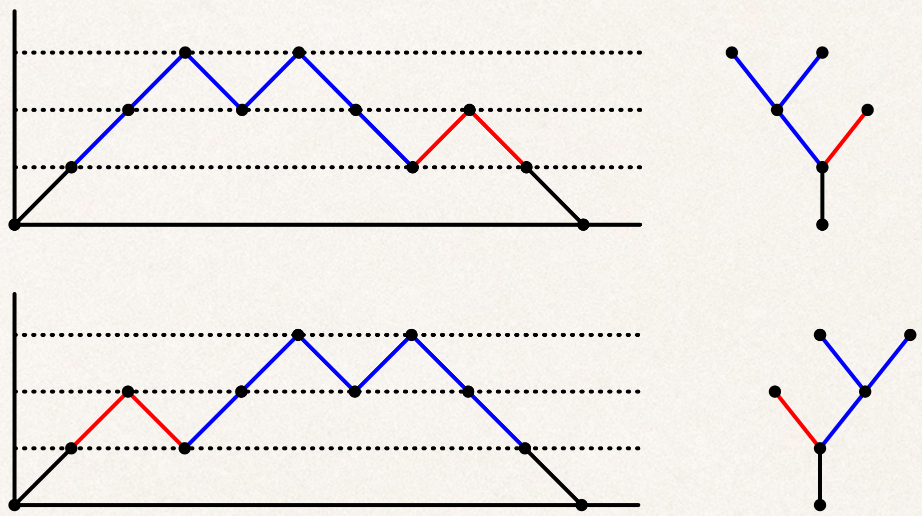
$\longrightarrow$



$$\left(\frac{1}{\sqrt{2n}} C_n(2nt)\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(\mathrm{d})} (\mathbb{e}_t)_{0\le t\le 1}$$

The Continuum Random Tree

Note: many different contour functions code the same metric tree.



CRT

- Limit of metric trees distributed according to the uniform distribution on Dyck paths.
- A random metric space. Not directly a limit of random planar maps.

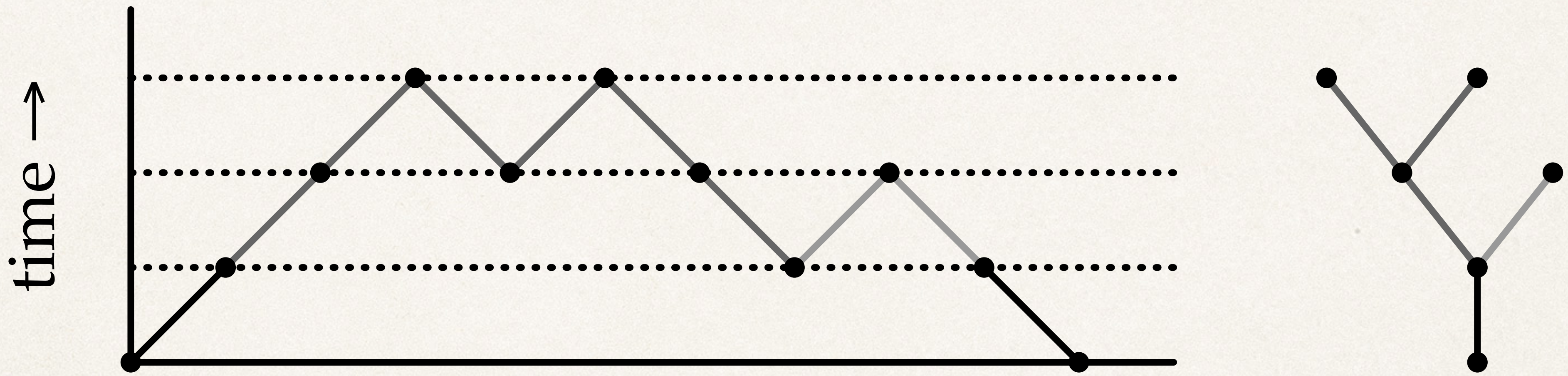
Goal: Take the scaling limit of embedded plane trees to get an embedding of the CRT.

Plane trees as growth processes

- Our approach: think of trees as growth processes

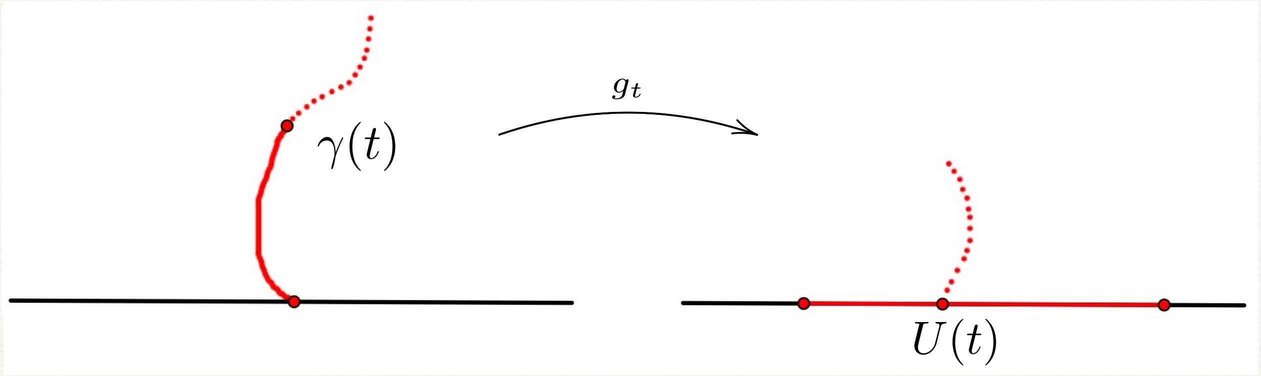
- Graph distance from the root = time parameter:

  branching processes $\Longleftrightarrow$ plane trees.



The Loewner Equation & SLE

Let $\gamma : (0, T] \to \mathbb{H}$ be a simple curve with $\gamma(0) \in \mathbb{R}$.



Loewner (1920s): g_t satisfies the initial value problem

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z.$$

The Loewner Equation & SLE

Generalized version: Let $g_t(z)$ denote the solution to the initial value problem

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \quad g_0(z) = z.$$

Let $H_t = \{z \in \mathbb{H}\}$ for which $g_t(z) \in \mathbb{H}$ is well defined.

Then g_t is the unique conformal map from H_t onto $\mathbb{H}$ with the hydrodynamic normalization.

Hull generated by μ_t: $K_t = \mathbb{H} \setminus H_t$.

Idea: The measure is supported on points that are escaping $\mathbb{H}$.

Geometry of H_t? Need to know fine properties of μ_t.

The Loewner Equation & SLE

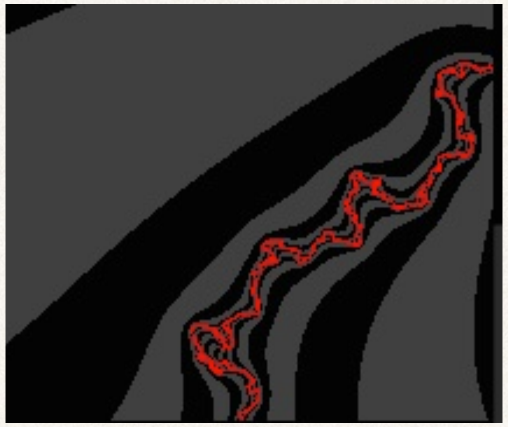
We consider the generalized Loewner equation for discrete driving measures μ_t:

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \quad g_0(z) = z.$$

<u>Examples:</u>

1) $\mu_t = \delta_{U(t)}$

2) $\mu_t = \delta_{\sqrt{\kappa} B_t}$ generates SLE$_\kappa$.



3) $\mu_t = \sum_{i=1}^{N} \delta_{U_i(t)}$ produces the multislit equation:

$$\dot{g}_t(z) = \sum_{i=1}^{N} \frac{1}{g_t(z) - U_i(t)}.$$

<u>Goal</u>: Piece together simple curves in multislit equation. (N is varying.)

Tree Embedding: (α, β)-approach

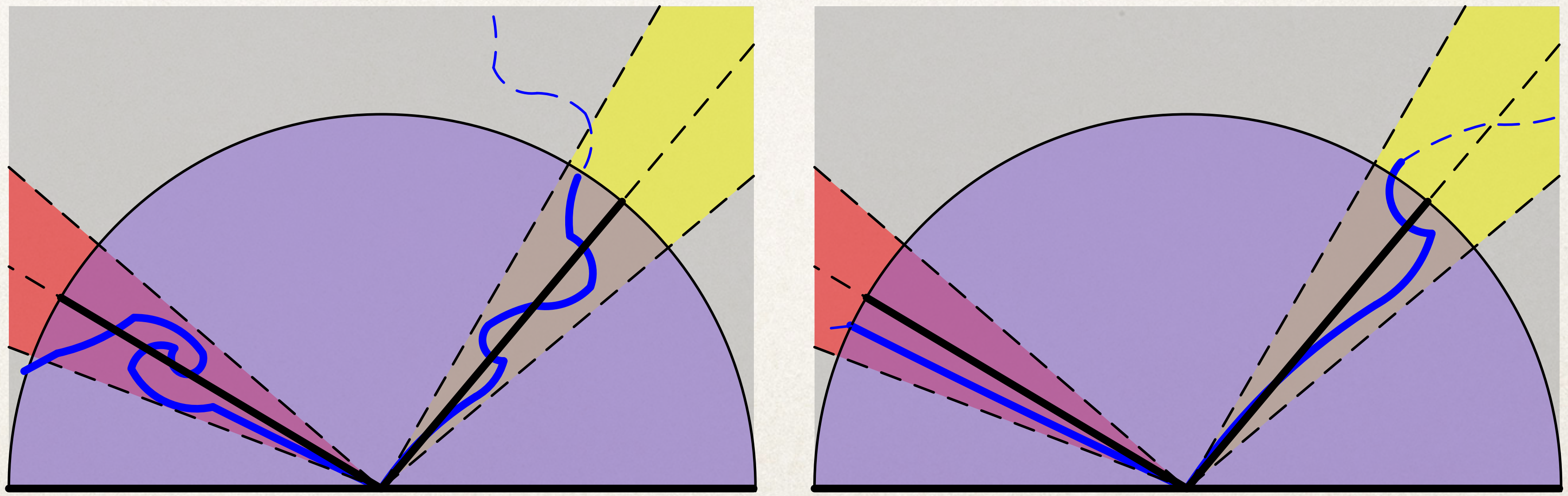
Setup:

- $U_1, \ldots, U_n$ continuous functions $U_i : [0,T] \to \mathbb{R}$
- $U_j(0) = U_{j+1}(0)$
- mutually nonintersecting: $U_i(t) < U_{i+1}(t)$, $i = 1, \ldots, n$, $t \in [0,T]$

$$\mu_t = c \sum_{i=1}^{n} \delta_{U_i(t)}$$

Local behavior: want hulls ρK_t to converge in the Hausdorff metric to $\boldsymbol{V}_{\alpha,\beta}$ inside the disc D_R centered at 0. (Call this property (α,β)-approach.)

(Motivated by Schleissinger '12.)



Tree Embedding: (α, β)-approach

Theorem (H, Menon, 2017): In the setting above, if

$$\lim_{t\searrow 0}\frac{U_j(t)-U_j(0)}{\sqrt{t}}=\phi_1(\alpha,\beta)-\phi_2(\alpha,\beta)$$

$$\lim_{t\searrow 0}\frac{U_{j+1}(t)-U_{j+1}(0)}{\sqrt{t}}=\phi_1(\alpha,\beta)+\phi_2(\alpha,\beta),$$

for $\phi_1(\alpha, \beta)$ and $\phi_2(\alpha, \beta)$ given below, then the hulls K_t approach $\mathbb{R}$ in (α, β)-direction at $U_j(0)$.

$$\phi_1(\alpha,\beta)=\sqrt{c}\frac{(1+x-3a-3bx)}{\sqrt{a(1-a)-2abx+b(1-b)x^2}}$$

$$\phi_2(\alpha,\beta)=\sqrt{c}\sqrt{\frac{(1-a)^2+2x(a+b+ab-1)+x^2(1-b)^2}{a(1-a)-2abx+b(1-b)x^2}},$$

where $\alpha=a\pi$, $\beta=b\pi$, and $x=x(a,b)$ is the unique negative root of

$$-a+a^3+3ax-3a^2x-3abx+3a^2bx+3bx^2-3abx^2-3b^2x^2+3ab^2x^2-bx^3+b^3x^3.$$

Tree Embedding: (α, β)-approach

<u>Balanced case</u>: If $0 < \alpha = \beta < \pi/2$, then ϕ_1 and ϕ_2 simplify to

$$\phi_1(\alpha, \alpha) = 0 \quad \text{and} \quad \phi_2(\alpha, \alpha) = \sqrt{2c}\sqrt{\frac{\pi - 2\alpha}{\alpha}}.$$

<u>Intuitively:</u> Loewner scaling

- If μ_t generates hulls K_t, then $\rho\mu_{t/\rho^2}$ generates the hulls ρK_t.
- So, expect to see $\sqrt{t}$ whenever a hull is preserved under dilation.

<u>Advantage of explicit expressions for ϕ_1 and ϕ_2:</u>

- Gives the precise angles.

Tree Embedding: (α, β)-approach

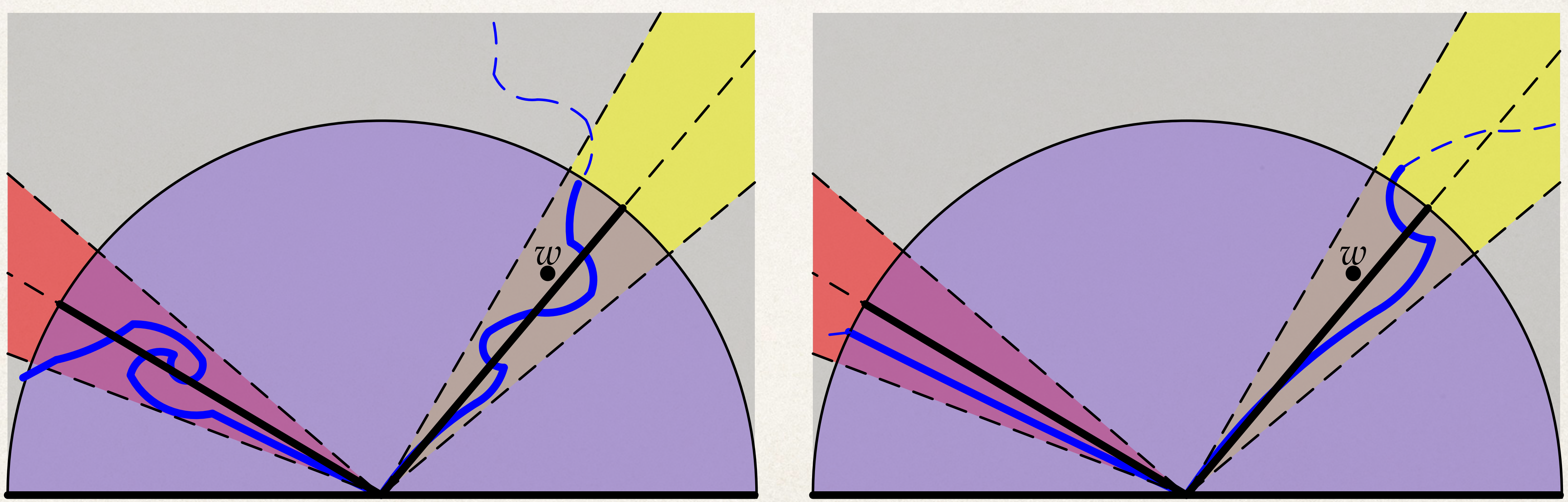
Proof idea:

Use estimates on conformal radius (comparable to Euclidean distance)

$$\mathrm{rad}(w, \mathbb{H} \setminus \rho K_t) = \frac{2\Im(g_t^\rho(w))}{\left|(g_t^\rho)'(w)\right|}$$

Need to uniformly bound $\left|\frac{\partial}{\partial z}(h_t(z)) - g_t^\rho(z))\right|$.

(Show contribution of other driving points is negligible.)



K_t ρK_t

A Specific Tree Embedding

Let $\mathcal{T} = \{v, h(v)\}$ be a marked plane tree.

- Think: $h(v)$ = time of death of v

Let μ_t be supported on elements of $\mathcal{T}$ alive at t:
$$\mu_t = c \sum_{v \in \Delta_t \mathcal{T}} \delta_{U_v(t)}.$$

On time intervals without branching, how should the U_v evolve?

- Dyson Brownian motion? We'll come back to this at the end.

A Specific Tree Embedding

Let $\mathcal{T} = \{v, h(v)\}$ be a marked plane tree.

- Think: $h(v)$ = time of death of v

Let μ_t be supported on elements of $\mathcal{T}$ alive at t: $$\mu_t = c \sum_{v \in \Delta_t \mathcal{T}} \delta_{U_v(t)}.$$

On time intervals without branching:

$$\dot{U}_v(t) = \sum_{v \neq \eta \in \Delta_t \mathcal{T}} \frac{c_1}{U_v(t) - U_\eta(t)}.$$

Theorem (H, Menon, 2017): Let $\mathcal{T}$ be a binary tree with $h_v \neq h_\eta$.
Let $\{K_s\}$ be the hulls generated by the Loewner equation driven by μ.
Then each K_s is a graph embedding of the subtree

$$\mathcal{T}_s = \{v \in \mathcal{T} : h(p(v)) < s\}$$

in $\mathbb{H}$, with the image of the root on the real line, and $K_s \subset K_{s'}$ if $s < s'$.

A Specific Tree Embedding

Proof (idea):

The proof relies on analyzing the interacting particle system

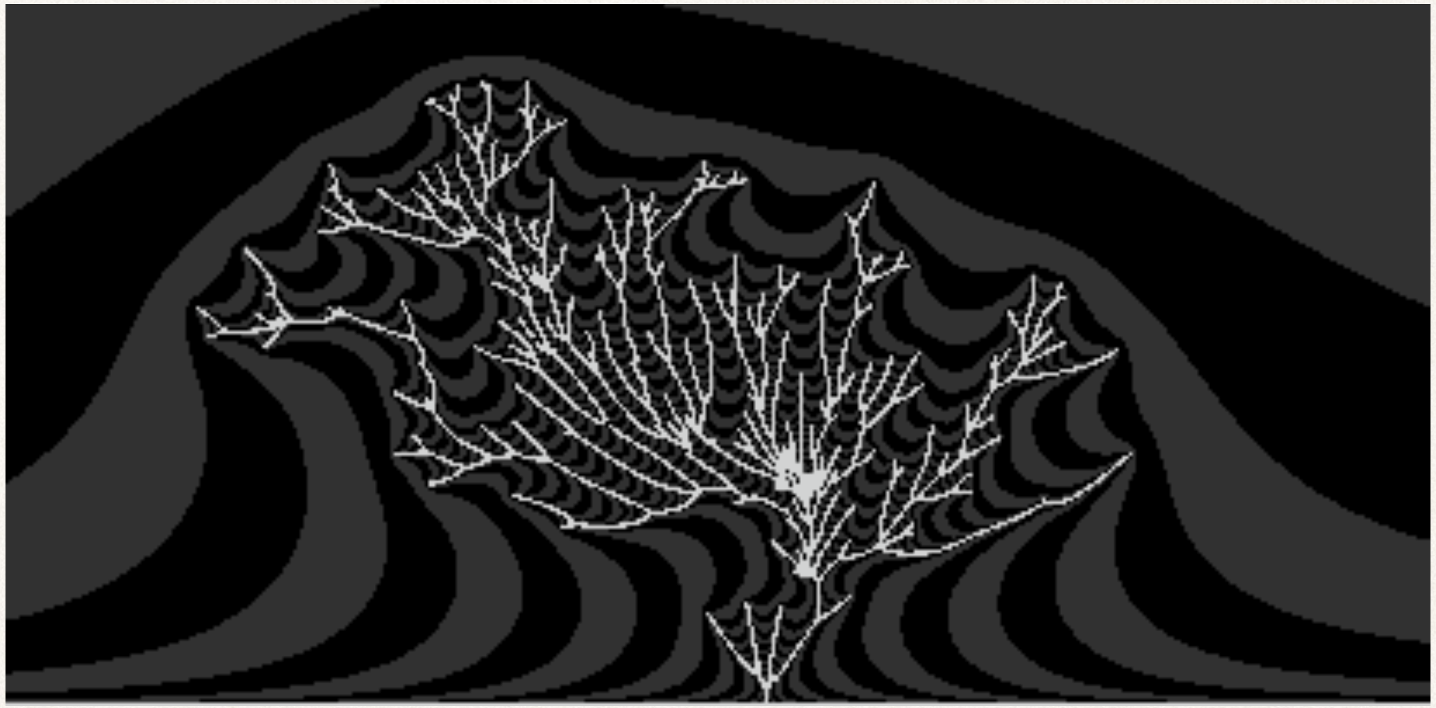
$$\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t \mathcal{T}} \frac{c_1}{U_\nu(t) - U_\eta(t)}.$$

- Extend the solution backward to the initial condition $U_j(0) = U_{j+1}(0)$.
- Show that the solution gives simple curves away from $t = 0$.
  (Use Marshall & Rohde '05, Lind '05, Schleissinger '13)
- Show that the generated hull approaches $\mathbb{R}$ in (α, α)-direction for

$$\alpha = \frac{\pi}{2 + \frac{c_1}{2c}}.$$

$\square$

Application: Galton-Watson Trees



Resulting embedding of a sample of a binary Galton-Watson tree with exponential lifetimes.

(Simulation courtesy of Brent Werness.)

Scaling Limit?

Question 2 (geometric scaling limit):

- Let $\{T_k\}$ be a sequence of random trees that (when appropriately rescaled) converges in distribution to the CRT when T_k is conditioned on having k edges.
- Does the law of the generated hulls converge to a scaling limit?

Question 2a (first step):

- Find the scaling limit of the corresponding sequence of **measure-valued processes.**

Choosing a Sequence of Measures

Let $\{\mathcal{T}_k\}$ be a sequence of random trees.

Let $\{c^k\}$ and $\{c_1^k\}$ be two sequences in $\mathbb{R}^+$. For each k, define

$$\mu_t^k = c^k \sum_{\nu \in \Delta_t \mathcal{T}_k} \delta_{U_\nu(t)},$$

where the $U_\nu(t)$ evolve according to

$$\dot{U}_\nu(t) = \sum_{\nu \neq \eta \in \Delta_t \mathcal{T}_k} \frac{c_1^k}{U_\nu(t) - U_\eta(t)}.$$

- Same setting as tree embedding theorem.
- Remains to choose random trees $\{\mathcal{T}_k\}$ and constants $\{c^k\}$ and $\{c_1^k\}$.

The Scaling Limit

Choose: T_k distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges.

Theorem (Aldous '91): T_k converges in distribution to the CRT as $k \to \infty$.

The Scaling Limit

<u>Theorem</u> (H, Menon '17): For each k, let T_k be distributed as a critical binary Galton-Watson tree with exponential lifetimes of mean $\frac{1}{2\sqrt{k}}$, conditioned to have k edges, and let $\{\mu^k\}$ be the corresponding sequence of measures.
If the scaling constants are

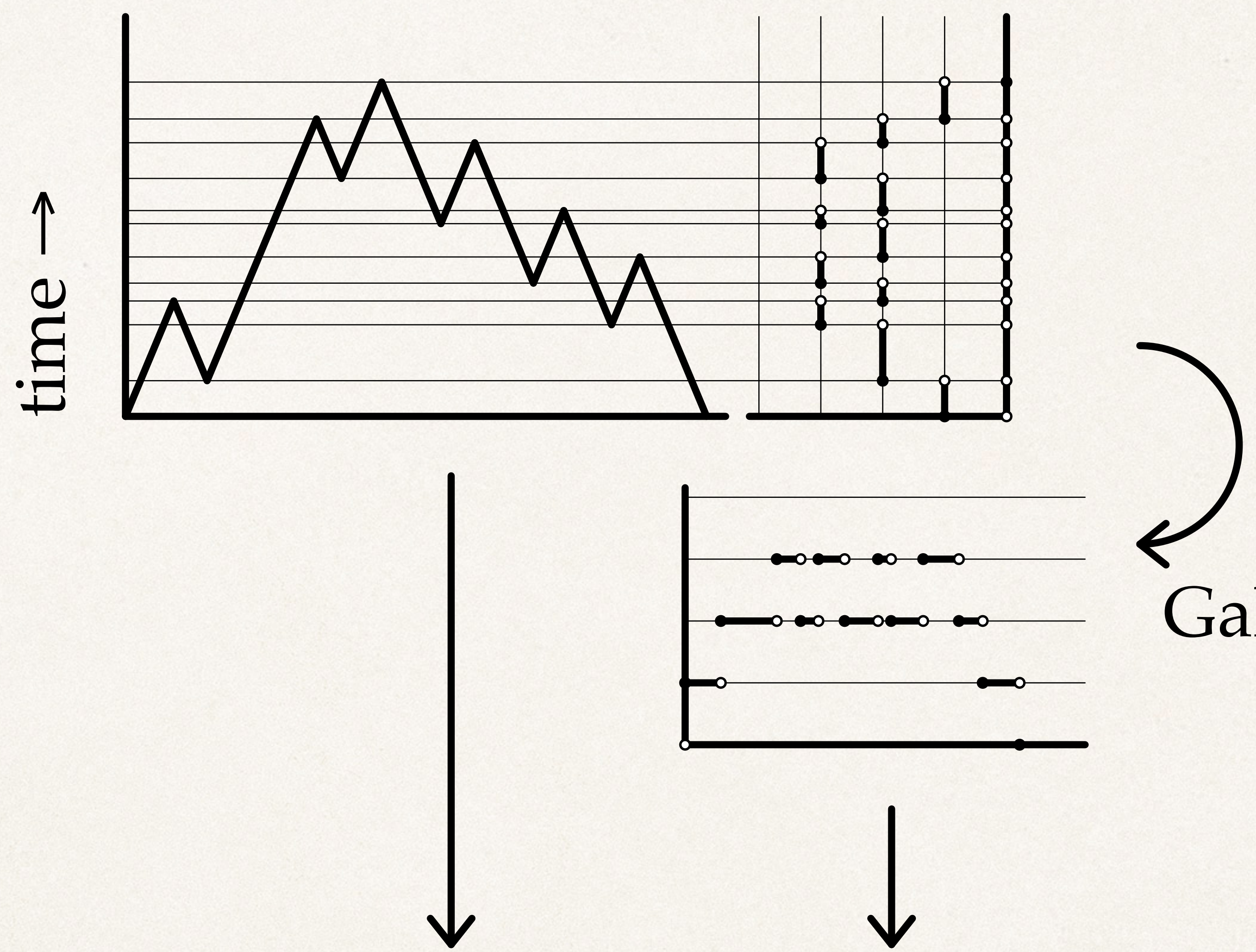
$$c^k = c_1^k = \frac{1}{\sqrt{k}}$$

then the sequence $\{\mu^k\}$ is tight in $D_{\mathcal{M}_f(\hat{\mathbb{R}})}[0, \infty)$.

- Choose $c^k = c_1^k$, since the ratio c_1^k / c^k determines the branching angle.
- $c^k = 1/\sqrt{k}$ is the rescaling for which the total population process of T_k converges to $L_{\mathbb{e}}^t$, the local time at level t of the normalized Brownian excursion.

The Scaling Limit

Contour function



Galton-Watson process

Normalized Brownian excursion $(\mathbb{e})_{0\leq t\leq 1}$

Local time at level t of normalized Brownian excursion: $L^t_{\mathbb{e}}$ (Pitman)

The Flow of the Stieltjes Transform

- To understand the limit, reframe problem in terms of the evolution of the Stieltjes transform:

$$f(t,z) = \int_{\mathbb{R}} \frac{1}{z-x} \mu_t(dx).$$

Setup:

- N_t be a critical binary Galton-Watson process with exponential lifetimes.
- $\mathcal{T}$ is the genealogical tree of N_t, so that $N_t = |\Delta_t \mathcal{T}|$.
- μ_t be defined as before, with indexing tree $\mathcal{T}$.
- $f(t, z)$ denotes the Stieltjes transform of μ_t.

Theorem (H, Menon '17):

Then $f(t, z)$ has the distribution of the solution to the equation

$$\partial_t f = -\frac{c_1}{c} f \partial_z f - \frac{c_1}{2} \partial_z^2 f + \frac{c}{z - Y_t} \partial_t N_t,$$

where $Y_t \sim \dfrac{\mu_{t-}}{|\mu_{t-}|}$.

The Flow of the Stieltjes Transform

For the sequence of constants $c^k = c_1^k = \frac{1}{\sqrt{k}}$, we have equations

$$\partial_t f_k = -f_k \, \partial_z f_k - \frac{1}{2\sqrt{k}} \partial_z^2 f_k + \frac{1}{z - Y_t^k} \partial_t \frac{N_t^k}{\sqrt{k}}.$$

<u>Question 3:</u> Can we identify a limiting equation?

Conjectural Limiting Equation

<u>Conjecture (H, Menon):</u> In the unconditioned case, the limit

$$f = \lim_{k\to\infty} f_k = \int_{\mathbb{R}} \frac{1}{z-x}\mu_t^\infty(dx)$$

exists, and there is a real constant $\sigma > 0$ such that f has the same distribution as the solution to the equation

$$\partial_t f = -f\partial_z f + \sigma h(z,t),$$

where $h(z,t)$ is the Gaussian analytic function with covariance kernel

$$\mathbb{E}\left(h(z,t)h(\bar{w},t')\right) = \delta(t-t')\int_{\mathbb{R}} \frac{1}{z-x}\frac{1}{\bar{w}-x}\mu_t^\infty(dx).$$

<u>Motivation:</u> jumps happen very quickly compared to the diffusion of μ_t.

Conjectural Limiting Equation

<u>Further evidence:</u> Let $\rho(x,t)$ denote the density of the limiting measure. (We don't know a density exists, but suppose it does.)

<u>Dawson-Watanabe superprocess</u> (superbrownian motion)
Scaling limit of branching Brownian motion.
The limiting density ρ satisfies

$$\frac{\partial}{\partial t}\rho(x,t) = \underbrace{\frac{1}{2}\partial_x^2\rho(x,t)}_{\text{motion term}} + \underbrace{\sqrt{\sigma^2\rho(x,t)}\cdot\dot{W}}_{\text{branching term}},$$

where $\dot{W}$ is space-time white noise. [Dawson '75, LeGall '99]

Motion term is time derivative of density of particle motion. (Density of Brownian motion satisfies the heat equation.)

Conjectural Limiting Equation

Dawson-Watanabe superprocess:

$$\frac{\partial}{\partial t}\rho(x,t) = \underbrace{\frac{1}{2}\partial_x^2\rho(x,t)}_{\text{motion term}} + \underbrace{\sqrt{\sigma^2\rho(x,t)}\cdot\dot{W}}_{\text{branching term}},$$

Our case: Motion given by the complex Burgers equation:

$$\partial_t f = -f\partial_z f,$$

where $f(z,t)$ is the Stieltjes transform, from which we can derive

$$\frac{\partial}{\partial t}\rho(x,t) = -\partial_x\left(\rho\cdot\mathcal{H}\rho\right) + \sqrt{\sigma^2\rho(x,t)}\cdot\dot{W},$$

where $\mathcal{H}\rho$ is the Hilbert transform of ρ. $\left(\mathcal{H}\rho(x,t) = \frac{\text{p.v.}}{\pi}\int_{\mathbb{R}}\frac{1}{x-\xi}\rho(\xi,t)d\xi\right)$

Exactly the boundary limit of the equation in the conjecture!

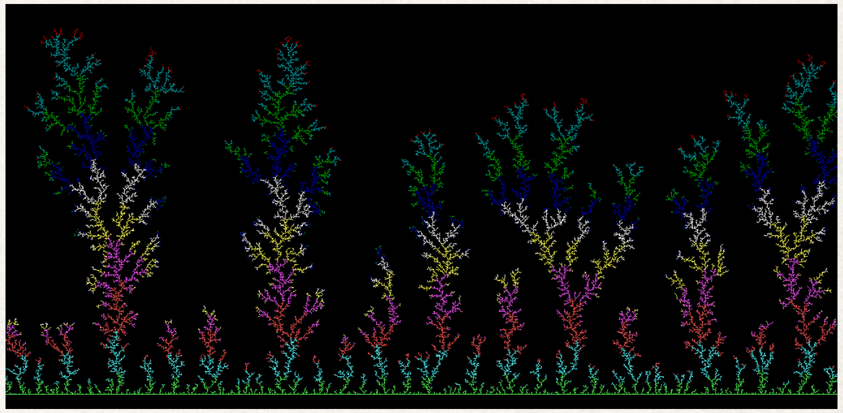
Open Problems and Applications

- Geometric limit of these embeddings

- Repulsion force = deterministic part of Dyson Brownian motion.
  Dyson BM with branching?

$$dU_i = dB_i + \sum_{j \neq i} \frac{dt}{U_i - U_j}$$

- Applications to growth processes that exhibit branching behavior?



- Scaling limit of discrete models?

Thank you!

Simulation for Dyson Brownian motion

Thank you!

Questions?