# Annulus SLE partition functions and martingale-observables

Joint work with Nam-Gyu Kang, Hee-Joon Tak

Sung-Soo Byun
Seoul National University

Random Conformal Geometry and Related Fields

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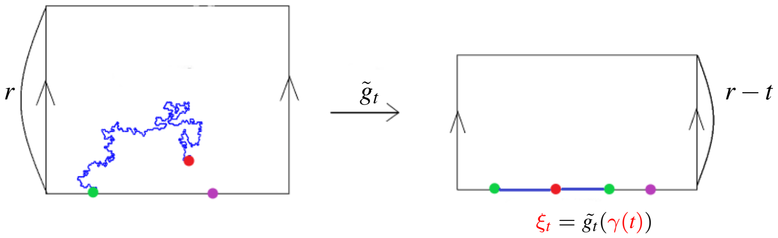
# Outline

1. *Annulus SLE partition functions*
   - Annulus SLE$(\kappa, \Lambda)$
   - Null-vector equation
2. *CFT of GFF in a doubly connected domain*
   - GFF with Dirichlet and Excursion-Reflected boundary conditions
   - Eguchi-Ooguri and Ward's equations
   - Coulomb gas formalism
3. *Connection to SLE theory*
   - One-leg operator and Insertion
   - Martingale-observables for annulus SLE
   - Screening
4. *Work in progress: Multiple SLEs in the annulus*

# Chordal type annulus SLE($\kappa, \Lambda$)




- Loewner Flow: $\partial_t \tilde{g}_t(z) = H(r - t, \tilde{g}_t(z) - \xi_t), \quad H(r, z) := 2\partial_z \log \Theta(r, z)$
- Driving process: For $\kappa > 0$, $\quad d\xi_t = \sqrt{\kappa} dB_t + \Lambda(r - t, \xi_t - \tilde{g}_t(q))dt.$
- Annulus SLE partition function $Z(r, x)$:

$$\Lambda(r, x) = \kappa \partial_x \log Z(r, x)$$

- Null-vector equation (Zhan):

$$\partial_r Z = \frac{\kappa}{2} Z'' + H Z' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) H' Z$$

# The null-vector equation

- Null-vector equation (Zhan):

$$\partial_r Z = \frac{\kappa}{2} Z'' + H Z' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) H' Z$$

- Lawler used Brownian loop measures to define annulus SLE($\kappa, \Lambda$) and proved that the SLE partition function (total mass) satisfies the null-vector equation.
- (B.-Kang-Tak) For each $\kappa > 0$,

$$Z(r,x) := \Theta(r,x)^{\frac{2}{\kappa}} \oint_\gamma \Theta(r, x-\zeta)^{-\frac{4}{\kappa}} \Theta(r,\zeta)^{-\frac{4}{\kappa}} \, d\zeta$$

solves the null-vector equation.

- *Examples*

| $\kappa$ | $Z(r,\cdot)$ |
|---|---|
| 4 | $\Theta^{-1/2}$ |
| 2 | $\Theta^{-1} H$ |
| 4/3 | $\Theta^{-3/2}\left(3H^2 - 2H' + 4\frac{\zeta_r(\pi)}{\pi}\right)$ |
| 1 | $\Theta^{-2}\left(4H^3 - 6HH' + H'' + 12\frac{\zeta_r(\pi)}{\pi} H\right)$ |

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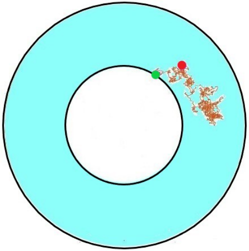


Figure: BM

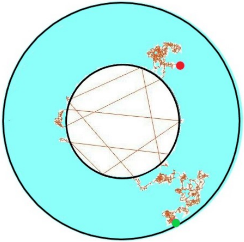


Figure: ERBM (Drenning, Lawler)

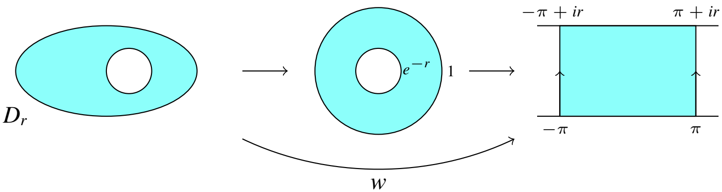
• In the cylinder $\mathcal{C}_r := \{z : 0 < \operatorname{Im} z < r\}/\langle z \mapsto z + 2\pi\rangle$, the Green's function $G_r$ is represented as

$$G_r(\zeta, z) = \begin{cases} \log \left| \dfrac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)} \right| - \dfrac{\operatorname{Im}\zeta \cdot \operatorname{Im} z}{r} & \text{for Dirichlet b.c.} \\[2ex] \log \left| \dfrac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)} \right| & \text{for ER b.c.} \end{cases}$$

# GFF in a doubly connected domain

## Dirichlet and ER boundary conditions

$\Phi_{(0)}$: GFF

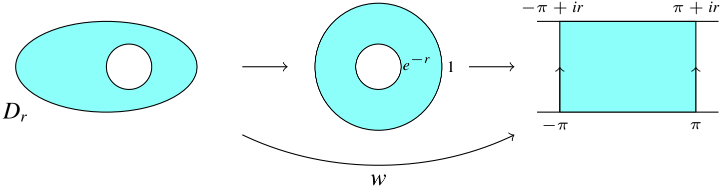



$$\boxed{\mathbf{E}[\Phi(\zeta)\Phi(z)] = 2G_{D_r}(\zeta, z) = 2G_r(w(\zeta), w(z))}$$

$$G_r(\zeta, z) = \begin{cases} \log \left| \dfrac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)} \right| - \dfrac{\operatorname{Im} \zeta \cdot \operatorname{Im} z}{r} & \text{for Dirichlet b.c.} \\ \log \left| \dfrac{\Theta(r, \zeta - \bar{z})}{\Theta(r, \zeta - z)} \right| & \text{for ER b.c.} \end{cases}$$

# Central charge modification of GFF

$\Phi_{(0)}$: GFF




- Fix a real parameter $b$ $(=\sqrt{\kappa/8} - \sqrt{2/\kappa})$ and define

$$\Phi_{(b)}(z) := \Phi_{(0)}(z) - 2b \arg w'(z).$$

- The central charge is given as $\boxed{c = 1 - 12b^2 = (6-\kappa)(3\kappa-8)/2\kappa}$
- The *Fock space fields* are obtained from the GFF by applying basic operations: 1. derivatives; 2. Wick's product $\odot$; 3. multiplying by scalar functions and taking linear combinations.

# OPE family of GFF

- Fix a real parameter $b$ (=$\sqrt{\kappa/8} - \sqrt{2/\kappa}$) and define

$$\Phi_{(b)}(z) := \Phi_{(0)}(z) - 2b \arg w'(z).$$

- The central charge is given as $c = 1 - 12b^2 = (6-\kappa)(3\kappa-8)/2\kappa$

- The *Fock space fields* are obtained from the GFF by applying basic operations: 1. derivatives; 2. Wick's product $\odot$; 3. multiplying by scalar functions and taking linear combinations.

- Operator product expansion (OPE) of two (holomorphic) fields $X(\zeta)$ and $Y(z)$ are given as

$$X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \quad \zeta \to z.$$

  In particular, the *OPE multiplication* $X * Y := C_0$.

- *OPE family* $\mathcal{F}_{(b)}$ of the $\Phi_{(b)}$:
  the algebra (over $\mathbb{C}$) spanned by the generators 1, mixed derivatives of $\Phi_{(b)}$, those of OPE exponentials $e^{*\alpha\Phi_{(b)}}$ $(\alpha \in \mathbb{C})$

# Ward's equation in doubly connected domain

*Stress energy tensor* $A_{(b)}$:

$$A_{(b)} := -\frac{1}{2}J_{(0)} \odot J_{(0)} + \Big(ib\partial - \mathbf{E}[J_{(b)}]\Big)J_{(0)}, \quad J_{(b)} = \partial\Phi_{(b)}.$$

## Theorem (B.-Kang-Tak)

For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{(b)}$, we have

$$2\mathbf{E}\left[A_{(b)}(\zeta)\mathcal{X}\right] = \left(\mathcal{L}^{+}_{v_\zeta} + \mathcal{L}^{-}_{v_{\bar\zeta}}\right)\mathbf{E}\left[\mathcal{X}\right] + \partial_r\mathbf{E}\left[\mathcal{X}\right],$$

where all fields are evaluated in the identity chart of $\mathcal{C}_r$ and the Loewner vector field $v_\zeta$ is given by

$$(v_\zeta \| \mathrm{id}_{\bar{\mathcal{C}}_r})(z) = H(r, \zeta - z) = 2\frac{\Theta'(r, \zeta - z)}{\Theta(r, \zeta - z)}.$$

**Cf.** On a complex torus of genus one, similar form of Ward's equation holds.

- Eguchi-Ooguri: *Conformal and current algebras on a general Riemann surface*, Nuclear Phys. B, 282(2):308-328, 1987.
- Kang-Makarov: *Calculus of conformal fields on a compact Riemann surface,* arXiv:1708.07361, 86 pp.

# Eguchi-Ooguri's type equation in a doubly connected domain

**Lemma**

For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{(b)}$, in $\mathcal{C}_r$,

$$\frac{1}{\pi}\oint_{[-\pi+ir,\pi+ir]} \mathbf{E}\left[A(\zeta)\mathcal{X}\right] d\zeta = \partial_r \mathbf{E}\left[\mathcal{X}\right].$$

**Ingredients of proof**

1. Heat equation of Jacobi theta function:

$$\partial_r \Theta(r,z) = \Theta(r,z)''.$$

2. Frobenius-Stickelberger's pseudo-addition theorem for Weierstrass $\zeta$-function:

$$\big(\zeta(z_1)+\zeta(z_2)+\zeta(z_3)\big)^2 + \zeta'(z_1)+\zeta'(z_2)+\zeta'(z_3) = 0, \qquad (z_1+z_2+z_3=0).$$

# Neutrality condition and multi-vertex field

- Given divisors $\boldsymbol{\sigma} = \sum_{j=1}^{n} \sigma_j \cdot z_j$, $\boldsymbol{\sigma}_* = \sum_{j=1}^{n} \sigma_{j*} \cdot z_j$, we set

$$\Phi_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] := \sum_{j=1}^{n} \sigma_j \Phi_{(b)}^{+}(z_j) - \sigma_{*j} \Phi_{(b)}^{-}(z_j),$$

  where $\Phi_{(b)} = \Phi_{(b)}^{+} + \Phi_{(b)}^{-}$, $\Phi_{(b)}^{-} = \overline{\Phi_{(b)}^{+}}$.

- Then $\Phi_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is a well-defined Fock space field if and only if the following *neutrality condition* ($\mathrm{NC}_0$) holds:

$$\boxed{\sum_{j=1}^{n} (\sigma_j + \sigma_{*j}) = 0}$$

- We define the *multi-vertex field* $\mathcal{O}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] \equiv \mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ by

$$\mathcal{O}_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] = C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*] e^{\odot i \Phi_{(0)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]}$$

  where $C_{(b)}[\boldsymbol{\sigma}, \boldsymbol{\sigma}_*]$ is *Coulomb gas correlation function*.

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# One-leg operator

- We choose real parameters $a$ and $b$ in terms of SLE parameter $\kappa$ as

$$a = \sqrt{2/\kappa}, \qquad b = \sqrt{\kappa/8} - \sqrt{2/\kappa}.$$

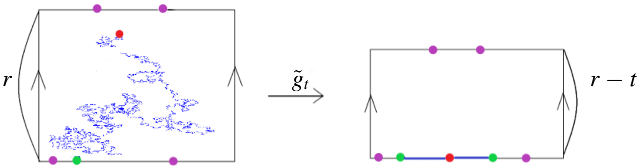
- Given a divisor $\boldsymbol{\beta} = \sum_{j=1}^{N} \beta_j \cdot q_j$ define the one-leg operator $\Psi \equiv \Psi_{\boldsymbol{\beta}}$ by

$$\Psi_{\boldsymbol{\beta}}(p, \boldsymbol{q}) := \mathcal{O}\left[a \cdot p + \boldsymbol{\beta},\, \mathbf{0}\right].$$

- Now we consider SLE$(\kappa, \Lambda)$, where $\Lambda$ is given by

$$\boxed{\Lambda(r, p, \boldsymbol{q}) := \kappa\, \partial_\xi\Big|_{\xi=p} \log \mathbf{E}\left[\Psi(\xi, \boldsymbol{q})\right]}$$

i.e., $$d\xi_t = \sqrt{\kappa}\, dB_t + \Lambda(r - t, \xi_t - \tilde{g}_t(q_1), \cdots, \xi_t - \tilde{g}_t(q_N))dt.$$

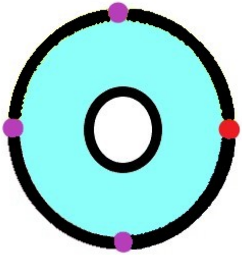


# Insertion

Using $\Psi$ as an insertion field, set

$$\widehat{\mathbf{E}}[\mathcal{X}] := \frac{\mathbf{E}[\Psi(p,q)\mathcal{X}]}{\mathbf{E}\left[\Psi(p,q)\right]} = \mathbf{E}\left[e^{\odot ia\Phi^+_{(0)}(p)+i\sum\beta_j\Phi^+_{(0)}(q_j)}\mathcal{X}\right].$$

**Example.** Suppose that $q_j$'s are on the outer boundary component.



- In the cylinder $\mathcal{C}_r$,

$$\mathbf{E}[\Phi_{(b)}](z) = 0$$

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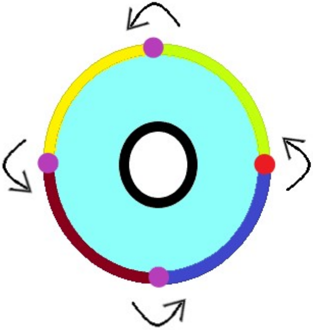
**Example.** Suppose that $q_j$'s are on the outer boundary component.

- In the cylinder $\mathcal{C}_r$,

$$\widehat{\mathbf{E}}[\Phi_{(b)}](z) = 2a\arg\Theta(r,p-z) + \sum 2\beta_j\arg\Theta(r,q_j-z).$$

- $\widehat{\mathbf{E}}[\Phi_{(b)}]$ has piecewise Dirichlet boundary condition with jump $2a\pi$ at $p$, $2\pi\beta_j$ at $q_j$ and by $\mathrm{NC}_0$ all jumps add up to 0.
- Izyurov-Kytölä: *Hadamard's formula and couplings of SLEs with free field*. Probab. Theory Related Fields, 155(1-2):35-69, 2013.



# Martingale-observables for annulus SLE

By definition, a non-random field $M$ is a *martingale-observable* for annulus SLE$(\kappa, \Lambda)$ if for any $z_1, \cdots, z_n$, the process

$$M_t(z_1, \cdots, z_n) := \left(M \| \tilde{w}_t^{-1}\right)(z_1, \cdots, z_n), \quad \tilde{w}_t = \tilde{g}_t - \xi_t$$

is a local martingale on the SLE probability space.

**Theorem (B-Kang-Tak)**

For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{(b)}$ of $\Phi_{(b)}$, the non-random fields $M = \widehat{\mathbf{E}}\mathcal{X}$ are martingale-observables for SLE$(\kappa, \Lambda)$.

**Idea of proof**

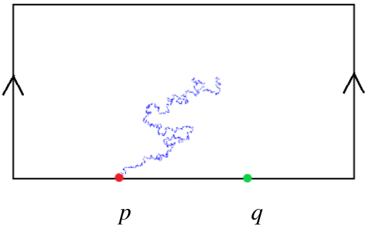
Ward's equation + level 2 degeneracy equation for $\Psi$
$\Rightarrow$ BPZ-Cardy equation $\Leftrightarrow$ $M_t$ is driftless.

# Screening

**Goal** To find explicit solutions of Zhan's PDE for $Z$

**Idea**: $Z \Longleftrightarrow \mathbf{E}\Psi$

Consider an chordal type annulus SLE$(\kappa, \Lambda)$ in cylinder from $p$ to $q$.



# Screening

**Goal** To find explicit solutions of Zhan's PDE for $Z$

**Idea**: $Z \Longleftrightarrow \mathbf{E}\Psi$

- The conformal dim $\lambda_z$ at $z$ with charge $\sigma$ is given as $\lambda_z = \sigma^2/2 - \sigma b$.
- To satisfy level-2 degeneracy eq., $\lambda_p = a^2/2 - ab$.

$$\Psi(p,q) = \mathcal{O}[? \cdot p + ? \cdot q]$$

| | Candidate 1 | Candidate 2 |
|---|---|---|
| $p$ | $a$ | $2b - a$ |

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- Due to $NC_0$, total sum of charge should vanish.

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| $q$ | $a$ | $2b - a$ |

Works in $\mathbb{H}$ with *background charge* $2b$ at $q$.

**Cf.** $Z_{\mathbb{H}}(x) = x^{1-6/\kappa}$

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~~$\Psi(p,q) = \mathcal{O}[? \cdot p + ? \cdot q]$~~

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Does not Work in the annulus

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- To satisfy level-2 degeneracy eq., $\lambda_p = a^2/2 - ab$.
- Due to reversibility, $\lambda_p = \lambda_q$.
- Due to $NC_0$, total sum of charge should vanish.
- Consider additional node $\zeta$ s.t. $\lambda_\zeta = 1$ and integrate out $\zeta$ along the proper contour $\gamma$.

$$\Psi(p,q) = \oint_\gamma \mathcal{O}[? \cdot p + ? \cdot q + ? \cdot \zeta] d\zeta$$

| | Candidate 1 | Candidate 2 |
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| $p$ | $a$ | $2b - a$ |
| $q$ | $a$ | $2b - a$ |
| $\zeta$ | $2a + 2b$ | $-2a$ |

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$$\Psi(p,q) = \oint_\gamma \mathcal{O}[a \cdot p + a \cdot q - 2a \cdot \zeta] d\zeta$$

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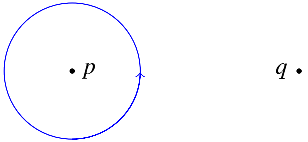
# Screening: Integration contour
## ER boundary conditions

$$\partial_r Z = \frac{\kappa}{2} Z'' + H Z' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) H' Z$$

$$Z(r, p-q) := \Theta(r, p-q)^{\frac{2}{\kappa}} \oint_\gamma \Theta(r, p-\zeta)^{-\frac{4}{\kappa}} \Theta(r, \zeta - q)^{-\frac{4}{\kappa}} d\zeta$$

**Examples of integration contour $\gamma$**

▷ If $4/\kappa \in \mathbb{Z}_+$,

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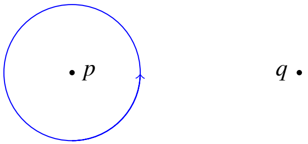
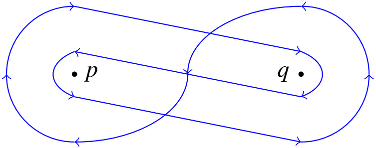



▷ For general $\kappa$,



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$$Z(r, p-q) := \Theta(r, p-q)^{\frac{2}{\kappa}} \oint_{\gamma} \Theta(r, p-\zeta)^{-\frac{4}{\kappa}} \Theta(r, \zeta - q)^{-\frac{4}{\kappa}} d\zeta$$

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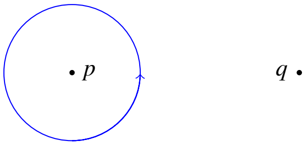
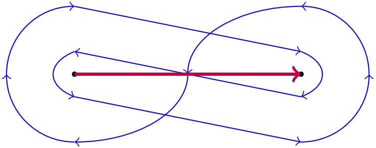



▷ For general $\kappa$, (if $\kappa > 4$)



**Theorem (B-Kang-Tak)**

For $p, q \in \mathbb{R}$, let $\gamma$ be a Pochhammer contour entwining $p$ and $q$. Then

$$Z(r, p-q) = \mathbf{E}\Psi(p,q) := C(\kappa) \oint_\gamma \mathbf{E}\mathcal{O}[a \cdot p + a \cdot q - 2a \cdot \zeta] d\zeta$$

is a non-trivial solution of the null-vector equation

$$\partial_r Z = \frac{\kappa}{2} Z'' + H Z' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) H' Z.$$

Moreover, for any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_{(b)}$, a non-random field

$$M = \widehat{\mathbf{E}}\mathcal{X} := \frac{\mathbf{E}\Psi(p,q)\mathcal{X}}{\mathbf{E}\Psi(p,q)}$$

is a martingale-observable for chordal type SLE($\kappa, \Lambda$).

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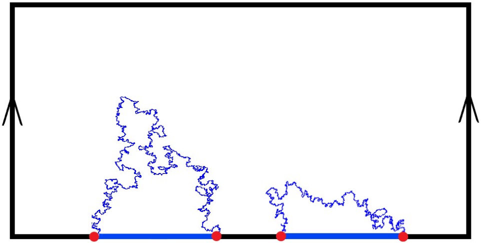
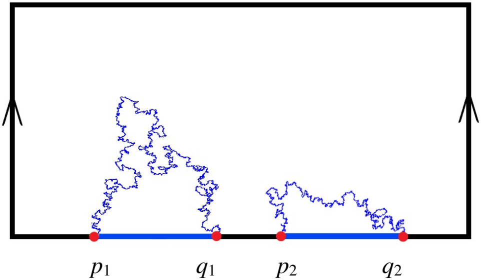
   - Screening
4. *Work in progress: Multiple SLEs in the annulus*



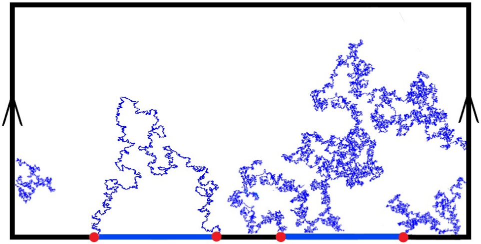

| | Candidate 1 | Candidate 2 |
|---|---|---|
| $p_1$ | $a$ | $2b-a$ |
| $q_1$ | $a$ | $2b-a$ |
| $p_2$ | $a$ | $2b-a$ |
| $q_2$ | $a$ | $2b-a$ |
| $\zeta_1$ | $2a+2b$ | $-2a$ |
| $\zeta_2$ | $2a+2b$ | $-2a$ |

$$Z = \oint \oint \mathbf{E}\mathcal{O}[? \cdot p_1 + ? \cdot q_1 + ? \cdot p_2 + ? \cdot q_2 + ? \cdot \zeta_1 + ? \cdot \zeta_2]\, d\zeta_1 d\zeta_2$$



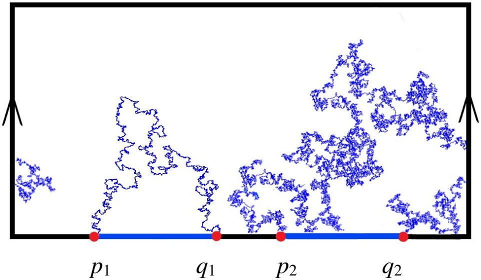

| | Candidate 1 | Candidate 2 |
|---|---|---|
| $p_1$ | $a$ | $2b - a$ |
| $q_1$ | $a$ | $2b - a$ |
| $p_2$ | $a$ | $2b - a$ |
| $q_2$ | $a$ | $2b - a$ |
| $\zeta_1$ | $2a + 2b$ | $-2a$ |
| $\zeta_2$ | $2a + 2b$ | $-2a$ |

$$Z = \oint \oint \mathbf{E}\mathcal{O}[a \cdot p_1 + a \cdot q_1 + a \cdot p_2 + a \cdot q_2 - 2a \cdot \zeta_1 - 2a \cdot \zeta_2]\, d\zeta_1 d\zeta_2$$




| | Candidate 1 | Candidate 2 |
|---|---|---|
| $p_1$ | $a$ | $2b - a$ |
| $q_1$ | $a$ | $2b - a$ |
| $p_2$ | $a + 3b$ | $-a - b$ |
| $q_2$ | $a + 3b$ | $-a - b$ |
| $\zeta_1$ | $2a + 2b$ | $-2a$ |
| $\zeta_2$ | $2a + 2b$ | $-2a$ |

$$Z = \oint \oint \mathbf{E}\mathcal{O}[? \cdot p_1 + ? \cdot q_1 + ? \cdot p_2 + ? \cdot q_2 + ? \cdot \zeta_1 + ? \cdot \zeta_2] \, d\zeta_1 d\zeta_2$$




| | Candidate 1 | Candidate 2 |
|---|---|---|
| $p_1$ | $a$ | $2b - a$ |
| $q_1$ | $a$ | $2b - a$ |
| $p_2$ | $a + 3b$ | $-a - b$ |
| $q_2$ | $a + 3b$ | $-a - b$ |
| $\zeta_1$ | $2a + 2b$ | $-2a$ |
| $\zeta_2$ | $2a + 2b$ | $-2a$ |

$$Z = \oint \oint \mathbf{E}\mathcal{O}[a \cdot p_1 + a \cdot q_1 - (a+b) \cdot p_2 - (a+b) \cdot q_2 - 2a \cdot \zeta_1 + (2a+2b) \cdot \zeta_2] \, d\zeta_1 d\zeta_2$$

# Thank you for your attention!