Complex Generalized Integral Means Spectrum of Whole-Plane SLE

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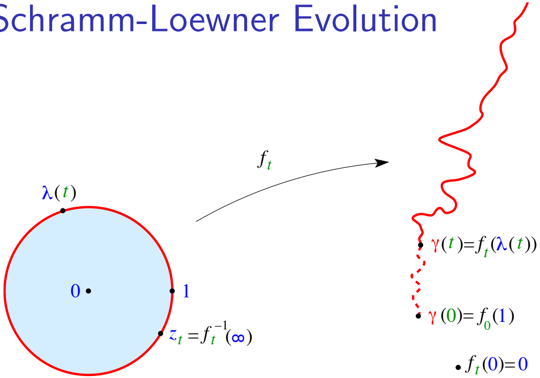
Whole-Plane Schramm-Loewner Evolution

- 

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$$z \in \mathbb{D}, \quad \frac{\partial}{\partial t} f_t(z) = z \frac{\partial}{\partial z} f_t(z) \frac{\lambda(t)+z}{\lambda(t)-z}, \quad \lambda(t) = \exp(i\sqrt{\kappa} B_t)$$

$$f_t(e^{-t}z) \to z,\ t \to +\infty; \quad \kappa = 0, \quad f_t(z) = \frac{e^t z}{(1-z)^2} \quad \text{(Koebe)}$$

- $f^{[-1]}(z) := 1/f(1/z)$ is the **bounded exterior version** from $\mathbb{C} \setminus \overline{\mathbb{D}}$ to the slit plane [Beliaev & Smirnov, Lawler].

Integral Means Spectrum

- Consider an injective Riemann map $\Phi \in \mathcal{S}$, i.e.,
$$\Phi : \mathbb{D} \to \mathbb{C},\ \Phi(0) = 0,\ \Phi'(0) = 1.$$

- The **integral means** of Φ are
$$\mathcal{I}(r, p, \Phi) := \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta,\ 0 < r < 1,\ p \in \mathbb{R};$$

- Φ **random**:
Expectation: $\mathbb{E}\,\mathcal{I}(r, p, \Phi) := \int_0^{2\pi} \mathbb{E}\left[|\Phi'(re^{i\theta})|^p\right] d\theta.$
- One then defines
$$\beta_\Phi(p) := \limsup_{r \to 1^-} \frac{\log(\mathcal{I}(r, p, \Phi))}{\log(\frac{1}{1-r})};$$

- If the limit exists,
$$\mathcal{I}(r, p, \Phi) \overset{r \to 1^-}{\asymp} \frac{1}{(1-r)^{\beta_\Phi(p)}}.$$

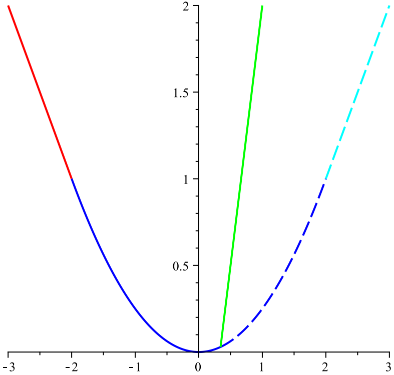
Integral means spectrum & harmonic measure

- The **integral means spectrum** is related to the **multifractal spectrum** of the **harmonic measure** ω on the boundary of the image domain.
- Define, for $\alpha \geq 1/2$, $\mathcal{E}_\alpha$ as being the set of points z on the boundary where
$$\omega(B(z,r)) \sim r^\alpha,$$
as $r \to 0$.
- The multifractal spectrum of ω is the function $f(\alpha) = D_{\mathrm{Hausdorff}}(\mathcal{E}_\alpha)$.
- One goes from the integral means spectrum β to f by a **Legendre transform**,
$$\frac{1}{\alpha} f(\alpha) = \inf_p \left\{ \beta(p) - p + 1 + \frac{1}{\alpha} p \right\},$$
$$\beta(p) = \sup_\alpha \left\{ \frac{1}{\alpha}(f(\alpha) - p) \right\} + p - 1.$$

Universal Integral Means Spectrum

- $B(p) = \sup\{\beta_\Phi(p),\ \Phi \in \mathcal{S}\}$.
- $B_{\mathrm{bd}}(p) = \sup\{\beta_\Phi(p),\ \Phi \in \mathcal{S},\ \Phi\ \text{bounded}\}$.
- Theorem (Makarov):

$$B(p) = \max\{B_{\mathrm{bd}}(p), 3p - 1\}.$$



Generalized Integral Means Spectrum

- Consider a (**random**) injective Riemann map $\Phi \in \mathcal{S}$, i.e.,

$$\Phi : \mathbb{D} \to \mathbb{C},\ \Phi(0) = 0,\ \Phi'(0) = 1.$$

- For $(p, q) \in \mathbb{R}^2$, define the generalized integral means

$$\mathcal{I}(r, p, q, \Phi) := \int_0^{2\pi} \frac{|\Phi'(re^{i\theta})|^p}{|\Phi(re^{i\theta})|^q} d\theta,\ \ 0 < r < 1;$$

- **Expected:** $\mathbb{E}\,\mathcal{I}(r, p, q, \Phi) := \int_0^{2\pi} \mathbb{E}\frac{|\Phi'(re^{i\theta})|^p}{|\Phi(re^{i\theta})|^q} d\theta,\ \ 0 < r < 1.$
- Define

$$\beta_\Phi(p, q) := \limsup_{r \to 1^-} \frac{\log(\mathcal{I}(r, p, q, \Phi))}{\log(\frac{1}{1-r})};$$

- If the limit exists,

$$\mathcal{I}(r, p, q, \Phi) \overset{r \to 1^-}{\asymp} \frac{1}{(1-r)^{\beta_\Phi(p,q)}}.$$

Generalized Integral Means Spectrum

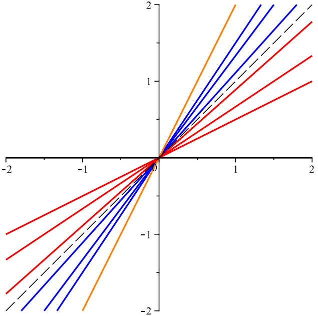
- Unified treatment of the bounded and the unbounded cases.
- $\Phi \in \mathcal{S} \Rightarrow \Psi = \frac{1}{\Phi}$ is bounded,
- $$|\Psi'|^p = \frac{|\Phi'|^p}{|\Phi|^{2p}}.$$
- m-fold transform of $f \in \mathcal{S}$: $f^{[m]}(z) := \sqrt[m]{f(z^m)}$, $m \in \mathbb{Z}_+$, holomorphic branch with derivative 1 at 0.
  For $m \in \mathbb{Z}_-$ and $z \in \mathbb{D}_- := \mathbb{C} \setminus \overline{\mathbb{D}}$, $f^{[m]}(z) := 1/f^{[-m]}(1/z)$.
  For $m < 0$, $f^{[m]}(\mathbb{D}_-)$ has bounded boundary.
  For $m = -1$, $f^{[-1]}(z) = 1/f(1/z)$, is the **bounded exterior whole-plane** of Beliaev & Smirnov.
- $$|(f^{[m]})'(z)|^p = |z|^{p(m-1)} \frac{|f'(z^m)|^p}{|f(z^m)|^{p(1-\frac{1}{m})}}.$$

Generalized Integral Means Spectrum

- One finds various **standard spectra** in the (p, q) plane:
- The **standard integral means spectrum** on the line $q = 0$,
- The **bounded** one on the **line** $q = 2p$,
- The spectrum for the ***m*-fold** $f^{[m]}(z) = (f(z^m))^{\frac{1}{m}}$, $m \in \mathbb{Z}_+$,

$$\beta^{[m]}(p) = \beta^{[1]}(p, q_m), \quad q_m := p(1 - 1/m);$$

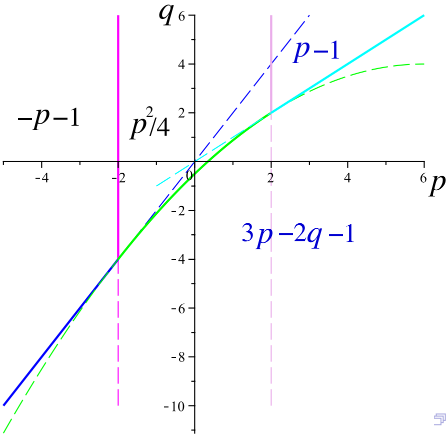
- The standard spectrum for the m-fold for $m \in \mathbb{Z}_-$.



Universal Generalized Integral Means Spectrum

- One can similarly define a **universal generalized** integral means spectrum.
- Theorem (Astala, D., Zinsmeister):

$$B(p,q) = \max\{B_{\mathrm{bd}}(p), 3p-2q-1\}.$$



Beliaev-Smirnov Generalized PDE

- Let f be a **whole-plane (inner)** SLE$_\kappa$, $z \in \mathbb{D}$, $(p.q) \in \mathbb{R}^2$

$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}}\left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right), \; G(z,\bar{z}) := \mathbb{E}\left(|f'(z)|^{p}\left|\frac{z}{f(z)}\right|^{q}\right).$$

- Using the SLE equation and Itô calculus, one derives a differential equation satisfied by F,

$$\mathcal{P}(\partial)[F(z)] = \Bigg[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z - \frac{p}{(1-z)^2} + \frac{q}{1-z} + p - q\Bigg]F(z) = 0,$$

- and a partial differential equation satisfied by G,

$$\mathcal{P}(D)[G(z,\bar{z})] = \Bigg[-\frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2 - \frac{1+z}{1-z}z\partial_z - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\partial_{\bar{z}} - \frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} + \frac{q}{1-z} + \frac{q}{1-\bar{z}} + 2(p-q)\Bigg]G(z,\bar{z}) = 0.$$

Integrable Probability

- Let f be a time 0 whole-plane (inner) SLE_κ, and $(p, q) \in \mathbb{R}^2$,

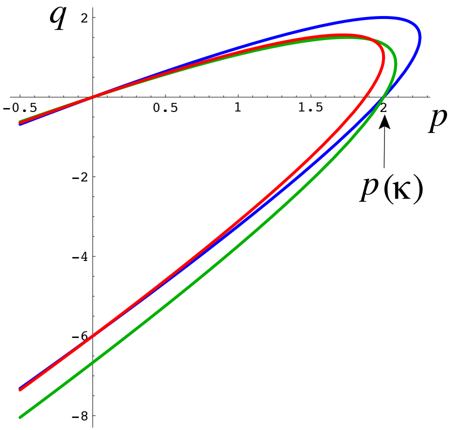
$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}}\left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right), \; G(z,\bar{z}) := \mathbb{E}\left(|f'(z)|^p \left|\frac{z}{f(z)}\right|^q\right).$$

- **Integrable parabola** with parameterization,

$$p(\gamma) := (2 + \frac{\kappa}{2})\gamma - \frac{\kappa}{2}\gamma^2, \;\; \gamma \in \mathbb{R},$$
$$q(\gamma) := (3 + \frac{\kappa}{2})\gamma - \kappa\gamma^2.$$

- **Theorem** [DHLZ '18]: If $p = p(\gamma)$ and $q = q(\gamma)$, then

$$F(z) = (1 - z)^\gamma, \;\; G(z_1, \bar{z}_2) = \frac{(1 - z_1)^\gamma (1 - \bar{z}_2)^\gamma}{(1 - z_1\bar{z}_2)^{\kappa\gamma^2/2}}.$$



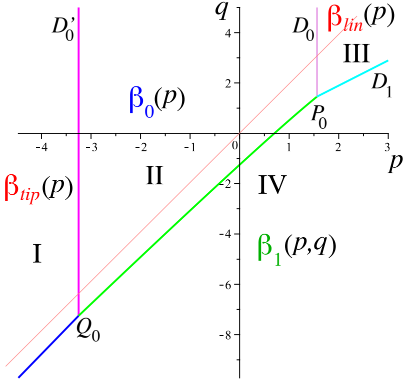
Integrable parabola for $\kappa \in \{2, 4, 6\}$
Other integrable parabolae.

Generalized Integral Means Spectrum of Whole-Plane SLE

- The generalized spectrum is [D., Ho, Le & Zinsmeister '18],

$$\beta_1(p, q; \kappa) := 3p - 2q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)}.$$

- Phase transition lines: **green parabola** & **blue quartic**



SLE Standard Integral Means Spectrum

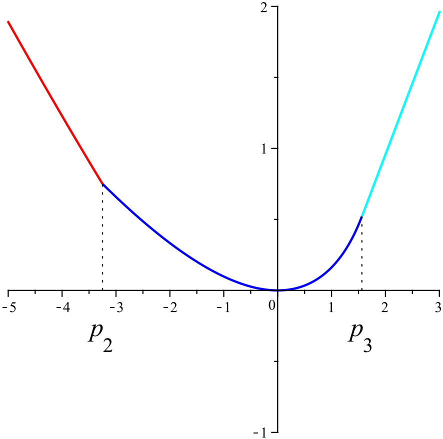
- As predicted in Lawler & Werner '99, D. '99, (BM), D.'00, and Hastings '02, and proven in Beliaev & Smirnov '05, and Beliaev, D. & Zinsmeister '17, the **average spectrum** of SLE_κ involves 3 phases:

$$\beta_{\mathrm{tip}}(p,\kappa) = -p - 1 + \frac{1}{4}\left(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\right),$$

$$\beta_0(p,\kappa) = -p + \frac{(4+\kappa)^2}{4\kappa} - \frac{(4+\kappa)}{4\kappa}\sqrt{(4+\kappa)^2-8\kappa p},$$

$$\beta_{\mathrm{lin}}(p,\kappa) = p - \frac{(4+\kappa)^2}{16\kappa}.$$

- a.s. β_{tip} [Johansson Viklund & Lawler '12]
- a.s. β_0 [Gwynne, Miller & Sun '18]
- a.s. boundary spectrum [Alberts, Binder & Viklund '16] [Schoug '18]



$$p_2 = -1 - \frac{3\kappa}{8}, \quad p_3 = \frac{3(4+\kappa)^2}{32\kappa}$$

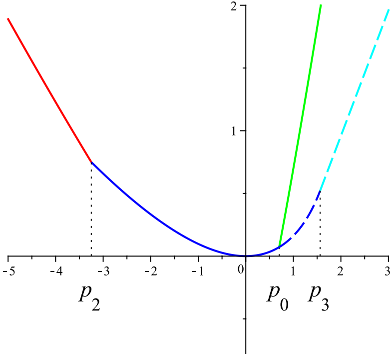
Average integral means spectrum for **bounded** whole-plane SLE.

Unbounded Whole-plane SLE

- In this case, [D., Nguyen, Nguyen & Zinsmeister '14] (see also [Loutsenko & Yermolayeva '13]) have shown the existence of a phase transition at $p_0 := \frac{(4+\kappa)^2 - 4 - 2\sqrt{2(4+\kappa)^2+4}}{16\kappa}$ to

$$\beta_1(p, 0; \kappa) := 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}.$$

(Related to SLE derivative exponents [Lawler, Schramm, Werner '01] and 'tip' quantum gravity ones [D.'03].)



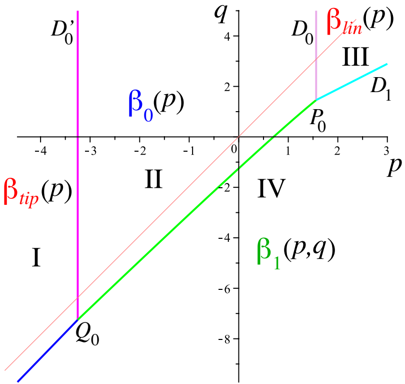
Remarks

- This β_1 spectrum for the **unbounded interior** case is proven in a finite p-interval above the transition point p_0.
- In the **bounded exterior** case, the original Beliaev-Smirnov proof has a **gap** for negative enough p, namely when

$$p \leq p_1 := -\frac{(4+\kappa)^2(8+\kappa)}{128},$$

a sub-/super solution to the PDE being **no longer positive**.
- This corresponds to a **phase transition** to a **'second tip' spectrum**, that requires a **new proof** [Beliaev, D. & Zinsmeister '17].

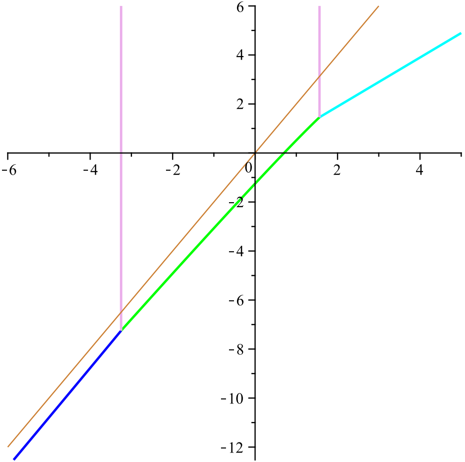
Phase Diagram

- 

- Phase transition lines: **green parabola** & **blue quartic**
-
$$\beta_1(p, q; \kappa) := 3p - 2q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)}.$$

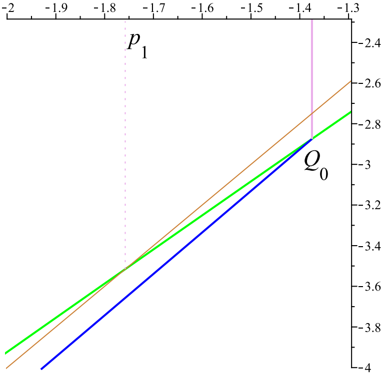
Bounded whole-plane SLE

- The **Beliaev-Smirnov line** $q = 2p$ does not intersect the **green parabola** part, and is asymptotically parallel to the **blue quartic**.

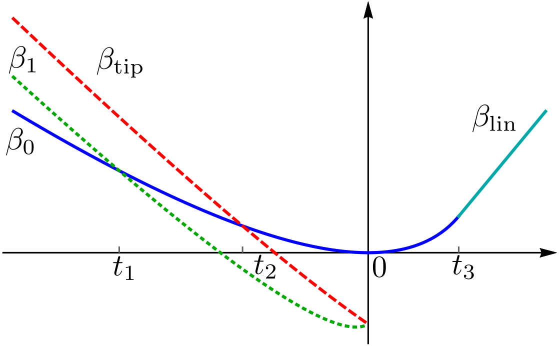


Subjacent β_1 spectrum

- Zooming below Q_0



The **bounded SLE line** intersects the **continuation** of the green parabola at p_1. For $p < p_1$, the β_1 **spectrum** dominates the **bulk one,** β_0, but not the **tip one,** $\beta_{\rm tip}$.



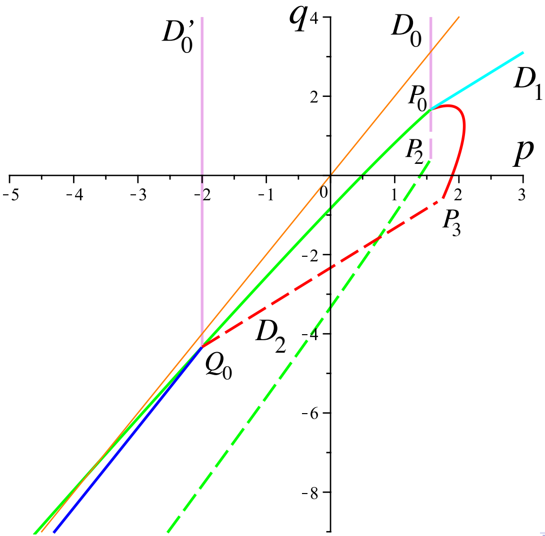
$$\beta_1(p, 2p; \kappa) := -p - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2\kappa p}.$$

'Second tip' spectrum

[Beliaev, D. & Zinsmeister '17]

Domain of Proof

- Domain where the form of the generalized integral means spectrum has been established:



Complex Generalized Moments

- Let f be a **whole-plane (inner)** SLE$_\kappa$, and $z \in \mathbb{D}$, $(p,q) \in \mathbb{C}^2$

$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}}\left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right),\ G(z,\bar{z}) := \mathbb{E}\left|(f'(z))^p\left(\frac{z}{f(z)}\right)^q\right|.$$

- The differential equation satisfied by F is the same,

$$\mathcal{P}(\partial)[F(z)] = \Big[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z - \frac{p}{(1-z)^2} + \frac{q}{1-z} + p - q\Big]F(z) = 0,$$

- while the partial differential equation satisfied by G becomes

$$\mathcal{P}(D)[G(z,\bar{z})] = \Big[-\frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2 - \frac{1+z}{1-z}z\partial_z - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\partial_{\bar{z}} - \frac{p}{(1-z)^2} - \frac{\bar{p}}{(1-\bar{z})^2} + \frac{q}{1-z} + \frac{\bar{q}}{1-\bar{z}} + 2\Re(p-q)\Big]G(z,\bar{z}) = 0.$$

Integrable Probability in $\mathbb{C}^2$

- Let f be a whole-plane (inner) SLE$_\kappa$, and $z \in \mathbb{D}$, $(p, q) \in \mathbb{C}^2$.

$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}} \left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right), \; G(z, \bar{z}) := \mathbb{E}\left|(f'(z))^{p} \left(\frac{z}{f(z)}\right)^{q}\right|.$$

- **Complex integrable parabola**, as parameterized in $\mathbb{C}^2$,

$$p(\gamma) := (2 + \frac{\kappa}{2})\gamma - \frac{\kappa}{2}\gamma^2, \;\; \gamma \in \mathbb{C},$$
$$q(\gamma) := (3 + \frac{\kappa}{2})\gamma - \kappa\gamma^2.$$

- **Theorem** [DHLZ '18$^+$]: If $p = p(\gamma)$ and $q = q(\gamma)$, then

$$F(z) = (1 - z)^{\gamma}, \; G(z, \bar{z}) = \frac{(1 - z)^{\gamma}(1 - \bar{z})^{\bar{\gamma}}}{(1 - z\bar{z})^{\kappa\gamma\bar{\gamma}/2}}.$$

Mixed Bulk Spectrum of SLE$_\kappa$

- Recall the SLE standard **bulk** spectrum :

$$\beta_0(p,\kappa) = -p + \frac{(4+\kappa)^2}{4\kappa} - \frac{(4+\kappa)}{4\kappa}\sqrt{(4+\kappa)^2 - 8\kappa p}$$

Packing spectrum

$$s_0(p,\kappa) := \beta_0(p,\kappa) - p + 1 = 1 + 2\tau - \sqrt{b\tau}$$

$$\tau := b - p := \frac{(4+\kappa)^2}{8\kappa} - p, \quad \tau \in \mathbb{R}^+$$

- **Complex moments**

$$s_0(p,\kappa) := \beta_0(p,\kappa) - \Re(p) + 1 = 1 + 2\tau' - \sqrt{b\tau'}$$

$$\tau' := \frac{1}{2}\left[\Re\,\tau + |\tau|\right], \quad \tau \in \mathbb{C}$$

- LQG, KPZ & Coulomb Gas [D. & Binder '02]
- C.G. [D. & Binder '08] [Belikov, Gruzberg & Rushkin '08]
- SLE (Expected) [Aru '15] [Binder & D. '18$^+$]

Mixed Tip Spectrum of SLE$_\kappa$

- SLE **tip** spectrum :

$$\beta_{\mathrm{tip}}(p,\kappa) = -p - 1 + \frac{1}{4}\left(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\right)$$

$$\beta_{\mathrm{tip}}(p,\kappa) - p = 2\tau - \sqrt{b\tau} + \sqrt{8\kappa^{-1}\tau} - 2(2+\kappa)\kappa^{-1}$$

$$\tau := b - p := \frac{(4+\kappa)^2}{8\kappa} - p, \quad \tau \in \mathbb{R}^+$$

- **Complex moments**

$$\beta_{\mathrm{tip}}(p,\kappa) - \Re(p) = 2\tau' - \sqrt{b\tau'} + \sqrt{8\kappa^{-1}\tau'} - 2(2+\kappa)\kappa^{-1}$$

$$\tau' := \frac{1}{2}\left[\Re\,\tau + |\tau|\right], \quad \tau \in \mathbb{C}$$

- LQG, KPZ, CG & Rev. Eng. [D., Sheffield, Sun & Viklund]
- SLE martingale [Binder & D. '18$^+$]

Complex Generalized Integral Means Spectrum

- The generalized **packing** spectrum is

$$\beta_1(p, q; \kappa) = 3p - 2q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)}$$

$$\begin{aligned} s_1(p, q; \kappa) &:= \beta_1(p, q; \kappa) - p + 1; \quad p, q \in \mathbb{R} \\ &= 2(p - q) + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)} \end{aligned}$$

- **Complex moments** [D., Ho, Le & Zinsmeister '18$^+$]

$$s_1(p, q; \kappa) := \beta_1(p, q; \kappa) - \Re(p) + 1; \quad p, q \in \mathbb{C}$$

$$= 2t + \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa t}$$

$$1 + 2\kappa t := \frac{1}{2}\left[1 + 2\kappa\Re(p - q) + |1 + 2\kappa(p - q)|\right].$$

Idea of proof

For $p, q \in \mathbb{R}$

$$F(z,\bar{z}) := \frac{1}{|z|^q} G(z,\bar{z}) = \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right)$$

satisfies

$$\begin{aligned}\mathcal{P}(D)[F(z,\bar{z})] = &-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 F - \frac{1+z}{1-z} z\partial F - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}F \\ &- p\left[\frac{1}{(1-z)^2} + \frac{1}{(1-\bar{z})^2} + \sigma - 1\right] F = 0,\end{aligned}$$

in terms of the parameter,

$$\sigma := q/p - 1.$$

- **Interior** w.-p. SLE: $q = 0$, $\sigma = -1$;
- **Exterior** w.-p. SLE: $q = 2p$, $\sigma = +1$.

Idea of proof

- Following Beliaev-Smirnov, one considers the action of $\mathcal{P}(D)$ on test functions of the form,

$$\psi(z,\bar{z}) := (1 - z\bar{z})^{-\beta} g(u),$$

  where g is a C^2 function of $u := |1-z|^2$, of the form $g(u) = u^{\gamma} g_0(u)$, $\gamma \in \mathbb{R}$.
- Let

$$\ell_\delta = \ell_\delta(z,\bar{z}) := [-\log(1 - z\bar{z})]^{\delta};$$

$$\frac{\mathcal{P}(D)(\psi\ell_\delta)}{\psi\ell_\delta} = \frac{\mathcal{P}(D)(\psi)}{\psi} - \frac{2\delta z\bar{z}}{u(-\log(1-z\bar{z}))}.$$

- Sub- and super-solutions $\psi\ell_\delta$ with ψ positive.

Boundary equation

- In terms of the quadratic polynomials,

$$\beta(\gamma) := \frac{\kappa}{2}\gamma^2 - C(p,\gamma),$$
$$C(p,\gamma) := -\kappa\frac{\gamma^2}{2} + \left(\frac{\kappa}{2} - 2\right)\gamma - p,$$
$$A^{\sigma} := A(p,q,\gamma) := -\frac{\kappa}{2}\gamma^2 + \gamma - \sigma p,$$

  the choice $\beta = \beta(\gamma)$ yields the **hypergeometric** boundary equation for $u = |1-z|^2 \in [0,4]$,

- $$A^{\sigma} g_0(u) + \left[\frac{\kappa}{2}(2-u) + (\kappa\gamma - 1)(4-u)\right] g_0'(u) + \frac{\kappa}{2}(4-u)u g_0''(u) = 0.$$

Solution space

- Define the **duality relation** $\gamma + \gamma' = \frac{4+\kappa}{2\kappa}$, s.t. $\beta(\gamma) = \beta(\gamma')$, and denote the zeroes of A^σ by $\gamma^\sigma_\pm := \frac{1}{\kappa}(1 \pm \sqrt{1 - 2\kappa\sigma p})$; g is the weighted combination of two hypergeometric functions,

$$g(u) := C_0\,(u/4)^{\gamma}\,{}_2F_1(a, b, c, u/4) - C_0'\,(u/4)^{\gamma'}\,{}_2F_1(a', b', c', u/4),$$

$$a = \gamma - \gamma^\sigma_+, \quad b = \gamma - \gamma^\sigma_-, \quad c = \frac{1}{2} + a + b,$$

$$a' = \frac{1}{2} - a, \quad b' = \frac{1}{2} - b, \quad c' = \frac{1}{2} + a' + b'.$$

- In this continuous γ-family of solutions, two play a **critical** role, that obtained for the choice $\gamma = \gamma_0$ such that $C(p, \gamma_0) = 0$, as in Beliaev-Smirnov '09, and the **power law solution** u^{γ_1}, as obtained for the choice $\gamma = \gamma_1 := \gamma^\sigma_+$.
- Either take $\psi = u^\gamma g_0(u)(1 - |z|^2)^{-\beta(\gamma)}$ for γ in the neighborhood of γ_1 and use **duality**;
- or take $\psi = \psi_0 + \psi_1$ with $\psi_0 := u^{\gamma_0} g_0(u)(1 - |z|^2)^{-\beta_0}$, $\psi_1 := u^{\gamma_1}(1 - |z|^2)^{-\beta_1}$.

Domain of Proof